Lecture Notes on Classical Logic

15-317: Constructive Logic
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1 Introduction

In this lecture, we present a judgmental formulation of classical logic, giving a proof theory in terms of judgments A *ctrue* (A is classically true), A *false* (A is classically false), and # (Contradiction). We will see how to prove propositions classically true or false, and that most of these proofs involve proof by contradiction.

Classical logic is not, however, entirely separate from constructive logic, and we will make this precise by showing that a constructively provable proposition is also classically provable, and giving a way to translate classically provable propositions into constructively provable ones, via double negation.

Finally, we will present a system of proof terms for this classical proof theory. Proof terms for the judgment A *true* will be functional expressions as usual, while those for the judgment A *false* will be *continuations*. An expression and a continuation of the same type A can be combined to yield a contradiction. This can be thought of similar to a goto, where we jump to some continuation with a given expression as argument.

2 What is Classical Logic?

We have briefly discussed classical logic at several points in this course, primarily with a focus on defining what makes a logic constructive. One main theme in these discussions is that constructive logic does not believe that $A \vee \neg A$ *true* is provable for all A, while classical logic does. Thus, one way to present classical logic is to simply add to our existing rules for constructive logic the following axiom:

$$\frac{}{A \vee \neg A\ \mathit{true}}\ LEM$$

In fact, there are a variety of related rules we could add to our existing system to get classical logic, including proof by contradiction, double-negation elimination, and Peirce's law:

$$\frac{\begin{array}{c}\overline{\neg A\ \mathit{true}}^{\,u}\\ \vdots \\ F\ \mathit{true}\end{array}}{A\ \mathit{true}}\ PBC^u \qquad \frac{\neg\neg A\ \mathit{true}}{A\ \mathit{true}}\ DNE \qquad \frac{}{((A \supset B) \supset A) \supset A\ \mathit{true}}\ Peirce$$

Each of these can be used to prove the others. For instance, we may prove LEM using PBC^u as follows:

$$\dfrac{\dfrac{\overline{\neg(A \vee \neg A)\ \mathit{true}}^{\,u} \quad \dfrac{\dfrac{\dfrac{\overline{\neg(A \vee \neg A)\ \mathit{true}}^{\,u} \quad \dfrac{\overline{A\ \mathit{true}}^{\,v}}{A \vee \neg A\ \mathit{true}}\ \vee I_1}{F\ \mathit{true}}\ \supset E}{\neg A\ \mathit{true}}\ \supset I^v}{A \vee \neg A\ \mathit{true}}\ \vee I_2}{F\ \mathit{true}}\ \supset E}{A \vee \neg A\ \mathit{true}}\ PBC^u$$

One interesting observation about this proof is that the only place we use the classical rule PBC^u is at the start (or at the end, depending on your perspective on proofs). This hints at the idea (which we will make more precise in section 4.2) that every classically true proposition A can be translated into a constructively true proposition B such that A and B are classically equivalent. Of course, if A is not constructively true, then A and B will not be constructively equivalent.

While adding any of these rules to our logic will make it classical, all four violate a principle we have set out for rules: a rule should make use of at most one connective. As is, if we take PBC^u to define classical logic, does this mean that a logic must have falsehood and implication in order to be classical? If we take LEM instead, does that then mean that we must have negation and disjunction? Avoiding all of these issues, we will seek to characterize classical logic without using any connectives at all, instead working with some new judgements.

3 A Classical Proof Theory

We begin by examining the proposed rule PBC^u:

$$\dfrac{\dfrac{}{\neg A\ \mathit{true}}\ u \quad \vdots \quad F\ \mathit{true}}{A\ \mathit{true}}\ PBC^u*$$

This rule has three components, each of which will inform one of our new judgements. First, and most simply, the conclusion is that A is true. We will use a new judgement A *ctrue* to denote that A is *classically* true, in order to distinguish it from the constructive truth of A *true*.[1]

Now looking at the premise of the rule, we see a hypothetical judgement, saying that F *true* under the hypothesis $\neg A$ *true*. Thinking classically, we may note that $\neg A$ is true exactly when A is false, and this motivates our second new judgement A *false*, stating that the proposition A is classically false. Finally, F *true* denotes a contradiction, and we will introduce a new judgement # for contradiction to capture this. Using these new judgements, we can write PBC^u, now using no connectives:

$$\dfrac{\dfrac{}{A\ \mathit{false}}\ u \quad \vdots \quad \#}{A\ \mathit{ctrue}}\ PBC^u$$

So far, this looks good — we have put the classical rule PBC^u solely in terms of judgements. But what needs to change for our other rules? As is, we have no way to use a hypothesis A *false*, no rule that allows us to

[1] If it is clear that we are working in classical logic, we may just write A *true* for simplicity.

conclude contradiction #, and, for that matter, no rules that allow us to work with the new judgement A *ctrue*.

One approach would be to take all of our existing rules of constructive logic and then see what rules we need to add to get a working system. However, this will end up with many redundant rules, and so we will instead start from the ground up.[2]

For simplicity of presentation, and because it will be useful later on, we will use a variant of the notation for natural deduction with contexts. There, the usual judgement was of the form $\Gamma \vdash A\ true$, where Γ is a set of hypotheses of the form $B\ true$. We can already see from PBC^u that we will have some assumptions of the form B *false* as well as $B\ true$, and so, anticipating that, we will work with two contexts, writing $\Gamma\,;\Delta \vdash J$. Here, Γ consists of hypotheses $B\ ctrue$, Δ consists of hypotheses B *false*, and J may be A *ctrue*, A *false*, or #.

3.1 Contradiction

Rewritten in this notation, the rule PBC^u from before becomes

$$\frac{\Gamma\,;\Delta, A\ \mathit{false} \vdash \#}{\Gamma\,;\Delta \vdash A\ \mathit{ctrue}}\ T\#$$

If using A *false* as a hypothesis leads to a contradiction (along with the existing hypotheses Γ and Δ), then we may conclude A *ctrue*. Of course, there is no reason to restrict proof by contradiction to only proving truth — we can give a dual form of this rule to prove propositions false as well:

$$\frac{\Gamma, A\ \mathit{ctrue}\,;\Delta \vdash \#}{\Gamma\,;\Delta \vdash A\ \mathit{false}}\ F\#$$

These two rules allow us to use proof by contradiction to conclude that a proposition is true or false.

A key question then arises: How do we prove a contradiction? There are several approaches we could take here, but we will use the simplest, saying that we reach a contradiction if we can prove both A *ctrue* and A *false* for some proposition A — clearly, a proposition being both true and false is unreasonable. Turning this intuition into a rule, we get

$$\frac{\Gamma\,;\Delta \vdash A\ \mathit{ctrue} \quad \Gamma\,;\Delta \vdash A\ \mathit{false}}{\Gamma\,;\Delta \vdash \#}\ \#I$$

[2] The system we present here is due to William Lovas [LC06].

Observe that if we think of A *ctrue* as A *true*, A *false* as $\neg A$ *true*, and # as F *true*, then: $T\ \#$ is the PBC^u rule of classical logic, $F\ \#$ is a special case of $\supset I$, and $\#I$ is a special case of $\supset E$. These new rules are just different expressions of ideas we already have.[3]

Now, with the rules surrounding contradiction worked out, we move on to look at how we can actually prove a proposition true or false.

3.2 Truth and Falsity

Conjunction We will begin by looking at our usual connectives. We already know what it means for $A \wedge B$ to be true — precisely that both A and B are true. This yields the $\wedge T$ rule, analogous to the constructive $\wedge I$ rule:

$$\frac{\Gamma \,;\, \Delta \vdash A\ \mathit{ctrue} \quad \Gamma \,;\, \Delta \vdash B\ \mathit{ctrue}}{\Gamma \,;\, \Delta \vdash A \wedge B\ \mathit{ctrue}}\ \wedge T$$

Now, however, not only do we need to have rules to prove that $A \wedge B$ is true, we also need to be able to prove it false. Again following our intuition for classical logic, we might conclude that if A is false, then $A \wedge B$ is false, and likewise if B is false. This leads us to the following rules:

$$\frac{\Gamma \,;\, \Delta \vdash A\ \mathit{false}}{\Gamma \,;\, \Delta \vdash A \wedge B\ \mathit{false}}\ \wedge F_1 \qquad \frac{\Gamma \,;\, \Delta \vdash B\ \mathit{false}}{\Gamma \,;\, \Delta \vdash A \wedge B\ \mathit{false}}\ \wedge F_2$$

At this point, we can prove propositions of the form $A \wedge B$ either true or false, and we can nearly write some simple proofs. The only thing that remains is to give the *hypothesis* rules

$$\frac{}{\Gamma, A\ \mathit{ctrue} \,;\, \Delta \vdash A\ \mathit{ctrue}}\ hypT \qquad \frac{}{\Gamma \,;\, \Delta, A\ \mathit{false} \vdash A\ \mathit{false}}\ hypF$$

allowing us to use true or false hypotheses in proofs.

[3] We will see a bit later on that identifying A *ctrue* and A *true* is not quite the right approach, but it serves as good intuition here.

As an example, we can now show that $\wedge$ is commutative:

$$\dfrac{\dfrac{}{A \wedge B\ \mathit{ctrue}\ ;\ B\ \mathit{false} \vdash A \wedge B\ \mathit{ctrue}}\ hypT \quad \dfrac{\dfrac{}{A \wedge B\ \mathit{ctrue}\ ;\ B\ \mathit{false} \vdash B\ \mathit{false}}\ hypF}{A \wedge B\ \mathit{ctrue}\ ;\ B\ \mathit{false} \vdash A \wedge B\ \mathit{false}}\ \wedge F_2}{\dfrac{A \wedge B\ \mathit{ctrue}\ ;\ B\ \mathit{false} \vdash \#}{A \wedge B\ \mathit{ctrue}\ ;\ \cdot \vdash B\ \mathit{ctrue}}\ T\#} \#I$$

$$\dfrac{A \wedge B\ \mathit{ctrue}\ ;\ \cdot \vdash B\ \mathit{ctrue} \qquad \vdots}{A \wedge B\ \mathit{ctrue}\ ;\ \cdot \vdash B \wedge A\ \mathit{ctrue}}\ \wedge T$$

We only show the left branch of $\wedge T$ here, as the proof is already rather large, but the right branch is similar.[4]

The rules for other connectives are similar, with the rule for proving a connective true matching the constructive introduction rule, and the rule for proving it false coming from a classical understanding of the connective.

Disjunction For disjunction, for instance, we can prove $A \vee B$ false only when both A and B are false, giving us the following set of rules:

$$\dfrac{\Gamma\ ;\ \Delta \vdash A\ \mathit{ctrue}}{\Gamma\ ;\ \Delta \vdash A \vee B\ \mathit{ctrue}}\ \vee T_1 \qquad \dfrac{\Gamma\ ;\ \Delta \vdash B\ \mathit{ctrue}}{\Gamma\ ;\ \Delta \vdash A \vee B\ \mathit{ctrue}}\ \vee T_2 \qquad \dfrac{\Gamma\ ;\ \Delta \vdash A\ \mathit{false} \quad \Gamma\ ;\ \Delta \vdash B\ \mathit{false}}{\Gamma\ ;\ \Delta \vdash A \vee B\ \mathit{false}}\ \vee F$$

Negation Negation (which we will treat as a primitive connective here, rather than derived from implication and falsity) is even simpler — $\neg A$ is true when A is false, and false when A is true. Again, we can turn this simple intuition into correspondingly simple rules:

$$\dfrac{\Gamma\ ;\ \Delta \vdash A\ \mathit{false}}{\Gamma\ ;\ \Delta \vdash \neg A\ \mathit{ctrue}}\ \neg T \qquad \dfrac{\Gamma\ ;\ \Delta \vdash A\ \mathit{ctrue}}{\Gamma\ ;\ \Delta \vdash \neg A\ \mathit{false}}\ \neg F$$

With negation and disjunction both having rules, we can now prove the

[4] One might observe that here, we used proof by contradiction in order to show that $A \wedge B\ \mathit{ctrue}\ ;\ \cdot \vdash B\ \mathit{ctrue}$ — something that is almost immediate if we use $\wedge E_2$. In this system, however, we are not using elimination rules — we will see in section 4.1 that they would be redundant, and as we have already observed on several occasions, redundant rules make proof search harder.

law of the excluded middle in this classical system:

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      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{}{A\ \mathit{ctrue}\ ;\ A \vee \neg A\ \mathit{false} \vdash A\ \mathit{ctrue}}\ \mathit{hypT}
          }{A\ \mathit{ctrue}\ ;\ A \vee \neg A\ \mathit{false} \vdash A \vee \neg A\ \mathit{ctrue}}\ \vee T_1
          \qquad
          \dfrac{}{A\ \mathit{ctrue}\ ;\ A \vee \neg A\ \mathit{false} \vdash A \vee \neg A\ \mathit{false}}\ \mathit{hypF}
        }{A\ \mathit{ctrue}\ ;\ A \vee \neg A\ \mathit{false} \vdash \#}\ \#I
      }{\cdot\ ;\ A \vee \neg A\ \mathit{false} \vdash A\ \mathit{false}}\ F\#
    }{\cdot\ ;\ A \vee \neg A\ \mathit{false} \vdash \neg A\ \mathit{ctrue}}\ \neg T
  }{\cdot\ ;\ A \vee \neg A\ \mathit{false} \vdash A \vee \neg A\ \mathit{ctrue}}\ \vee T_2
  \qquad \vdots
}{
  \dfrac{\cdot\ ;\ A \vee \neg A\ \mathit{false} \vdash \#}{\cdot\ ;\ \cdot \vdash A \vee \neg A\ \mathit{ctrue}}\ T\#
}\ \#I
$$

The right branch can be closed simply with

$$
\dfrac{}{\cdot\ ;\ A \vee \neg A\ \mathit{false} \vdash A \vee \neg A\ \mathit{false}}\ \mathit{hypF}
$$

Implication A few connectives remain to address. Implication, just as the others, can be proven true in the same way classically as constructively. To prove an implication $A \supset B$ false, we again consider the cases for whether A and B are true or false. If B is true, then the implication is of course true. Likewise, if A is false, then the implication is true (vacuously). We see, therefore, that $A \supset B$ is false exactly when A is true and B is false. Turning this intuition into rules, we get

$$
\dfrac{\Gamma, A\ \mathit{ctrue}\ ;\ \Delta \vdash B\ \mathit{ctrue}}{\Gamma\ ;\ \Delta \vdash A \supset B\ \mathit{ctrue}} \supset T
\qquad\qquad
\dfrac{\Gamma\ ;\ \Delta \vdash A\ \mathit{ctrue} \quad \Gamma\ ;\ \Delta \vdash B\ \mathit{false}}{\Gamma\ ;\ \Delta \vdash A \supset B\ \mathit{false}} \supset F
$$

Truth and Falsity Finally, we come to the simplest connectives of truth and falsity. We can always prove T *ctrue* and F *false*, but there is no reasonable way to prove T *false* or F *ctrue* — either would immediately lead us to a proof of #, showing that our system would be inconsistent. The rules for these connectives are therefore:

$$
\dfrac{}{\Gamma\ ;\ \Delta \vdash T\ \mathit{ctrue}}\ TT
\qquad\qquad
\dfrac{}{\Gamma\ ;\ \Delta \vdash F\ \mathit{false}}\ FF
$$

At this point, we have all of the necessary rules for classical logic (See appendix A for a figure containing all of the rules together, as well as for a figure of the context-free forms of the rules). Now, we will move on to show how this system relates to the system of natural deduction for constructive logic that we are more familiar with.

4 Relating Classical and Constructive Logic

There are two key questions when we want to relate different logical systems, as we have seen when looking at verifications or the sequent calculus:

- Can system A prove everything that system B can?
- Can system B prove everything that system A can?

We already know in this case that there are some classically true propositions which are not constructively true (e.g. $A \vee \neg A$), and so the main goal remaining is then to show that every constructively true proposition is also classically true. We can see that the $\_T$ rules correspond exactly to the $\_I$ rules of classical natural deduction, so all that is necessary to show that every constructively true proposition is also classically true is to show that the elimination rules are classically valid.

4.1 Reconstructing elimination rules

An important first step is to consider how we translate an elimination rule into this system. Let us begin by considering $\wedge E_1$:

$$\frac{\Gamma \vdash A \wedge B \; \mathit{true}}{\Gamma \vdash A \; \mathit{true}} \wedge E_1$$

A natural translation of this would be

$$\frac{\Gamma \, ; \cdot \vdash A \wedge B \; \mathit{ctrue}}{\Gamma \, ; \cdot \vdash A \; \mathit{ctrue}} \; ?$$

However, if we look at the context-free form of the rule:

$$\frac{A \wedge B \; \mathit{true}}{A \; \mathit{true}} \wedge E_1$$

we see that the context-free classical version of this rule might be

$$\frac{A \wedge B\ \mathit{ctrue}}{A\ \mathit{ctrue}}\ ?$$

This corresponds to allowing arbitrary hypotheses to be used to prove $A \wedge B\ \mathit{ctrue}$, as long as we were entitled to use them to prove $A\ \mathit{ctrue}$, and so a better translation is

$$\frac{\Gamma\ ;\ \Delta \vdash A \wedge B\ \mathit{ctrue}}{\Gamma\ ;\ \Delta \vdash A\ \mathit{ctrue}}$$

Now, we want to give a derivation of this rule, meaning we want a derivation of the conclusion of the rule, in which we are allowed to assume the premise.

$$\begin{array}{c} \vdots \\ \Gamma\ ;\ \Delta \vdash A\ \mathit{ctrue} \end{array}$$

At this point, most of our rules do not apply — Γ, Δ, and A are all arbitrary, and cannot be further broken down without additional assumptions about their structure. We are left with only one rule to use: $T\,\#$.

$$\frac{\begin{array}{c} \vdots \\ \Gamma\ ;\ \Delta, A\ \mathit{false} \vdash \# \end{array}}{\Gamma\ ;\ \Delta \vdash A\ \mathit{ctrue}}\ T\,\#$$

We now reach the most uncertain part of any proof by contradiction: What proposition is contradictory? We observe that we have A *false* in the context, so we can easily prove A *false*, but we have no immediate way to prove A *ctrue*, so this is of limited use. However, we can use A *false* to prove $A \wedge B$ *false*, using $\wedge F_1$, and we are entitled to assume that $\Gamma\ ;\ \Delta \vdash A \wedge B\ \mathit{ctrue}$. We will therefore try to prove that $A \wedge B$ is contradictory (both true and false).

$$\frac{\dfrac{\begin{array}{c} \vdots \\ \Gamma\ ;\ \Delta, A\ \mathit{false} \vdash A \wedge B\ \mathit{ctrue} \end{array} \quad \dfrac{\dfrac{}{\Gamma\ ;\ \Delta, A\ \mathit{false} \vdash A\ \mathit{false}}\ \mathit{hypF}}{\Gamma\ ;\ \Delta, A\ \mathit{false} \vdash A \wedge B\ \mathit{false}}\ \wedge F_1}{\Gamma\ ;\ \Delta, A\ \mathit{false} \vdash \#}\ \#I}{\Gamma\ ;\ \Delta \vdash A\ \mathit{ctrue}}\ T\,\#$$

Now, our remaining proof goal, $\Gamma\ ;\ \Delta, A\ \mathit{false} \vdash A \wedge B\ \mathit{ctrue}$, is nearly our given premise $\Gamma\ ;\ \Delta \vdash A \wedge B\ \mathit{ctrue}$, but has an additional hypothesis A *false* available. If we assume weakening (allowing us to add extra hypotheses without affecting provability), then we are done:

$$\dfrac{\dfrac{\Gamma \,;\, \Delta \vdash A \wedge B \textit{ ctrue}}{\Gamma \,;\, \Delta, A \textit{ false} \vdash A \wedge B \textit{ ctrue}}\ \mathit{weaken} \qquad \dfrac{\dfrac{}{\Gamma \,;\, \Delta, A \textit{ false} \vdash A \textit{ false}}\ \mathit{hypF}}{\Gamma \,;\, \Delta, A \textit{ false} \vdash A \wedge B \textit{ false}}\ \wedge F_1}{\dfrac{\Gamma \,;\, \Delta, A \textit{ false} \vdash \#}{\Gamma \,;\, \Delta \vdash A \textit{ ctrue}}\ T\#}\ \#I$$

Weakening does in fact hold for this system, and can be proven in much the same way as for constructive sequent calculus.

Theorem 1 (Weakening) *If* $\Gamma \,;\, \Delta \vdash J$, *then for all* A,

- Γ, A *ctrue* $;\, \Delta \vdash J$ *with a structurally identical proof.*
- $\Gamma \,;\, \Delta, A$ *false* $\vdash J$ *with a structurally identical proof.*

Proof: By induction on the derivation that $\Gamma \,;\, \Delta \vdash J$. At each step, simply add the new hypothesis, but do not use it. □

Equipped with weakening, we can easily prove the other elimination rules — in each of them, we set out to prove a contradiction based on the proposition being eliminated. Disjunction and falsity look slightly different, due to their arbitrary conclusion — in a sense, the correct classical counterparts of $\vee E$ and FE should conclude arbitrary judgements, rather than just arbitrary truth judgements.

Falsity For falsity, the proof remains relatively simple, but splits into three cases depending on what judgment the goal is. We show here the case for $A\ ]jfalse$.

$$\dfrac{\dfrac{\Gamma \,;\, \Delta \vdash F \textit{ ctrue}}{\Gamma, A \textit{ ctrue} \,;\, \Delta \vdash F \textit{ ctrue}}\ \mathit{weaken} \qquad \dfrac{}{\Gamma, A \textit{ ctrue} \,;\, \Delta \vdash F \textit{ false}}\ FF}{\dfrac{\Gamma, A \textit{ ctrue} \,;\, \Delta \vdash \#}{\Gamma \,;\, \Delta \vdash A \textit{ false}}\ F\#}\ \#I$$

The case for A *ctrue* simply uses $T\,\#$ instead of $F\,\#$, and the case for # skips the first rule altogether, using $\#I$ immediately.

Disjunction Disjunction is perhaps the most complex case, as the $\vee E$ rule is our only three-premise rule in constructive natural deduction. However, our approach is the same as with the other elimination rules.

$$\frac{\Gamma \vdash A \vee B\ \mathit{true} \quad \Gamma, A\ \mathit{true} \vdash C\ \mathit{true} \quad \Gamma, B\ \mathit{true} \vdash C\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}} \vee E$$

becomes

$$\frac{\Gamma\ ;\ \Delta \vdash A \vee B\ \mathit{ctrue} \quad \Gamma, A\ \mathit{ctrue}\ ;\ \Delta \vdash J \quad \Gamma, B\ \mathit{ctrue}\ ;\ \Delta \vdash J}{\Gamma\ ;\ \Delta \vdash J} \ ?$$

If J is either A *ctrue* or A *false*, we can use T/F #, and we write $\Gamma\ ;\ \Delta\ ;\ \overline{J}$ for either $\Gamma\ ;\ \Delta, A\ \mathit{false}$ (if $J = A\ \mathit{ctrue}$) or $\Gamma, A\ \mathit{ctrue}\ ;\ \Delta$ (if $J = A\ \mathit{false}$).

$$\frac{\begin{array}{c}\vdots \\ \Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash \#\end{array}}{\Gamma\ ;\ \Delta \vdash J} \ T/F\ \#$$

If J is #, then we can skip this first step, and we will still let $\Gamma\ ;\ \Delta\ ;\ \overline{J}$ denote $\Gamma\ ;\ \Delta$ (when $J = \#$). In all three cases, then, we can complete the proof by giving a derivation of $\Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash \#$ from the premises of $\vee E$. We begin this derivation by setting up a contradiction at $A \vee B$:

$$\frac{\dfrac{\Gamma\ ;\ \Delta \vdash A \vee B\ \mathit{ctrue}}{\Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash A \vee B\ \mathit{ctrue}}\ \mathit{weaken} \quad \begin{array}{c}\vdots \\ \Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash A \vee B\ \mathit{false}\end{array}}{\Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash \#} \ \#I$$

The first branch is either already a premise of $\vee E$ (if $J = \#$), or becomes one with one weakening step. We proceed with the second branch, where we will apply the $\vee F$ rule and attempt to show that both A and B must be false (from the hypothesis that J does not hold).

$$\frac{\dfrac{\Gamma\ ;\ \Delta \vdash A \vee B\ \mathit{ctrue}}{\Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash A \vee B\ \mathit{ctrue}}\ \mathit{weaken} \quad \dfrac{\begin{array}{c}\vdots \\ \Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash A\ \mathit{false}\end{array} \quad \begin{array}{c}\vdots \\ \Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash B\ \mathit{false}\end{array}}{\Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash A \vee B\ \mathit{false}}\ \vee F}{\Gamma\ ;\ \Delta\ ;\ \overline{J} \vdash \#} \ \#I$$

We will focus on the first branch, but the second one can be proven in essentially the same way. At the moment, we know that J does not hold, and want to show that A *false*. We also have as a premise of $\lor E$ that $\Gamma, A\ \mathit{ctrue} ; \Delta \vdash J$. A natural approach, then, is to set up a contradiction on J:

$$\dfrac{\dfrac{\Gamma ; \Delta \vdash A \lor B\ \mathit{ctrue}}{\Gamma ; \Delta ; \overline{J} \vdash A \lor B\ \mathit{ctrue}}\ \mathit{weaken} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots \\ \Gamma, A\ \mathit{ctrue} ; \Delta ; \overline{J} \vdash \#\end{array}}{\Gamma ; \Delta ; \overline{J} \vdash A\ \mathit{false}}\ F\# \qquad \begin{array}{c}\vdots \\ \Gamma ; \Delta ; \overline{J} \vdash B\ \mathit{false}\end{array}}{\Gamma ; \Delta ; \overline{J} \vdash A \lor B\ \mathit{false}}\ \lor F}{\Gamma ; \Delta ; \overline{J} \vdash \#}\ \#I$$

Note that if J is itself #, then this branch is already done — $\Gamma, A\ \mathit{ctrue} ; \Delta ; \overline{J} \vdash \#$ is exactly $\Gamma, A\ \mathit{ctrue} ; \Delta \vdash J$, one of our premises. Otherwise, we can set up for a contradiction as planned:

$$\dfrac{\dfrac{\Gamma, A\ \mathit{ctrue} ; \Delta \vdash J}{\Gamma, A\ \mathit{ctrue} ; \Delta ; \overline{J} \vdash J}\ \mathit{weaken} \qquad \dfrac{}{\Gamma, A\ \mathit{ctrue} ; \Delta ; \overline{J} \vdash \overline{J}}\ \mathit{hyp}}{\Gamma, A\ \mathit{ctrue} ; \Delta ; \overline{J} \vdash \#}\ \#I$$

After filling in the remaining branch of the proof, we get a (very long) derivation of $\lor E$, and, moreover, versions of it where the conclusion is an arbitrary J, not just A *ctrue*.

We will not show the cases for the remaining connectives here, but they follow a similar pattern. Putting all of these together, we can justify the following theorem:

Theorem 2 *If* A_1 *true*$, \ldots, A_n$ *true* $\vdash A$ *true, then* A_1 *ctrue*$, \ldots, A_n$ *ctrue* $; \cdot \vdash A$ *ctrue.*

In other words, any constructively true proposition is also classically true.

4.2 Double Negation

We know that there are classically true propositions that are not constructively true, with $A \lor \neg A$ being the standard example. However, we observe that if we interpret A *false* as $\neg A$ *true*, # as F *true*, and A *ctrue* as A *true*, most of the rules of classical logic are either already rules of constructive logic (all of the $\_T$ rules), are special cases of rules of constructive logic (F # and #I), or are otherwise derived rules (all of the $\_F$ rules — more on this shortly). The one rule that stands out is T #.

$$\frac{\Gamma\ \textit{true}, \neg\Delta\ \textit{true}, \neg A\textit{true} \vdash F\ \textit{true}}{\Gamma\ \textit{true}, \neg\Delta\ \textit{true} \vdash A\ \textit{ctrue}}\ T\ \#$$

Examining T # more closely, we see that interpreting A *ctrue* as $\neg\neg A$ *true* makes T # a specific instance of $\supset I$. What, then, does this do to the other classical rules? The $\_F$ rules are unchanged (other than $\supset F$) since they do not mention the *Actrue* judgement. The $\_T$ rules no longer correspond so directly to constructive rules, and would need to be reexamined, but seem plausible. #I remains a special case of $\supset E$. The remaining odd case is F #:

$$\frac{\neg\neg\Gamma\ \textit{true}, \neg\neg A\textit{true}, \neg\Delta\ \textit{true},\ \vdash F\ \textit{true}}{\neg\neg\Gamma\ \textit{true}\ ;\ \neg\Delta\ \textit{true} \vdash \neg A\ \textit{true}}\ F\ \#$$

This looks suspect at first, but is actually derivable (omitting the unused Γ and Δ for simplicity):

$$\frac{\dfrac{\dfrac{\dfrac{\neg\neg A\ \textit{true} \vdash F\ \textit{true}}{A\ \textit{true}, \neg\neg A\ \textit{true} \vdash F\ \textit{true}}\ \textit{weaken}}{A\ \textit{true} \vdash \neg\neg\neg A\ \textit{true}}\supset I \qquad \dfrac{\dfrac{}{A\ \textit{true}, \neg A\ \textit{true} \vdash \neg A\ \textit{true}}\ \textit{hyp}}{A\ \textit{true} \vdash \neg\neg A\ \textit{true}}\supset I}{A\ \textit{true} \vdash F\ \textit{true}}\supset E}{\vdash \neg A\ \textit{true}}\supset I$$

With this translation, then, all three of our contradiction-related rules, including *both* forms of proof by contradiction, are constructively derivable. Since these were the rules that most exemplified classical reasoning, this is strong evidence that we are on the right path.

False rules The false rules are somewhat simpler to examine than the true rules, simply because they have fewer negations to unpack. We will look at a few cases here, again ignoring Γ and Δ for simplicity.

First, we examine one of the $\wedge F$ rules:

$$\frac{\vdash \neg A\ \textit{true}}{\vdash \neg(A \wedge B)\ \textit{true}}\ \wedge F_1$$

This rule can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\vdash \neg A\ \mathit{true}}{A \wedge B\ \mathit{true} \vdash \neg A\ \mathit{true}}\ \mathit{weaken} \qquad \dfrac{\dfrac{}{A \wedge B\ \mathit{true} \vdash A \wedge B\ \mathit{true}}\ \mathit{hyp}}{A \wedge B\ \mathit{true} \vdash A\ \mathit{true}}\ \wedge E_1}{A \wedge B\ \mathit{true} \vdash F\ \mathit{true}}\ \supset E}{\vdash \neg(A \wedge B)\ \mathit{true}}\ \supset I$$

The other $\wedge F$ rule, of course, is similar. Now, looking at $\vee F$:

$$\dfrac{\vdash \neg A\ \mathit{true} \quad \vdash \neg B\ \mathit{true}}{\vdash \neg(A \vee B)\ \mathit{true}}\ \vee F$$

becomes (with some weakening implicit in the derivation)

$$\dfrac{\dfrac{\dfrac{}{A \vee B\ \mathit{true} \vdash A \vee B\ \mathit{true}}\ \mathit{hyp} \quad \dfrac{\vdash \neg A\ \mathit{true} \quad \dfrac{}{A\ \mathit{true} \vdash A\ \mathit{true}}\ \mathit{hyp}}{A\ \mathit{true} \vdash F\ \mathit{true}}\ \supset E \quad \dfrac{\vdash \neg B\ \mathit{true} \quad \dfrac{}{B\ \mathit{true} \vdash B\ \mathit{true}}\ \mathit{hyp}}{B\ \mathit{true} \vdash F\ \mathit{true}}\ \supset E}{A \vee B\ \mathit{true} \vdash F\ \mathit{true}}\ \vee E}{\vdash \neg(A \vee B)\ \mathit{true}}\ \supset I$$

The $\neg F$ rule simply becomes a no-op — both its premise and its conclusion translate to $\neg\neg A$ *true*.

Finally, we consider the $\supset F$ rule:

$$\dfrac{\vdash \neg\neg A\ \mathit{true} \quad \vdash \neg B\ \mathit{true}}{\vdash \neg(A \supset B)\ \mathit{true}}\ \supset F$$

can be derived as (again leaving some weakening implicit)

$$\dfrac{\dfrac{\vdash \neg\neg A\ \mathit{true} \qquad \dfrac{\dfrac{\vdash \neg B\ \mathit{true} \qquad \dfrac{\dfrac{}{A \supset B\ \mathit{true} \vdash A \supset B\ \mathit{true}}\ \mathit{hyp} \quad \dfrac{}{A\ \mathit{true} \vdash A\ \mathit{true}}\ \mathit{hyp}}{A \supset B\ \mathit{true}, A\ \mathit{true} \vdash B\ \mathit{true}}\ \supset E}{A \supset B\ \mathit{true}, A\ \mathit{true} \vdash F\ \mathit{true}}\ \supset E}{A \supset B\ \mathit{true} \vdash \neg A\ \mathit{true}}\ \supset I}{A \supset B\ \mathit{true} \vdash F\ \mathit{true}}\ \supset E}{\vdash \neg(A \supset B)\ \mathit{true}}\ \supset E$$

The remaining F rule (for falsity) is also derivable, and can be done as an easy exercise.

True rules The true rules can be derived in a similar manner to the false rules, although because each A *ctrue* is translated with two negations, the proofs end up much larger. We will look at only a few cases this time, and will again omit Γ, Δ and implicitly use weakening in order to make proofs more readable.

The $\lor T_2$ rule after translation becomes

$$\frac{\vdash \neg\neg A\ \mathit{true}}{\vdash \neg\neg(A \lor B)\ \mathit{true}}\ \lor T_2$$

As before, we begin trying to derive this rule with $\supset I$ to remove a negation:

$$\frac{\begin{array}{c}\vdots\\ \neg(A \lor B)\ \mathit{true} \vdash F\ \mathit{true}\end{array}}{\vdash \neg\neg(A \lor B)\ \mathit{true}}\ \supset I$$

Now, we need to prove false, using some of the negations available to us. The most obvious option is to try to use $\neg(A \lor B)$, by proving $A \lor B$ using the premise $\vdash \neg\neg A$ *true*. While this seems (perhaps based on classical reasoning) likely, it will not work — $\neg\neg A$ is insufficiently strong to prove $A \lor B$. Instead, we will use $\neg(A \lor B)$ to prove $\neg A$, which we can then combine with $\neg\neg A$ to get F:

$$\dfrac{\dfrac{\vdash \neg\neg A\ \mathit{true} \qquad \dfrac{\dfrac{\dfrac{}{\neg(A \lor B)\ \mathit{true} \vdash \neg(A \lor B)\ \mathit{true}}\ hyp \qquad \dfrac{\dfrac{}{A\ \mathit{true} \vdash A\ \mathit{true}}\ hyp}{A\ \mathit{true} \vdash A \lor B\ \mathit{true}}\ \lor I_1}{\neg(A \lor B)\ \mathit{true}, A\ \mathit{true} \vdash F\ \mathit{true}}\ \supset E}{\neg(A \lor B)\ \mathit{true} \vdash \neg A\ \mathit{true}}\ \supset I}{\neg(A \lor B)\ \mathit{true} \vdash F\ \mathit{true}}\ \supset E}{\vdash \neg\neg(A \lor B)\ \mathit{true}}\ \supset I$$

The $\supset T$ rule after translation becomes

$$\frac{\neg\neg A\ \mathit{true} \vdash \neg\neg B\ \mathit{true}}{\vdash \neg\neg(A \supset B)\ \mathit{true}}\ \supset T$$

This is derivable, but is perhaps the longest of the rules to derive:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \neg(A\supset B)\ \mathit{true}\vdash\neg\neg\neg A\ \mathit{true}\end{matrix}\quad\begin{matrix}\mathcal{E}\\ \neg(A\supset B)\ \mathit{true}\vdash\neg\neg A\ \mathit{true}\end{matrix}}{\neg(A\supset B)\ \mathit{true}\vdash F\ \mathit{true}}\supset E}{\vdash\neg\neg(A\supset B)\ \mathit{true}}\supset I$$

where the derivation $\mathcal{D}$ is

$$\dfrac{\dfrac{\neg\neg A\ \mathit{true}\vdash\neg\neg B\ \mathit{true}\qquad\dfrac{\dfrac{\dfrac{}{\neg(A\supset B)\ \mathit{true}\vdash\neg(A\supset B)\ \mathit{true}}\mathit{hyp}\quad\dfrac{\dfrac{}{B\ \mathit{true}\vdash B\ \mathit{true}}\mathit{hyp}}{B\ \mathit{true}\vdash A\supset B\ \mathit{true}}\supset I}{\neg(A\supset B)\ \mathit{true},B\ \mathit{true}\vdash F\ \mathit{true}}\supset E}{\neg(A\supset B)\ \mathit{true}\vdash\neg B\ \mathit{true}}\supset I}{\neg(A\supset B)\ \mathit{true},\neg\neg A\ \mathit{true}\vdash F\ \mathit{true}}\supset E}{\neg(A\supset B)\ \mathit{true}\vdash\neg\neg\neg A\ \mathit{true}}\supset I$$

and $\mathcal{E}$ is

$$\dfrac{\dfrac{\dfrac{}{\neg(A\supset B)\ \mathit{true}\vdash\neg(A\supset B)\ \mathit{true}}\mathit{hyp}\qquad\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A\ \mathit{true}\vdash\neg A\ \mathit{true}}\mathit{hyp}\quad\dfrac{}{A\ \mathit{true}\vdash A\ \mathit{true}}\mathit{hyp}}{\neg A\ \mathit{true},A\ \mathit{true}\vdash F\ \mathit{true}}\supset E}{\neg A\ \mathit{true},A\ \mathit{true}\vdash B\ \mathit{true}}FE}{\neg A\ \mathit{true}\vdash A\supset B\ \mathit{true}}\supset I}{\neg(A\supset B)\ \mathit{true},\neg A\ \mathit{true}\vdash F\ \mathit{true}}\supset E}{\neg(A\supset B)\ \mathit{true}\vdash\neg\neg A\ \mathit{true}}\supset I$$

The remaining T rules are similarly derivable. Notably, $\neg T$ (like $\neg F$) is again a no-op, again with both premise and conclusion translating to $\neg\neg A$ *true*.

Completing all of these cases gives justification for the following theorem:

Theorem 3 *Define* $J^{\neg\neg}$ *by:*

- $(A\ \mathit{ctrue})^{\neg\neg} = \neg\neg A\ \mathit{true}$
- $(A\ \mathit{false})^{\neg\neg} = \neg A\ \mathit{true}$
- $\#^{\neg\neg} = F\ \mathit{true}$

and extend this definition to contexts pointwise. Suppose $\Gamma\ ;\ \Delta \vdash J$*. Then,* $\Gamma^{\neg\neg}, \Delta^{\neg\neg} \vdash J^{\neg\neg}$.

This theorem gives us the other direction of connection between classical and constructive logic. Not only is every constructively provable proposition classically true as well, but every classically true proposition can be transformed into a constructively true one that is classically equivalent. Put differently, every theorem of classical logic is a theorem of constructive logic as well — we just need to be precise enough about what the theorem actually is.

References

[LC06] William Lovas and Karl Crary. Structural normalization for classical natural deduction, 2006.

A Rules for Classical Logic

Rules for #

$$\frac{\Gamma\ ;\Delta \vdash A\ \mathit{ctrue} \quad \Gamma\ ;\Delta \vdash A\ \mathit{false}}{\Gamma\ ;\Delta \vdash \#}\ \#I$$

Rules for *A ctrue*

$$\frac{}{\Gamma, A\ \mathit{ctrue}\ ;\Delta \vdash A\ \mathit{ctrue}}\ hypT \qquad \frac{\Gamma\ ;\Delta, A\ \mathit{false} \vdash \#}{\Gamma\ ;\Delta \vdash A\ \mathit{ctrue}}\ T\#$$

$$\frac{\Gamma\ ;\Delta \vdash A\ \mathit{ctrue} \quad \Gamma\ ;\Delta \vdash B\ \mathit{ctrue}}{\Gamma\ ;\Delta \vdash A \wedge B\ \mathit{ctrue}}\ \wedge T$$

$$\frac{\Gamma\ ;\Delta \vdash A\ \mathit{ctrue}}{\Gamma\ ;\Delta \vdash A \vee B\ \mathit{ctrue}}\ \vee T_1 \qquad \frac{\Gamma\ ;\Delta \vdash B\ \mathit{ctrue}}{\Gamma\ ;\Delta \vdash A \vee B\ \mathit{ctrue}}\ \vee T_2$$

$$\frac{\Gamma\ ;\Delta \vdash A\ \mathit{false}}{\Gamma\ ;\Delta \vdash \neg A\ \mathit{ctrue}}\ \neg T \qquad \frac{\Gamma, A\ \mathit{ctrue}\ ;\Delta \vdash B\ \mathit{ctrue}}{\Gamma\ ;\Delta \vdash A \supset B\ \mathit{ctrue}}\ \supset T$$

$$\frac{}{\Gamma\ ;\Delta \vdash T\ \mathit{ctrue}}\ TT$$

Rules for *A false*

$$\frac{}{\Gamma\ ;\Delta, A\ \mathit{false} \vdash A\ \mathit{false}}\ hypF \qquad \frac{\Gamma, A\ \mathit{ctrue}\ ;\Delta \vdash \#}{\Gamma\ ;\Delta \vdash A\ \mathit{false}}\ F\#$$

$$\frac{\Gamma\ ;\Delta \vdash A\ \mathit{false}}{\Gamma\ ;\Delta \vdash A \wedge B\ \mathit{false}}\ \wedge F_1 \qquad \frac{\Gamma\ ;\Delta \vdash B\ \mathit{false}}{\Gamma\ ;\Delta \vdash A \wedge B\ \mathit{false}}\ \wedge F_2$$

$$\frac{\Gamma\ ;\Delta \vdash A\ \mathit{false} \quad \Gamma\ ;\Delta \vdash B\ \mathit{false}}{\Gamma\ ;\Delta \vdash A \vee B\ \mathit{false}}\ \vee F$$

$$\frac{\Gamma\ ;\Delta \vdash A\ \mathit{ctrue}}{\Gamma\ ;\Delta \vdash \neg A\ \mathit{false}}\ \neg F \qquad \frac{\Gamma\ ;\Delta \vdash A\ \mathit{ctrue} \quad \Gamma\ ;\Delta \vdash B\ \mathit{false}}{\Gamma\ ;\Delta \vdash A \supset B\ \mathit{false}}\ \supset F$$

$$\frac{}{\Gamma\ ;\Delta \vdash F\ \mathit{false}}\ FF$$

Figure 1: Rules for Classical Natural Deduction with Contexts

Rules for #

$$\frac{A\ \mathit{ctrue} \quad A\ \mathit{false}}{\#}\ \#I$$

Rules for *A ctrue*

$$\frac{\begin{array}{c}\overline{A\ \mathit{false}}\ u \\ \vdots \\ \#\end{array}}{A\ \mathit{ctrue}}\ T\#^u$$

$$\frac{A\ \mathit{ctrue} \quad B\ \mathit{ctrue}}{A \wedge B\ \mathit{ctrue}}\ \wedge T \qquad \frac{A\ \mathit{ctrue}}{A \vee B\ \mathit{ctrue}}\ \vee T_1 \qquad \frac{B\ \mathit{ctrue}}{A \vee B\ \mathit{ctrue}}\ \vee T_2$$

$$\frac{A\ \mathit{false}}{\neg A\ \mathit{ctrue}}\ \neg T \qquad \frac{\begin{array}{c}\overline{A\ \mathit{ctrue}}\ u \\ \vdots \\ B\ \mathit{ctrue}\end{array}}{A \supset B\ \mathit{ctrue}}\ \supset T \qquad \frac{}{T\ \mathit{ctrue}}\ TT$$

Rules for *A false*

$$\frac{\begin{array}{c}\overline{A\ \mathit{ctrue}}\ u \\ \vdots \\ \#\end{array}}{\Gamma\ ;\ \Delta \vdash A\ \mathit{false}}\ F\#^u$$

$$\frac{A\ \mathit{false}}{A \wedge B\ \mathit{false}}\ \wedge F_1 \qquad \frac{B\ \mathit{false}}{A \wedge B\ \mathit{false}}\ \wedge F_2 \qquad \frac{A\ \mathit{false} \quad B\ \mathit{false}}{A \vee B\ \mathit{false}}\ \vee F$$

$$\frac{A\ \mathit{ctrue}}{\neg A\ \mathit{false}}\ \neg F \qquad \frac{A\ \mathit{ctrue} \quad B\ \mathit{false}}{A \supset B\ \mathit{false}}\ \supset F \qquad \frac{}{F\ \mathit{false}}\ FF$$

Figure 2: Rules for Classical Natural Deduction without Contexts