Constructive Logic (15-317), Spring 2022
Assignment 5: Sequent Calculus, Cut, etc.

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Due: Submit to Gradescope by Thursday, Februrary 24, 2022, 11:59 pm

The assignments in this course must be submitted electronically through Gradescope. Written homework PDFs and Dcheck code files will both go to Gradescope. The written is required to be typeset. We recommend using LaTeX but other suitable software are also acceptable.

- `hw5.deriv` (your coding solutions)
- `hw5.pdf` (your written solutions)

The coding portion will use the experimental Dcheck derivation checker. You can find documentation and examples on the Software page at the course web site (`andrew.cmu.edu/user/kpruiksm/15317s22/software.html`).

Sequent Calculus Proofs

Provide Dcheck derivations of the following Sequent Calculus judgements using the `SC` system. Your solutions should go in `hw5.deriv`.

Task 1 (4 points). Define a derivation named `task1` that derives:

$$\Longrightarrow ((P \supset F) \vee Q) \supset (P \supset Q)$$

Task 2 (4 points). Define a derivation named `task2` that derives:

$$\Longrightarrow ((P \supset R) \wedge (Q \supset R)) \supset ((P \vee Q) \supset R)$$

Task 3 (4 points). Define a derivation named `task3` that derives:

$$R \supset P \vee Q, (P \wedge Q) \supset R \Longrightarrow P \supset (Q \supset R)$$

Note that Dcheck takes P, Q, and R to be atomic propositions.

Cut for a New Connective

Recall the ♣ connective from HW3, replicated below for convenience.

$$\dfrac{\begin{array}{cc} \overline{A\ true}^{\,u} & \overline{A\ true}^{\,v} \\ \vdots & \vdots \\ B\ true & C\ true \end{array}}{\clubsuit(A, B, C)\ true}\ \clubsuit I^{u,v} \qquad \dfrac{\clubsuit(A, B, C)\ true \quad A\ true \quad \begin{array}{c} \overline{B\ true}^{\,u} \\ \vdots \\ D\ true \end{array}}{D\ true}\ \clubsuit E_1^u \qquad \dfrac{\clubsuit(A, B, C)\ true \quad A\ true \quad \begin{array}{c} \overline{C\ true}^{\,u} \\ \vdots \\ D\ true \end{array}}{D\ true}\ \clubsuit E_2^u$$

The sequent calculus rules for this connective are as follows: [1]

$$\dfrac{\Gamma, A \Rightarrow B \quad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow \clubsuit(A, B, C)}\ \clubsuit R \qquad \dfrac{\Gamma, \clubsuit(A, B, C) \Rightarrow A \quad \Gamma, \clubsuit(A, B, C), B \Rightarrow D}{\Gamma, \clubsuit(A, B, C) \Rightarrow D}\ \clubsuit L_1 \qquad \dfrac{\Gamma, \clubsuit(A, B, C) \Rightarrow A \quad \Gamma, \clubsuit(A, B, C), C \Rightarrow D}{\Gamma, \clubsuit(A, B, C) \Rightarrow D}\ \clubsuit L_2$$

Task 4 (8 points). If we add this connective (and the sequent calculus rules above) to our logic, we would need to prove additional cases of the cut theorem. List the new cases that would be needed, using as few distinct cases as possible. You do not need to provide proofs for this task, just list the cases.

For example, for $\supset$, the cases needed are as follows:

- $\supset R / \supset L$
- $\supset L / *$
- $* / \supset L$
- $* / \supset R$

Task 5 (3 points). Choose a principal case of cut for ♣. What are A, $\mathcal{D}$, and $\mathcal{E}$ in this case?

[1] You may find it useful to think about how these rules relate to their natural deduction counterparts.

Applications of Cut

The central theorem of structural proof theory is the closure of sequent calculus under the principle of *cut*; the statement of cut depends on the logic, but for our purposes it can be stated as follows.

Theorem 1 (Cut)**.** If $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$ then $\Gamma \Longrightarrow C$.

This theorem can be used to prove many difficult properties about a proof system, including consistency, constructivity, and others. In mathematics, the same technique is also used to establish difficult coherence theorems for higher-dimensional structures.

Another theorem about intuitionistic sequent calculus is closure under the principle of *weakening*, stated as follows.

Theorem 2 (Weakening)**.** If $\Gamma \Longrightarrow C$ then $\Gamma, A \Longrightarrow C$.

Task 6 (10 points)**.** Using Theorem 1 and/or Theorem 2, prove:

If $\Gamma, A \wedge B \Longrightarrow C$ *then* $\Gamma, A, B \Longrightarrow C$.

In particular, you should not use any induction in your argument.