Constructive Logic (15-317), Spring2022
Assignment 3: Proofs as Programs + Verifications and Uses

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Due: Thursday, February 10, 2022, 11:59 pm

The assignments in this course must be submitted electronically through Gradescope. Written homework PDFs and coding Dcheck files will both go to Gradescope. For this homework, submit two files:

- `hw3.pdf` (your written solutions)
- `hw3.deriv` (your coding solutions)

Trees are Programs

Task 1 (18 points)**.** Provide proof terms for the following theorems using the proof-as-program logic. Your solution should go in `hw3.pdf`.

a. `prove deMorgagain:` $\neg A \land \neg B \supset \neg(A \lor B)$

b. `prove toptobottom:` $(A \supset \top) \land (\bot \supset A)$

c. `prove reuse:` $((A \supset B) \land (A \supset C)) \supset (A \supset B \land C)$

d. `prove ormap:` $((A \lor B) \supset C) \supset (A \supset C) \land (B \supset C)$

e. `prove curry:` $(A \land B \supset C) \supset A \supset B \supset C$

f. `prove exclusion:` $((A \lor B) \land \neg A) \supset B$

I thunk therefore I am

Task 2 (8 points)**.** Consider a unary connective $\circ$ defined by the following rules:

$$\dfrac{\dfrac{}{\top\ \mathit{true}}\ u \\ \vdots \\ A\ \mathit{true}}{\circ A\ \mathit{true}}\ \circ I^u \qquad \dfrac{\circ A\ \mathit{true} \quad \top\ \mathit{true}}{A\ \mathit{true}}\ \circ E$$

1. Can you prove a simple relationship between A *true* and $\circ A$ *true*?
2. Using **thunk**$(u.M)$ as the proof term for the intro rule (aka introduction form), give the appropriate intro rule. for **thunk**$(u.M) : \circ A$.
3. Using $M << N$ as the proof term for the elim rule (aka elimination form), give the appropriate elim rule for $M << N : A$.
4. Does $\circ$ have a reduction rule[1]? Write out a reduction rule for $\circ$ if one exists. Otherwise, show that no reduction rule is possible.
5. Why might a programming language or programmer want to use thunks in code?[2]

[1] Remember that a contraction rule shows how to reduce the elimination form of a connective to a simpler term
[2] Any reasonable guess is fine

Verifications

Consider the ♣ connective.

$$\dfrac{\dfrac{}{A\ true}\,u \quad \dfrac{}{A\ true}\,v \qquad \vdots \qquad \vdots}{\dfrac{B\ true \qquad C\ true}{\clubsuit(A,B,C)\ true}\,\clubsuit I^{u,v}} \qquad \dfrac{\clubsuit(A,B,C)\ true \quad A\ true \quad \dfrac{}{B\ true}\,u \qquad \vdots \qquad D\ true}{D\ true}\,\clubsuit E_1^u \qquad \dfrac{\clubsuit(A,B,C)\ true \quad A\ true \quad \dfrac{}{C\ true}\,u \qquad \vdots \qquad D\ true}{D\ true}\,\clubsuit E_2^u$$

Task 3 (5 points)**.** Give rules for forming the judgments that $\clubsuit(A, B, C)$ has a verification and that $\clubsuit(A, B, C)$ can be used.

Task 4 (4 points)**.** Give a verification for the following proposition in Dcheck, using the `VU` system.

$$(\neg P \wedge Q) \supset ((P \supset Q) \supset (\neg P \supset \neg Q)) \supset \bot$$

Your solution should go in `hw3.deriv`. For clarification on how to write a verifications-and-uses proof, please look at the course website[3]. Note that we use P and Q here to denote atomic propositions.

Task 5 (13 points)**.** For each of the following propositions, give a verification-and-uses proof and its **corresponding** proofs-as-programs term. Your proof should go in `hw3.deriv` and the proof term should go in `hw3.pdf`.

1. $\bot \supset \top$
2. $\bot \supset \top$ **(Do not use the same verification/proof term as part a. Use a new one.)**
3. $(P \supset Q) \supset (\neg Q \supset \neg P)$
4. $(P \supset Q) \supset (Q \supset R) \supset (P \supset R)$
5. $((P \supset R) \wedge (Q \supset R)) \supset ((P \vee Q) \supset R)$

[3] `https://www.andrew.cmu.edu/user/kpruiksm/15317s22/example.deriv`