Constructive Logic (15-317), Spring2022
Assignment 2: Harmony

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Due: Thursday, Feburary 3, 2022, 11:59 pm

The assignments in this course must be submitted electronically through Gradescope. Written homework PDFs and coding Dcheck files will both go to Gradescope. For this homework, you will be submitting two files:

- `hw2.deriv` (your natural deduction proofs)
- `hw2.pdf` (your written solutions)

More Proofs? Deduce that!

Task 1 (10 points)**.** Prove the following theorems using natural deduction logic in Dcheck, using the `ND` system. Your solution should go in `hw2.deriv`, and will be automatically graded when you submit to Gradescope. Remember that *not A* is syntatic sugar for *A implies False*. You can look at the examples natural deduction proofs on the course website [1].

a. `absurdity:` $A \wedge \neg A \supset B$

b. `sCombinator:` $(A \supset B) \supset (A \supset B \supset C) \supset (A \supset C)$

c. `deMorgin:` $\neg(A \vee B) \supset \neg A \wedge \neg B$

d. `deMorgout:` $\neg A \wedge \neg B \supset \neg(A \vee B)$

e. `contrapositive:` $(A \supset B) \supset (\neg B \supset \neg A)$

[1] `https://www.andrew.cmu.edu/user/kpruiksm/15317s22/example.deriv`

Harmony

Your solutions to the following two tasks should go in `hw2.pdf`.

Task 2 (11 points)**.** Consider a connective $\triangle$ with the following elimination rules:

$$\dfrac{A \triangle B\ \textit{true} \qquad \begin{array}{c} \overline{A\ \textit{true}}^{\,u} \quad \overline{B\ \textit{true}}^{\,v} \\ \vdots \\ C\ \textit{true} \end{array}}{C\ \textit{true}}\ \triangle E^{u,v}$$

(Normally we take the verificationist perspective that introduction rules come first to define a connective, but this time we'll go in the opposite direction.)

a. Come up with a set of zero or more introduction rules for this connective.

b. Show that the connective is locally sound for your choice of introduction rules.

c. Show that the connective is locally complete for your choice of introduction rules.

d. Is it possible to come up with a notational definition $A \triangle B \triangleq$ ______________ so that both your defined introduction rule(s) as well as the elimination rule given above are merely derived rules?

Task 3 (10 points)**.** Consider a connective $\circledcirc$ defined by the following rules.

$$\dfrac{A\ \textit{true}}{A \circledcirc B\ \textit{true}}\ \circledcirc I1 \qquad \dfrac{B\ \textit{true}}{A \circledcirc B\ \textit{true}}\ \circledcirc I2 \qquad \dfrac{A \circledcirc B\ \textit{true} \quad \begin{array}{c} \overline{A\ \textit{true}}^{\,u} \\ \vdots \\ C\ \textit{true} \end{array}}{C\ \textit{true}}\ \circledcirc E1^{u} \qquad \dfrac{A \circledcirc B\ \textit{true} \quad \begin{array}{c} \overline{B\ \textit{true}}^{\,v} \\ \vdots \\ C\ \textit{true} \end{array}}{C\ \textit{true}}\ \circledcirc E2^{v}$$

a. Is this connective locally sound? If so, provide the local reduction(s); if not, briefly explain why.

b. Is this connective locally complete? If so, provide the local expansion(s); if not, briefly explain why.

Hype for *hyp*s

Consider the following notations for a proofs of $A \supset B \supset A \wedge B$.

Floating Hypothesis Notation

$$\dfrac{\dfrac{\dfrac{\overline{A\ true}^{\,x} \quad \overline{B\ true}^{\,y}}{A \wedge B\ true} \wedge I}{B \supset (A \wedge B)\ true} \supset I^y}{A \supset (B \supset (A \wedge B))\ true} \supset I^x$$

Context Notation

$$\dfrac{\dfrac{\dfrac{\overline{A\ true, B\ true \vdash A\ true}\ hyp \quad \overline{A\ true, B\ true \vdash B\ true}\ hyp}{A\ true, B\ true \vdash A \wedge B\ true} \wedge I}{A\ true \vdash B \supset (A \wedge B)\ true} \supset I}{\vdash A \supset (B \supset (A \wedge B))\ true} \supset I$$

The Γ notation of hypotheses can be created by slightly modifying the rules of Natural Deduction to carry a context. (A context is a set of hypotheses.)

$$\dfrac{\Gamma \vdash A\ true \quad \Gamma \vdash B\ true}{\Gamma \vdash A \wedge B\ true} \wedge I \qquad \dfrac{\Gamma \vdash A \wedge B\ true}{\Gamma \vdash A\ true} \wedge E_1 \qquad \dfrac{\Gamma \vdash A \wedge B\ true}{\Gamma \vdash B\ true} \wedge E_2$$

$$\dfrac{\Gamma \vdash A\ true}{\Gamma \vdash A \vee B\ true} \vee I_1 \qquad \dfrac{\Gamma \vdash B\ true}{\Gamma \vdash A \vee B\ true} \vee I_2 \qquad \dfrac{\Gamma \vdash A \vee B\ true \quad \Gamma, A\ true \vdash C\ true \quad \Gamma, B\ true \vdash C\ true}{\Gamma \vdash C\ true} \vee E$$

$$\dfrac{\Gamma, A\ true \vdash B\ true}{\Gamma \vdash A \supset B\ true} \supset I \qquad \dfrac{\Gamma \vdash A \supset B\ true \quad \Gamma \vdash A\ true}{\Gamma \vdash B\ true} \supset E$$

$$\dfrac{}{\Gamma \vdash \top\ true} \top I \qquad \dfrac{\Gamma \vdash \bot\ true}{\Gamma \vdash C\ true} \bot E$$

Finally, the hypothesis rule, which allows you to conclude a hypothesis. We view contexts as unordered sets, so the J in this rule need not be written last.[2]

$$\dfrac{}{\Gamma, J \vdash J}\ hyp$$

Task 4 (4 points)**.** Consider the following proof and write the exact corresponding proof using context notation in Dcheck. Your solution should go in `hw2.deriv` with a derivation named `copyProof`. Since this problem focuses on the notations, please don't shorten the proof. For clarification on how to write proofs in context natural deduction in Dcheck, please look at Dcheck documentation[3].

[2]In the logic rule sheet we gave, this rule is written as $\dfrac{J \in \Gamma}{\Gamma \vdash J}\ hyp$. It should be clear that those two definitions are equivalent.

[3]`https://www.andrew.cmu.edu/user/kpruiksm/15317s22/dcheck.pdf`

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \supset A) \wedge (A \supset A)\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\,\wedge E_2 \qquad \dfrac{\dfrac{\dfrac{}{(A \supset A) \wedge (A \supset A)\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\,\wedge E_1 \quad \dfrac{}{A\ \mathit{true}}\,w}{A\ \mathit{true}}\,{\supset}E}{A\ \mathit{true}}\,{\supset}E}{(A \supset A)\ \mathit{true}}\,{\supset}I^w}{(A \supset A) \wedge (A \supset A) \supset (A \supset A)\ \mathit{true}}\,{\supset}I^u
$$

Task 5 (4 points)**.** Write derivations using Dcheck for the following judgements, using the `NDC` system. Your solutions should go in `hw2.deriv`, and will be automatically graded when you submit to Gradescope.

a. `implies`: $\cdot \vdash (\neg A \vee B) \supset (A \supset B)\ \mathit{true}$ [4]

b. `transitivity`: $\cdot \vdash (A \supset B) \supset (B \supset C) \supset (A \supset C)\ \mathit{true}$

[4] The other direction is, in fact, not constructively true. Think about it.