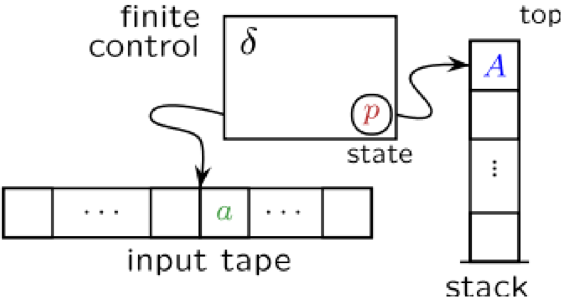
Formal Languages, Automata and Computation

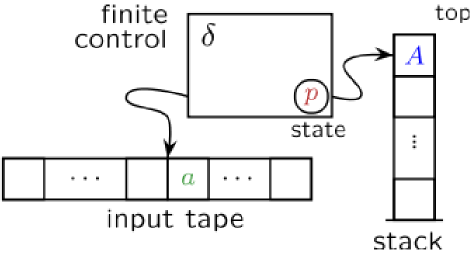
Pushdown Automata

PUSHDOWN AUTOMATA

- Pushdown automata (PDA) are abstract automata that accept all context-free languages.
- PDAs are essentially NFAs with an additional infinite stack memory.
  - (Or NFAs are PDAs with no additional memory!)



Pushdown Automata



- Input is read left-to-right
- Control has finite memory (NFA)
- State transition depends on input and top of stack
- Control can push and pop symbols to/from the infinite stack

- Let's look at $L = \{a^n b^n \mid n \geq 0\}$
- How can we use a stack to recognize $w \in L$?
  1. Push a special bottom of stack symbol \$ to the stack
  2. As long as you are seeing a's in the input, push an a onto the stack.
  3. While there are b's in the input AND there is a corresponding a on the top of the stack, pop a from the stack
  4. If at any point there is no a on the stack (hence you encounter \$), you should reject the string – not enough a's!
  5. If at the end of w, the top of the stack is NOT \$ reject the string – not enough b's.
  6. Otherwise accept the string.

Pushdown Automata – Informal

- How can we use a PDA to recognize $L = \{w \mid n_a(w) = n_b(w)\}$
- Remember how we argued that the grammar generates such strings
  - Keep track of the difference of counts
- We do something similar but using the stack.
  - Push a special bottom of stack symbol \$ to the stack
  - An a in the input "cancels" a b on the top of the stack, otherwise pushes an a
  - A b in the input "cancels" an a on the top of the stack, otherwise pushes a b
  - At the end nothing should be left on the stack except for the \$, if not reject.

PUSHDOWN AUTOMATA –FORMAL DEFINITION

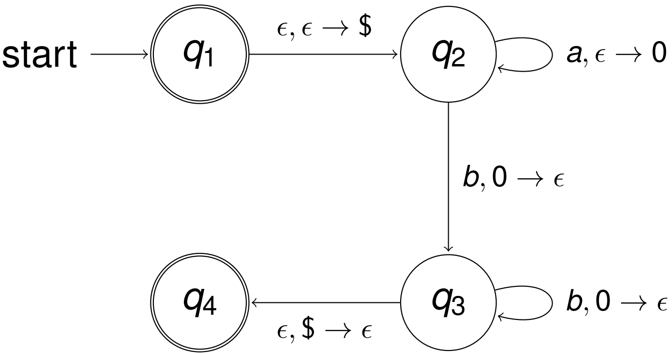
- We have two alphabets Σ for symbols of the input string and Γ for symbols for the stack. They need not be disjoint.
- Define $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$
- A pushdown automaton is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ, and F are finite sets, and
  - Q is the set of states
  - Σ is the input alphabet, Γ is the stack alphabet
  - $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \rightarrow \mathcal{P}(Q \times \Gamma_\epsilon)$ is the state transition function. ($\mathcal{P}(S)$is the power set of S. Earlier we used 2^S.)
  - $q_0 \in Q$ is the start state, and
  - $F \subseteq Q$ is the set of final or accepting states.

Computation on a PDA

- A PDA computes as follows:
  - Input w can be written as $w = w_1 w_2 \cdots w_m$ where each $w_i \in \Sigma_\epsilon$. So some w_i can be ϵ.
  - There is a sequence of states $r_0, r_1, \cdots, r_m$, $r_i \in Q$.
  - There is a sequence of strings $s_0, s_1, \cdots, s_m$, $s_i \in \Gamma^*$. These represent sequences of stack contents along an accepting branch of M's computation.
- $r_0 = q_0$ and $s_0 = \epsilon$.
- $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a), i = 0, 1, \cdots, m - 1$
- a is popped, b is pushed, t is the rest of the stack.
  - $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\epsilon$ and $t \in \Gamma^*$
- $r_m \in F$

EXAMPLE PDA

- PDA for $L = \{a^n b^n \mid n \geq 0\}$
  - $\Sigma = \{a, b\}, \Gamma = \{0, \$\}$
  - \$ keeps track of the "bottom" of the stack

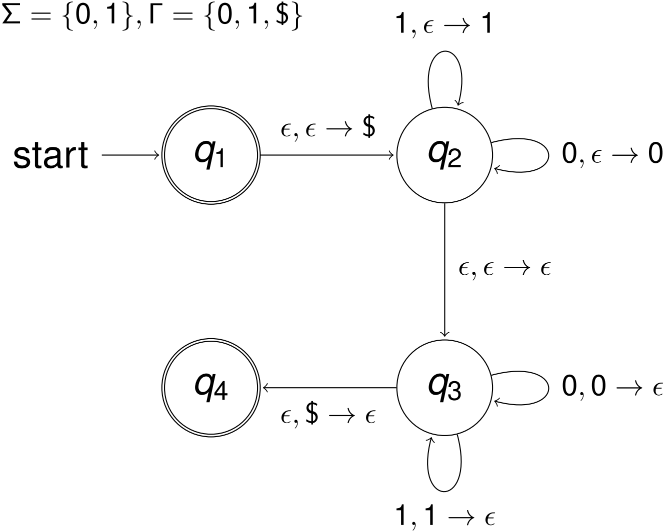


Example PDA

- PDA for $L = \{ww^R \mid w \in \{0,1\}^*\}$
- Palindromes: See `http://norvig.com/palindrome.html` for interesting examples:
- A 17,826 word palindrome starts and ends as:
  - *A man, a plan, a cameo, Zena, Bird, Mocha, Prowel, a rave, Uganda, Wait, a lobola, Argo, Goto, Koser, Ihab, Udall, a revocation, Ebarta, Muscat, eyes, Rehm, a cession, Udella, E-boat, OAS, a mirage, IPBM, a caress, Etam, ..., a lobo, Lati, a wadna, Guevara, Lew, Orpah, Comdr, Ibanez, OEM, a canal, Panama*

Example PDA

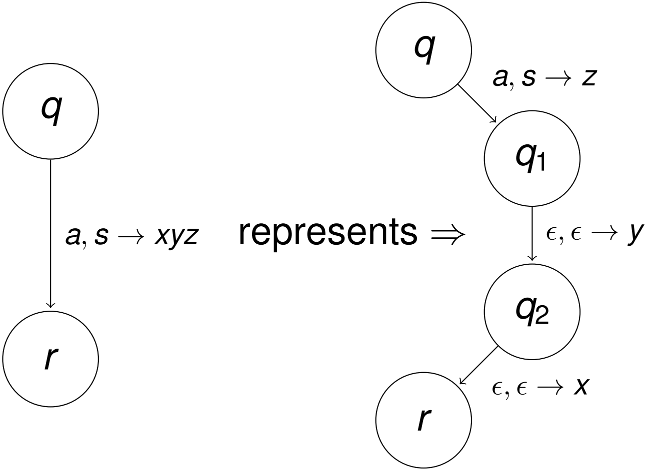
- PDA for $L = \{ww^R \mid w \in \{0,1\}^*\}$
  - $\Sigma = \{0,1\}, \Gamma = \{0,1,\$\}$



EXAMPLE PDA

- Let's construct a PDA for $L = \{w \mid n_a(w) = n_b(w)\}$

- It is usually better and more succinct to represent a series of PDA transitions using a shorthand



PDAs and CFGs

- PDAs and CFGs are equivalent in power: they both describe context-free languages.

Theorem

A language is context free if and only if some pushdown automaton recognizes it.

PDAs and CFGs

Lemma

If a language is context free, then some pushdown automaton recognizes it.

Proof Idea

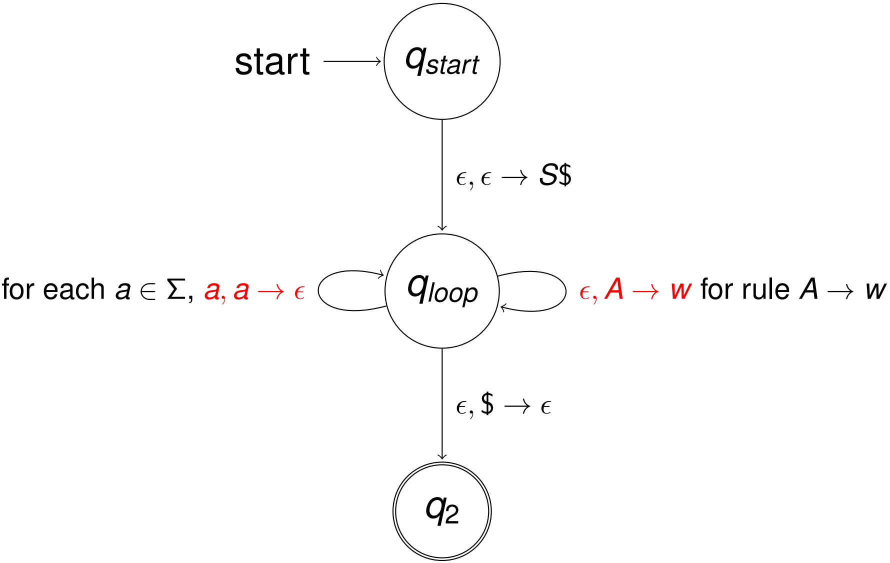
If A is a CFL, then it has a CFG G for generating it. Convert the CFG to an equivalent PDA.

- Each rule maps to a transition.

CFGs to PDAs

- We simulate the leftmost derivation of a string using a 3-state PDA with $Q = \{q_{start}, q_{loop}, q_{accept}\}$
- One transition from q_{start} pushes the start symbol S onto the stack (along with \$).
- Transitions from q_{loop} simulate either a rule expansion, or matching an input symbol.
  - $\delta(q_{loop}, \epsilon, A) = \{(q_{loop}, w) \mid A \rightarrow w \text{ is a production in } G\}$
    - If the top of the stack is A, nondeterministically expand it in all possible ways.
  - $\delta(q_{loop}, a, a) = \{(q_{loop}, \epsilon)\}$, for all $a \in \Sigma$.
    - If the input symbol matches the top of the stack, consume the input and pop the stack.
- One transition takes the PDA from q_{loop} to q_{accept} when \$ is seen on the stack.

CFGs to PDAs


start
q_{start}
$\epsilon, \epsilon \rightarrow S\$$
for each $a \in \Sigma$, $a, a \rightarrow \epsilon$
q_{loop}
$\epsilon, A \rightarrow w$ for rule $A \rightarrow w$
$\epsilon, \$ \rightarrow \epsilon$
q_2

CFG TO PDA EXAMPLE

- Let's convert the following grammar for $L = \{w \mid n_a(w) = n_b(w)\}$.
  - $S \to aSb$
  - $S \to bSa$
  - $S \to SS$
  - $S \to \epsilon$