FORMAL LANGUAGES, AUTOMATA AND COMPUTATION

REGULAR LANGUAGES

NONDETERMINISTIC FINITE STATE AUTOMATA

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SUMMARY

- Symbols, Alphabet, Strings, Σ^*, Languages, 2^{Σ^*}
- Deterministic Finite State Automata
  - States, Labels, Start State, Final States, Transitions
  - Extended State Transition Function
  - DFAs accept regular languages

Regular Languages

- Since regular languages are sets, we can combine them with the usual set operations
  - Union
  - Intersection
  - Difference

Theorem

If L_1 and L_2 are regular languages, so are $L_1 \cup L_2$, $L_1 \cap L_2$ and $L_1 - L_2$.

Proof Idea

Construct cross-product DFAs

Cross-product DFAs

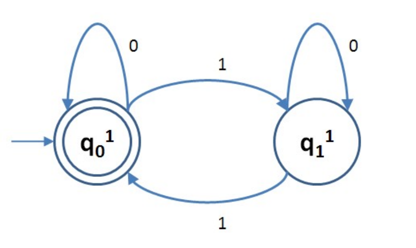
- A single DFA which simulates operation of two DFAs in parallel!
- Let the two DFAs be M_1 and M_2 accepting regular languages L_1 and L_2
  1. $M_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$
  2. $M_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$
- We want to construct DFAs $M = (Q, \Sigma, \delta, q_0, F)$ that recognize
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 - L_2$

Constructing the Cross-Product DFA M

- We need to construct $M = (Q, \Sigma, \delta, q_0, F)$
- $Q =$pairs of states, one from M_1 and one from M_2
  $Q = \{(q_1, q_2) | q_1 \in Q_1 \text{ and } q_2 \in Q_2\}$
  $Q = Q_1 \times Q_2$
- $q_0 = (q_0^1, q_0^2)$
- $\delta((q_i^1, q_j^2), x) = (\delta_1(q_i^1, x), \delta_2(q_j^2, x))$
- Union: $F = \{(q_1, q_2) | q_1 \in F_1 \text{ or } q_2 \in F_2\}$
- Intersection: $F = \{(q_1, q_2) | q_1 \in F_1 \text{ and } q_2 \in F_2\}$
- Difference: $F = \{(q_1, q_2) | q_1 \in F_1 \text{ and } q_2 \notin F_2\}$

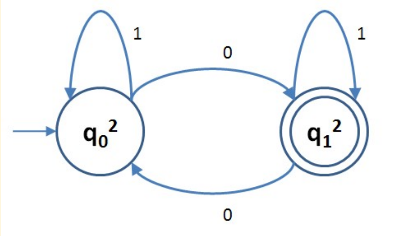
CROSS-PRODUCT DFA EXAMPLE

STRINGS WITH EVEN NUMBER OF 1S



M_1

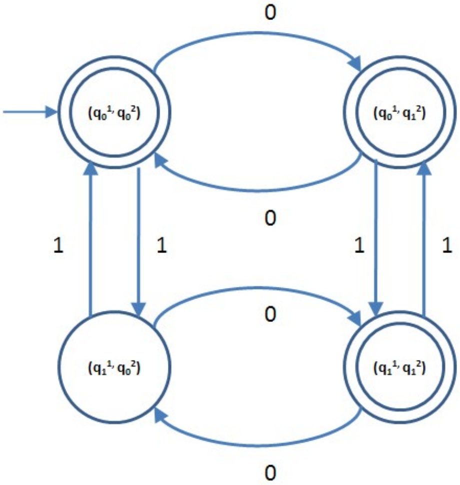
STRINGS WITH ODD NUMBER OF 0S



M_2

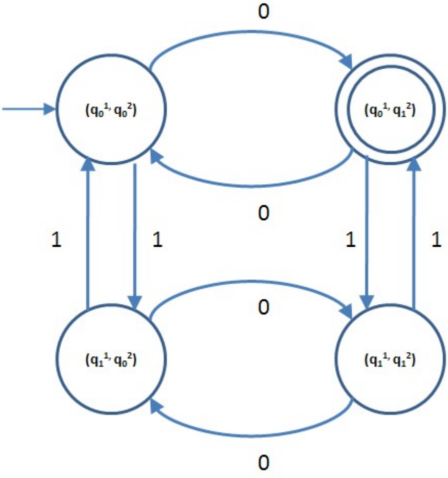
DFA FOR $L_1 \cup L_2$

- DFA for $L_1 \cup L_2$ accepts when either M_1 or M_2 accepts.



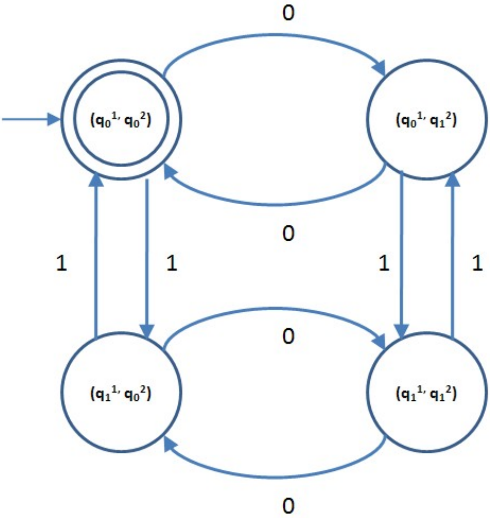
DFA FOR $L_1 \cap L_2$

- DFA for $L_1 \cap L_2$ accepts when both M_1 and M_2 accept.



DFA FOR $L_1 - L_2$

- DFA for $L_1 - L_2$ accepts when M_1 accepts and M_2 rejects.



Another example: Find the cross-product DFA for

- DFA for binary numbers divisible by 3
- DFA for binary numbers divisible by 2

Other Regular Operations

- Reverse: $L^R = \{\omega = a_1 \ldots a_n | \omega^R = a_n \ldots a_1 \in L\}$
- Concatenation: $L_1 \cdot L_2 = \{\omega\nu | \omega \in L_1 \text{ and } \nu \in L_2\}$
- Star Closure: $L^* = \{\omega_1\omega_2 \ldots \omega_k | k \geq 0 \text{ and } \omega_i \in L\}$

The Reverse of a Regular Language

Theorem

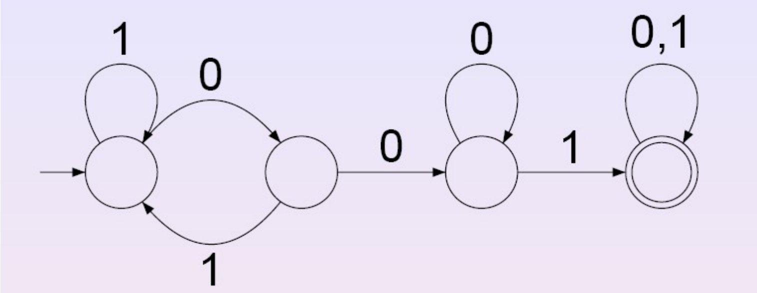
The reverse of a regular language is also a regular language.

- If a language can be recognized by a DFA that reads strings from right to left, then there is an "normal" DFA (one that reads from left to right) that accepts the same language.
- Counter-intuitive! DFAs have finite memory...

Reversing a DFA

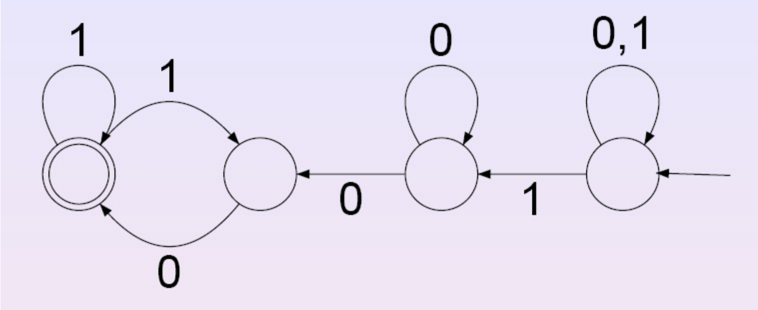
- Assume L is a regular language. Let M be a DFA that recognizes L
- We will build a machine M^R that accepts L^R
- If M accepts ω, then ω describes a directed path, in M, from the start state to a final state.
- First attempt: Try to define M^R as M as follows
  - Reverse all transitions
  - Turn the start state to a final state
  - Turn the final states to start states!
- But, as such, M^R is not always a DFA.
  - It could have many start states.
  - Some states may have too many outgoing transitions or none at all!

EXAMPLE



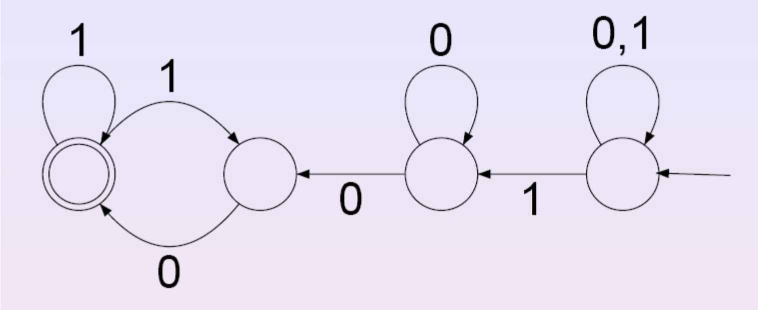
- What language does this DFA recognize?
  - All strings that contain a substring of 2 or more 0s followed by a 1.

Reversing the DFA



- What happens with input 100?
  - There are multiple transitions from a state labeled with the same symbol.
  - State transitions are not deterministic any more: the next state is not uniquely determined by the current state and the current input. $\rightarrow$ Nondeterminism

REVERSING THE DFA

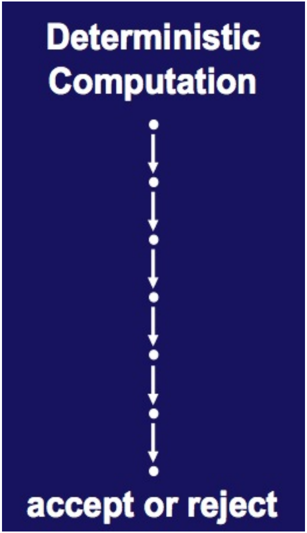


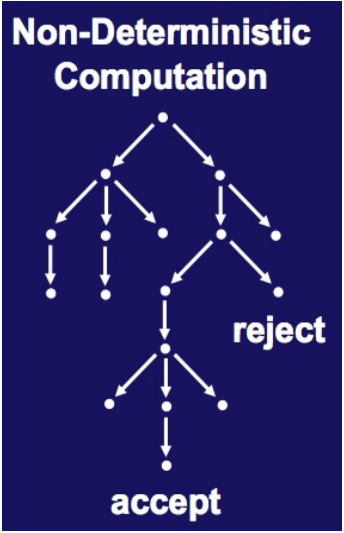
- We will say that this machine accepts a string if there is some path that reaches an accept state from a start state.

How does nondeterminism work?

- When a nondeterministic finite state automaton (NFA) reads an input symbol and there are multiple transitions labeled with that symbol
  - It splits into multiple copies of itself, and
  - follows all possibilities in parallel.

Deterministic vs Nondeterministic Computation





How does nondeterminism work?

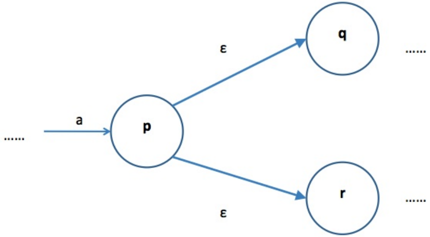
- When a nondeterministic finite state automaton (NFA) reads an input symbol and there are multiple transitions with labeled with that symbol
  - It splits into multiple copies of itself, and
  - follows all possibilities in parallel.
- Each copy of the machine takes one of the possible ways to proceed and continues as before.
- If there are subsequent choices, the machine splits again.
  - We have an unending supply of these machines that we can boot at any point to any state!

DFAs and NFAs – Other differences

- A state need not have a transition with every symbol in Σ
  - No transition with the next input symbol? $\Rightarrow$ that copy of the machine dies, along with the branch of computation associated with it.
  - If any copy of the machine is in a final state at the end of the input, the NFA accepts the input string.
- NFAs can have transitions labeled with ϵ – the empty string.

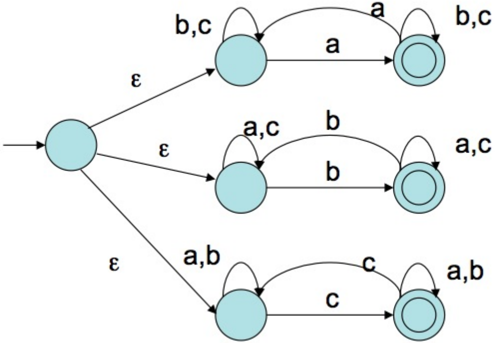
ϵ-TRANSITIONS

- If a transition with ϵ label is encountered, something similar happens:
    - The machine does not read the next input symbol.
    - It splits into multiple copies, one following each ϵ transition, and one staying at the current state.



- What the NFA arrives at p (say after having read input *a*, it splits into 3 copies

NFA Example

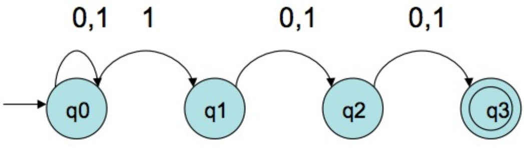


- Accepts all strings over $\Sigma = \{a, b, c\}$ with at least one of the symbols occuring an odd number of times.
- For example, the machine copy taking the upper ϵ transition guesses that there are an odd number of a's and then tries to verify it.

Nondeterminism

- So nondeterminism can also be viewed as
  - guessing the future, and
  - then verifying it as the rest of the input is read in.
- If the machine's guess is not verifiable, it dies!

NFA EXAMPLE



- Accepts all strings over $\Sigma = \{0, 1\}$ where the 3^{rd} symbol from the end is a 1.
  - How do you know that a symbol is the 3^{rd} symbol from the end?
- The start state guesses every 1 is the 3^{rd} from the end, and then the rest tries to verify that it is or it is not.
  - The machine dies if you reach the final state and you get one more symbol.

NFA–Formal Definition

- A Nondeterministic Finite State Acceptor (NFA) is defined as the 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$ where
  - Q is a finite set of states
  - Σ is a finite set of symbols – the alphabet
  - $\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^Q$, is the next-state function
    - $2^Q = \{P | P \subseteq Q\}$
  - $q_0 \in Q$ is the (label of the) start state
  - $F \subseteq Q$ is the set of final (accepting) states
- δ maps states and inputs (including ϵ) to a set of possible next states
- Similarly $\delta^* : Q \times \Sigma^* \rightarrow 2^Q$
  - $\delta^*(q, \epsilon) = \{q\}$
  - $\delta^*(q, \omega \cdot a) = \{p | \exists r \in \delta^*(q, \omega) \text{ such that } p \in \delta(r, a)\}$
    - a could be ϵ

HOW AN NFA ACCEPTS STRINGS

- An NFA accepts a string $\omega = x_1 x_2 \cdots x_n$ if a sequence of states $r_0 r_1 r_2 \cdots r_n$, $r_i \in Q$ exist such that
  1. $r_0 = q_0$ (Start in the initial state)
  2. $r_i \in \delta(r_{i-1}, x_i)$ for $i = 1, 2, \ldots n$ (Move from state to state – nondeterministically: r_i is one of the allowable next states)
  3. $r_n \in F$ (End up in a final state)

NONDETERMINISTIC VS DETERMINISTIC FA

- We know that DFAs accept regular languages.
- Are NFAs strictly more powerful than DFAs?
  - Are there languages that some NFA will accept but no DFA can accept?
- It turns out that NFAs and DFAs accept the same set of languages.
  - Q is finite $\Rightarrow |2^Q| = |\{P | P \subseteq Q\}| = 2^{|Q|}$ is also finite.

NFAs and DFAs are equivalent

Theorem

Every NFA has an equivalent DFA.

Proof Idea

- Convert the NFA to an equivalent DFA that accepts the same language.
- If the NFA has k states, then there are 2^k possible subsets (still finite)
- The states of the DFA are labeled with subsets of the states of the NFA
- Thus the DFA can have up to 2^k states.

NFAs and DFAs are equivalent

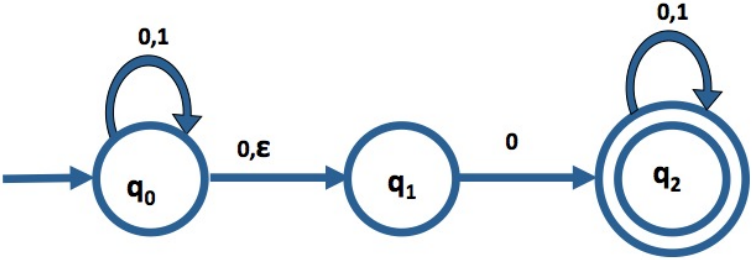
Theorem

Every NFA has an equivalent DFA.

Construction

- Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA. We construct $M = (Q', \Sigma, \delta', q_0', F')$.
  1. $Q' = 2^Q$, the power set of Q
  2. For $R \in Q'$ and $a \in \Sigma$, let $\delta'(R, a) = \{q \in Q | q \in \epsilon(\delta(r, a))$ for some $r \in R\}$
     - For $R \in Q$, the ϵ-closure of R, is defined as $\epsilon(R) = \{q | q \text{ is reachable from some } r \in R$ $\text{by traveling along zero or more } \epsilon - \text{transitions}\}$
  3. $q_0' = \epsilon(\{q_0\})$
  4. $F' = \{R \in Q' | R \cap F \neq \phi\}$: at least one of the states in R is a final state of N

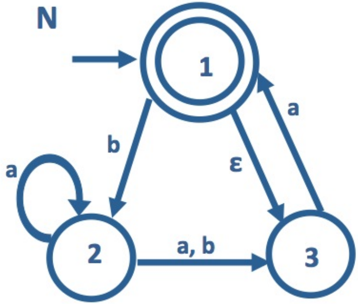
NFA EXAMPLE



- Note that q_0 has an ϵ-transition
- Some states (e.g., q_1) do not have a transition for some of the symbols in Σ. Machine dies if it sees input 1 when it is in state q_1.
- $\epsilon(\{q_0\}) = \{q_0, q_1\}$

NFA to DFA Conversion Example

- Given $N = (\{1,2,3\},\{a,b\},\delta,1,\{1\})$, construct $M = (Q', \Sigma, \delta', q_0', F')$.

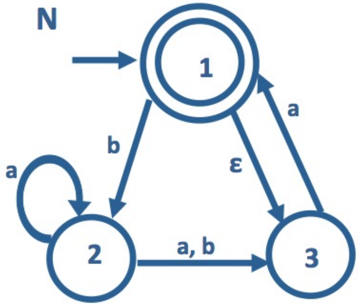


$\epsilon(\{1\}) = \{1,3\}$

	δ'	a	b
	ϕ	ϕ	ϕ
	$\{1\}$	ϕ	$\{2\}$
	$\{2\}$	$\{2,3\}$	$\{3\}$
	$\{3\}$	$\{1,3\}$	ϕ
	$\{1,2\}$	$\{2,3\}$	$\{2,3\}$
	$\{1,3\}$	$\{1,3\}$	$\{2\}$
	$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
	$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$

NFA TO DFA CONVERSION EXAMPLE

- Given $N = (\{1,2,3\},\{a,b\},\delta,1,\{1\})$, construct $M = (Q',\Sigma,\delta',q'_0,F')$.



$\epsilon(\{1\}) = \{1,3\}$

$\{1,3\}$ is the start state of M

	δ'	a	b
	ϕ	ϕ	ϕ
	$\{1\}$	ϕ	$\{2\}$
	$\{2\}$	$\{2,3\}$	$\{3\}$
	$\{3\}$	$\{1,3\}$	ϕ
	$\{1,2\}$	$\{2,3\}$	$\{2,3\}$
	$\{1,3\}$	$\{1,3\}$	$\{2\}$
	$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
	$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$

NFA to DFA Conversion Example

- Given $N = (\{1,2,3\}, \{a,b\}, \delta, 1, \{1\})$, construct $M = (Q', \Sigma, \delta', q_0', F')$.

  - States $\{1\}$ and $\{1,2\}$ do not appear as the next state in any transition! They can be removed
  - States with labels $\{1,3\}$ and $\{1,2,3\}$ are the final states of M.
  - We can now relabel the states as we wish!

	δ'	**a**	**b**
q_5	ϕ	ϕ	ϕ
q_2	$\{2\}$	$\{2,3\}$	$\{3\}$
q_1	$\{3\}$	$\{1,3\}$	ϕ
q_0	$\{1,3\}$	$\{1,3\}$	$\{2\}$
q_3	$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
q_4	$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$

NFA TO DFA CONVERSION EXAMPLE

- Given $N = (\{1,2,3\}, \{a,b\}, \delta, 1, \{1\})$, construct $M = (Q', \Sigma, \delta', q_0', F')$.

- States $\{1\}$ and $\{1,2\}$ do not appear as the next state in any transition! They can be removed
- States with labels $\{1,3\}$ and $\{1,2,3\}$ are the final states of M.
- We can now relabel the states as we wish!

δ'	**a**	**b**
q_5	q_5	q_5
q_2	q_3	q_1
q_1	q_0	q_5
$\mathbf{q_0}$	q_0	q_2
q_3	q_4	q_1
q_4	q_4	q_3