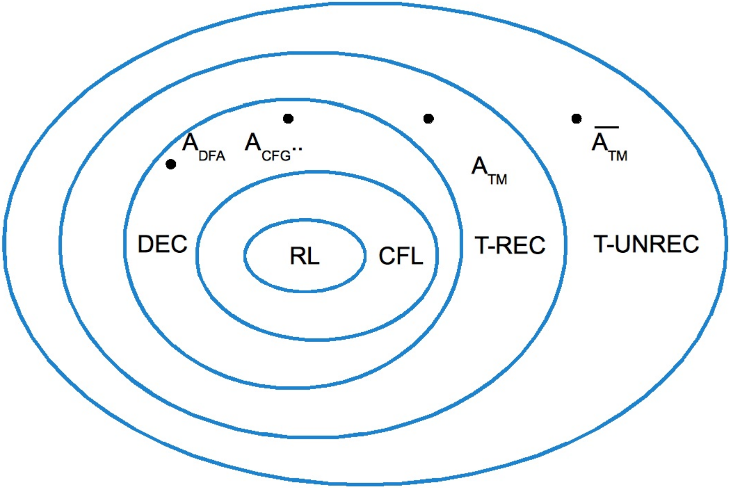
Formal Languages, Automata and Computation

Post Correspondence Problem

Review of Decidability and Reductions



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- If A is reducible to B and B is decidable, then A is also decidable.
- If A is undecidable and reducible to B, then B is undecidable.

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- Now we can check if $L(M_1) = \Phi$.

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- Note that M_1 either accepts w only or nothing!

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- So, if R decides $L(M_1)$ is empty,
  - then M does NOT accept w,
  - else M accepts w.
- If R decides E_{TM} then S decides A_{TM} – Contradiction.

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- Deterministic v.s nondeterministic computation histories.

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- The set of dominos $\left\{ \left[\frac{abc}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{acc}{ba}\right], \right\}$ does not have a solution.

Post Correspondence Problem

An instance of the PCP

A PCP instance over Σ is a finite collection P of dominos

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \cdots, \left[\frac{t_k}{b_k}\right] \right\}$$

where for all $i, 1 \leq i \leq k, t_i, b_i \in \Sigma^*$.

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where for all i, $1 \leq i \leq k$, $t_i, b_i \in \Sigma^*$.

Match

Given a PCP instance P, a match is a nonempty sequence

$$i_1, i_2, \ldots, i_\ell$$

of numbers from $\{1, 2, \ldots, k\}$ (with repetition) such that
$t_{i_1} t_{i_2} \cdots t_{i_\ell} = b_{i_1} b_{i_2} \cdots b_{i_\ell}$

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LANGUAGE FORMULATION:

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THEOREM 5.15

PCP is undecidable.

Proof: By reduction using computation histories. If PCP is decidable then so is A_{TM}. That is, if PCP has a match, then M accepts w.

PCP – The Structure of the Undecidability Proof

The reduction works in two steps:

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MPCP as a language problem

$MPCP = \{\langle P\rangle \mid P$ is a PCP instance and it has a match which starts with index 1$\}$

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$MPCP = \{\langle P \rangle \mid P$ is a PCP instance and it has a match which starts with index 1$\}$

- So the solution to MPCP starts with the domino $\left[\frac{t_1}{b_1}\right]$. We later remove this restriction in the second part of the proof.

PCP – The Structure of the Undecidability Proof

The reduction works in two steps:

1. We reduce A_{TM} to Modified PCP (MPCP).
2. We reduce MPCP to PCP.

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- So the solution to MPCP starts with the domino $\left[\frac{t_1}{b_1}\right]$. We later remove this restriction in the second part of the proof.
- We also assume that the decider for M never moves its head to the left of the input w.

PCP – The Proof

For input $\langle M, w \rangle$ of A_{TM}, construct an MPCP instance such that M accepts w iff P' has a match starting with domino 1

PCP – THE PROOF

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- The first part of the proof proceeds in 7 stages where we add different types of dominos to P' depending on the TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$.
- Using the dominos, we try to construct an accepting computation history for M accepting w.

PCP – ADDING THE RIGHT KIND OF DOMINOS

1. The first domino kicks of the computation history

$$\left[\frac{t_1}{b_1}\right] = \left[\frac{\#}{\#q_0 w_1 w_2 \cdots w_n \#}\right],$$

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3. Handle left moving transitions. For every $a, b, c \in \Gamma$ and every $q, r \in Q$ where $q \neq q_{reject}$

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PCP - How the dominos work

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- What exactly is going on ?
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- The process continues until M reaches a halting state and we then pad the upper string.

PCP – More domino types

6. For every $a \in \Gamma$,

$$\text{put } \left[\frac{\mathbf{a}\mathbf{q}_{\mathbf{accept}}}{\mathbf{q}_{\mathbf{accept}}}\right] \text{ and } \left[\frac{\mathbf{q}_{\mathbf{accept}}\mathbf{a}}{\mathbf{q}_{\mathbf{accept}}}\right] \text{into } P'$$

These dominos “clean-up” by adding any symbols to the top string while adding just the state symbol to the lower string.

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Just before these apply the upper and lower strings are like

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7. Finally we add the domino

$$\left[\frac{q_{accept}\#\#}{\#}\right]$$

to complete the match.

PCP PROOF – SUMMARY OF PART 1

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- Suppose we have the MPCP instance

$$P' = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \dots, \left[\frac{t_k}{b_k}\right] \right\}$$

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- We let P be the collection

$$P = \left\{ \left[\frac{\star t_1}{\star b_1 \star}\right], \left[\frac{\star t_2}{b_2 \star}\right], \ldots, \left[\frac{\star t_k}{b_k \star}\right] \left[\frac{\star \diamond}{\diamond}\right] \right\}$$

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Conclusion

PCP is undecidable!

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We know that language A is undecidable. By reducing A to B we want to show that the language B is also undecidable.

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4. B is undecidable.

Computable Functions

Idea

Turing Machines can also compute function $f : \Sigma^* \longrightarrow \Sigma^*$.

COMPUTABLE FUNCTIONS

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- Let $f(\langle n_1, n_2 \rangle) \stackrel{def}{=} \langle n \rangle$ where n_1 and n_2 are integers and $n = n_1 * n_2$. Then f is computable.

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The function f is called a reduction of A to B.

Theorem 5.22

If $A <_m B$ and B is decidable, then A is decidable.

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If $A <_m B$ and B is decidable, then A is decidable.

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If $A <_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

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PROOF

Essentially the same as the previous proof.

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Useful observation:

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- Suppose you can show $A_{TM} <_m \overline{B}$
- This means $\overline{A_{TM}} <_m B$
- Since $\overline{A_{TM}}$ is Turing-unrecognizable then B is Turing-unrecognizable.

Example of Use

Theorem 5.30

$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\}$ is neither Turing recognizable nor co-Turing-recognizable.

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PROOF IDEA

We show

- $\overline{A_{TM}} <_m EQ_{TM}$
- $\overline{A_{TM}} <_m \overline{EQ_{TM}}$
- These then imply the theorem.

Proof for $\overline{A_{TM}} <_m EQ_{TM}$

We show $A_{TM} <_m \overline{EQ_{TM}}$ (and hence $\overline{A_{TM}} <_m EQ_{TM}$) with the following f:

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