FORMAL LANGUAGES, AUTOMATA AND COMPUTATION

TURING MACHINES

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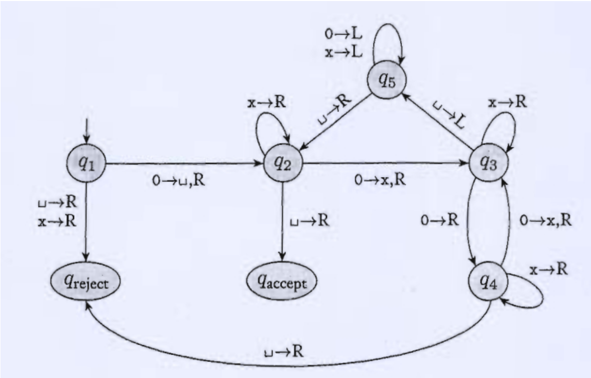
Turing Machines-Synopsis

- The most general model of computation
- Computations of a TM are described by a sequence of configurations.
  - Accepting Configuration
  - Rejecting Configuration
- Turing-recognizable languages
  - TM halts in an accepting configuration if w is in the language.
  - TM may halt in a rejecting configuration or go on indefinitely if w is not in the language.
- Turing-decidable languages
  - TM halts in an accepting configuration if w is in the language.
  - TM halts in a rejecting configuration if w is not in the language.

Example TM-2

- A Turing machine that decides $A = \{0^{2^n} \mid n \geq 0\}$
- $M =$ "On Input string w
  1. Sweep left-to-right across the tape, crossing off every other 0.
  2. If in 1) that tape has one 0 left, *accept* (Why?)
  3. If in 1) tape has more than one 0, and the number of 0's is odd, *reject*. (Why?)
  4. Return the head to the left end of the tape.
  5. Go to 1)"
- Basically every sweep cuts the number of 0's by two.
- At the end only 1 should remain and if so the original number of zeroes was a power of 2.'

Example TM-2



Configurations for input 0000.

1. $q_1 0000\sqcup$
2. $\sqcup q_2 000\sqcup$
3. $\sqcup x q_3 00\sqcup$
4. $\sqcup x0q_4 0\sqcup$
5. $\sqcup x0xq_3\sqcup$
6. $\sqcup x0q_5 x\sqcup$
7. $\sqcup xq_5 0x\sqcup$
8. $\sqcup q_5 x0x\sqcup$
9. $q_5 \sqcup x0x\sqcup$
10. $\sqcup q_2 x0x\sqcup$
11. $\sqcup xq_2 0x\sqcup$
12. $\sqcup xxq_3 x\sqcup$
13. $\sqcup xxxq_3\sqcup$
14. $\sqcup xxq_5 x\sqcup$
15. $\sqcup xq_5 xx\sqcup$
16. $\sqcup q_5 xxx\sqcup$
17. $q_5 \sqcup xxx\sqcup$
18. $\sqcup q_2 xxx\sqcup$
19. $\sqcup xq_2 xx\sqcup$
20. $\sqcup xxq_2 x\sqcup$
21. $\sqcup xxxq_2\sqcup$
22. $\sqcup xxx \sqcup q_{accept}$

EXAMPLE TM-3

- A TM to add 1 to a binary number (with a 0 in front)
- M = "On input w
  1. Go to the right end of the input string
  2. Move left as long as a 1 is seen, changing it to a 0.
  3. Change the 0 to a 1, and halt."
- For example, to add 1 to $w = 0110011$
  - Change all the ending 1's to 0's $\Rightarrow$ 0110000
  - Change the next 0 to a 1 $\Rightarrow$ 0110100
- Now let's design a TM for this problem.

Variants of TMs

- We defined the basic Turing Machine
  - Single tape (infinite in one direction)
  - Deterministic state transitions
- We could have defined many other variants:
  - Ordinary TMs which need not move after every move.
  - Multiple tapes – each with its own independent head
  - Nondeterministic state transitions
  - Single tape infinite in both directions
  - Multiple tapes but with a single head
  - Multidimensional tape (move up/down/left/right)

Equivalence of Power

- A computational model is robust if the class of languages it accepts does not change under variants.
  - We have seen that DFA's are robust for nondeterminism.
  - But not PDAs!
- The robustness of Turing Machines is by far greater than the robustness of DFAs and PDAs.
- We introduce several variants on Turing machines and show that all these variants have equal computational power.
- When we prove that a TM exists with some properties, we do not deal with questions like
  - How large is the TM? or
  - How complex is it to "program" that TM?
- At this point we only seek existential proofs.

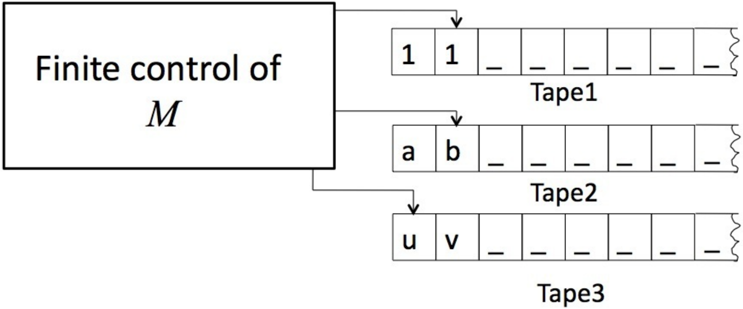
Turing Machines with the Stay Option

- Suppose in addition moving Left or Right, we give the option to the TM to stay (S) on the current cell, that is:

$$\delta : Q \times \Gamma = Q \times \Gamma \times \{L, R, S\}$$

- Such a TM can easily simulate an ordinary TM: just do not use the S option in any move.
- An ordinary TM can easily simulate a TM with the stay option.
  - For each transition with the S option, introduce a new state, and two transitions
    - One transition moves the head right, and transits to the new state.
    - The next transition moves the head back to left, and transits to the previous state.

Multitape Turing Machines



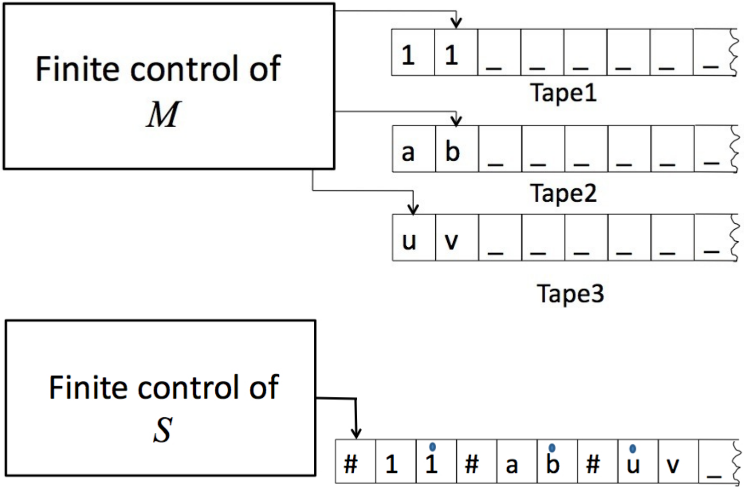
Multitape Turing Machines

- A multitape Turing Machine is like an ordinary TM
  - There are k tapes
  - Each tape has its own independent read/write head.
- The only fundamental difference from the ordinary TM is δ – the state transition function.

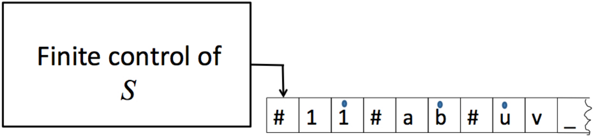
$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k$$

- The δ entry $\delta(q_i, a_1, \ldots, a_k) = (q_j, b_1, \ldots, b_k, L, R, L, ...L)$ reads as :
  - If the TM is in state q_i and
  - the heads are reading symbols a_1 through a_k,
  - Then the machine goes to state q_j, and
  - the heads write symbols b_1 through b_k, and
  - Move in the specified directions.

Simulating a Multitape TM with an Ordinary TM



Simulating a Multitape TM with an Ordinary TM



- We use # as a delimiter to separate out the different tape contents.
- To keep track of the location of heads, we use additional symbols
  - Each symbol in Γ has a "dotted" version.
  - A dotted symbol indicates that the head is on that symbol.
  - Between any two #'s there is only one symbol that is dotted.
- Thus we have 1 real tape with *k* "virtual' tapes, and
- 1 real read/write head with *k* "virtual" heads.

Simulating a Multitape TM with an Ordinary TM

- Given input $w = w_1 \cdots w_n$, S puts its tape into the format that represents all k tapes of M

$$\# \dot{w}_1 \; w_2 \cdots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \cdots \#$$

- To simulate a single move of M, S starts at the leftmost $\#$ and scans the tape to the rightmost $\#$.
  - It determines the symbols under the "virtual" heads.
  - This is remembered in the finite state control of S. (How many states are needed?)
- S makes a second pass to update the tapes according to M.
- If one of the virtual heads, moves right to a $\#$, the rest of tape to the right is shifted to "open up" space for that "virtual tape". If it moves left to a $\#$, it just moves right again.

Simulating a Multitape TM with an Ordinary TM

- Thus from now on, whenever needed or convenient we will use multiple tapes in our constructions.
- You can assume that these can always be converted to a single tape standard TM.

Nondeterministic Turing Machines

- We defined the state transition of the ordinary TM as

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

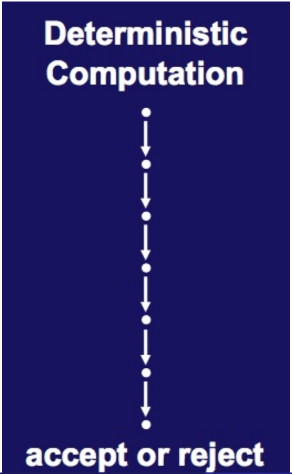
- A nondeterministic TM would proceed computation with multiple next cnfigurations. δ for a nondeterministic TM would be

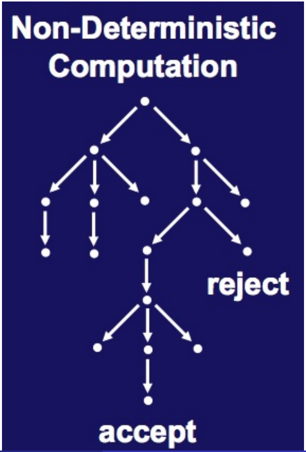
$$\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

  ($\mathcal{P}(S)$ is the power set of S.)
- This definition is analogous to NFAs and PDAs.

NONDETERMINISTIC TURING MACHINES

- A computation of a Nondeterministic TM is a tree, where each branch of the tree is looks like a computation of an ordinary TM.





Nondeterministic Turing Machines

- If a single branch reaches the accepting state, the Nondeterministic TM accepts, even if other branches reach the rejecting state.
- What is the power of Nondeterministic TMs?
  - Is there a language that a Nondeterministic TM can accept but no deterministic TM can accept?

Nondeterministic Turing Machines

Theorem

Every nondeterministic Turing machine has an equivalent deterministic Turing Machine.

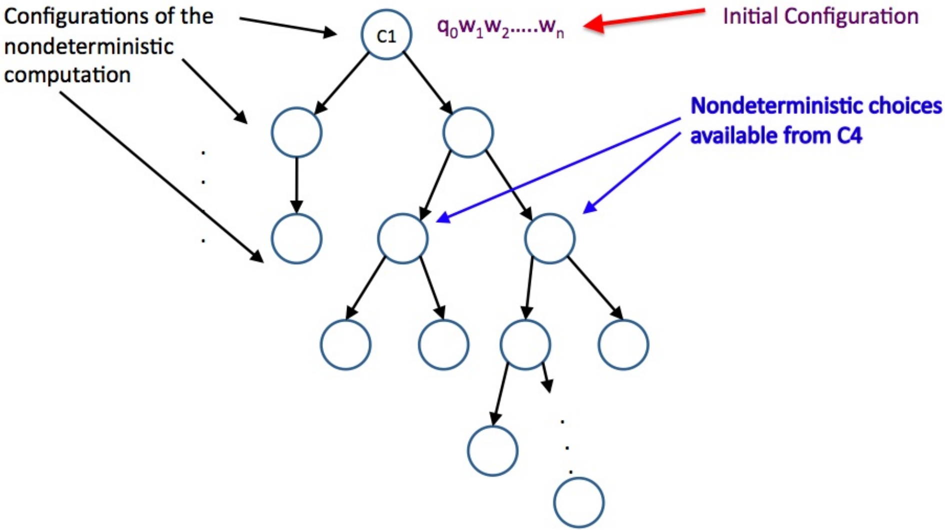
Proof Idea

- Timeshare a deterministic TM to different branches of the nondeterministic computation!
- Try out all branches of the nondeterministic computation until an accepting configuration is reached on one branch.
- Otherwise the TM goes on forever.

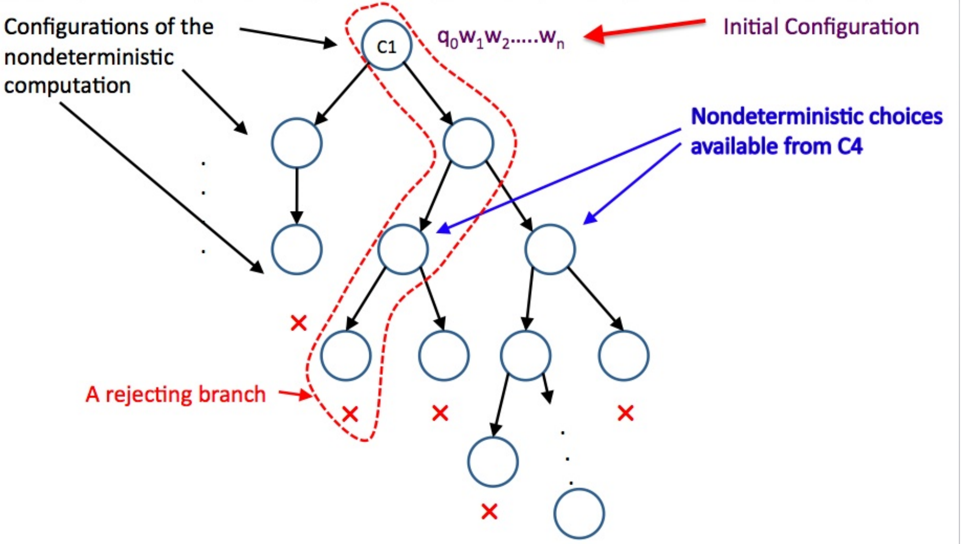
Nondeterministic Turing Machines

- Deterministic TM *D* simulates the Nondeterministic TM *N*.
- Some of branches of the *N*'s computations may be infinite, hence its computation tree has some infinite branches.
- If *D* starts its simulation by following an infinite branch, *D* may loop forever even though *N*'s computation may have a different branch on which it accepts.
- This is a very similar problem to processor scheduling in operating systems.
  - If you give the CPU to a (buggy) process in an infinite loop, other processes "starve".
- In order to avoid this unwanted situation, we want *D* to execute all of *N*'s computations concurrently.

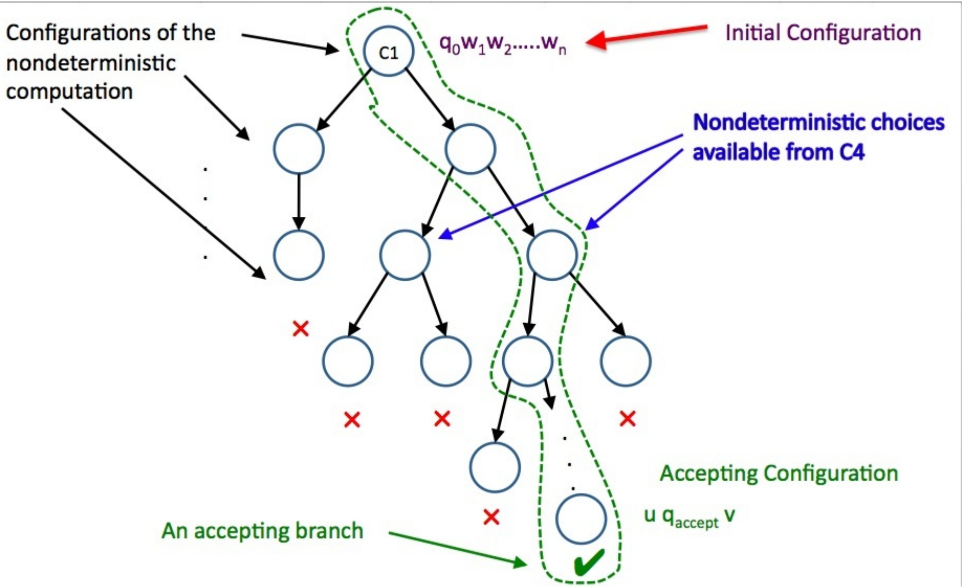
NONDETERMINISTIC COMPUTATION



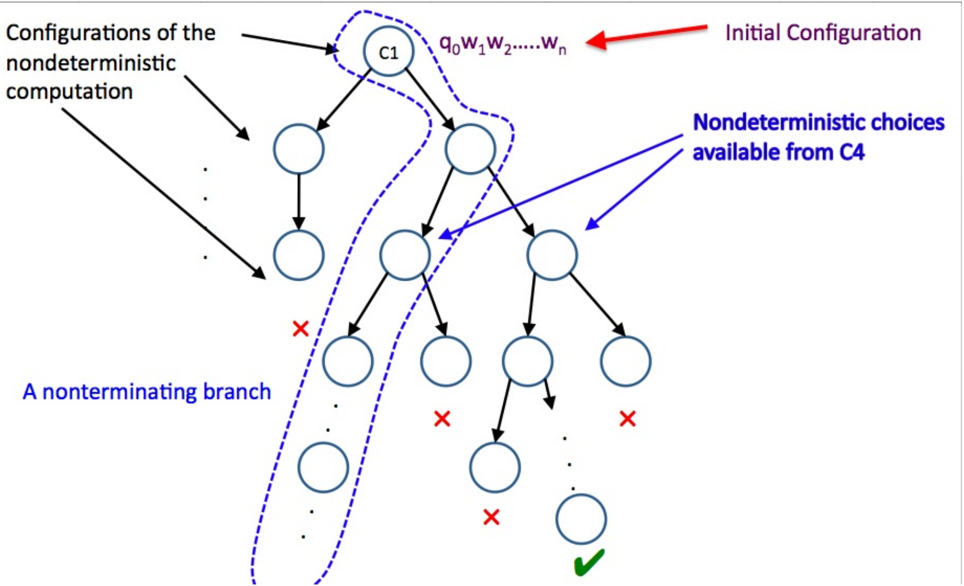
Nondeterministic Computation



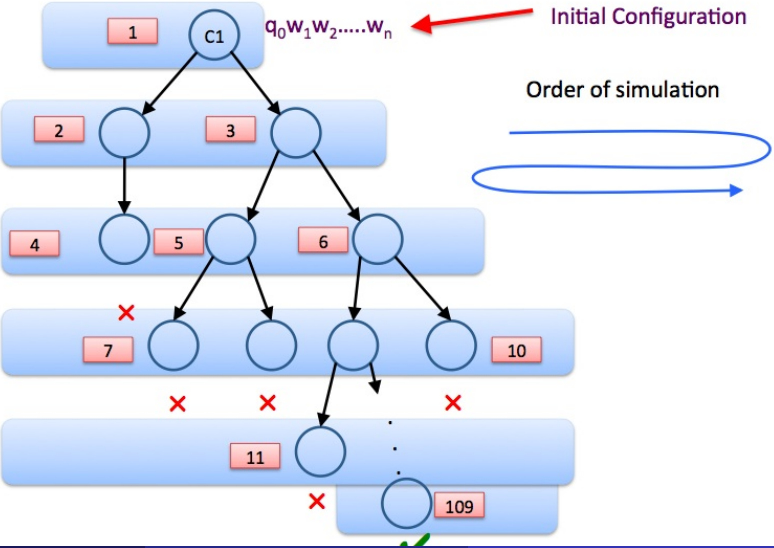
Nondeterministic Computation



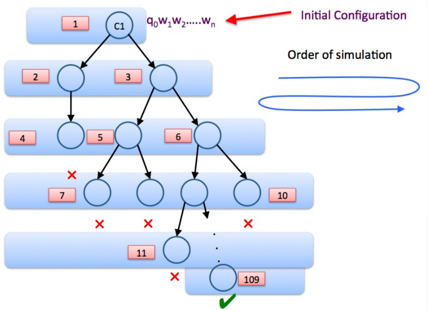
Nondeterministic Computation



Simulating Nondeterministic Computation



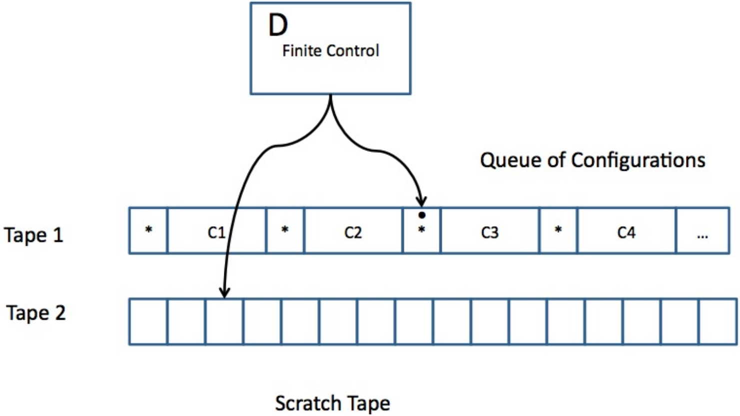
Simulating Nondeterministic Computation



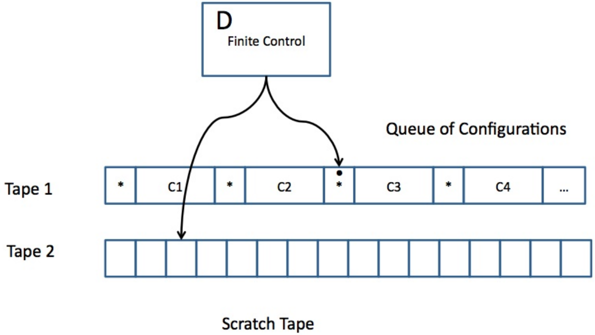
- During simulation, *D* processes the configurations of *N* in a breadth-first fashion.
- Thus *D* needs to maintain a queue of *N*'s configurations (Remember queues?)
- *D* gets the next configuration from the head of the queue.
- *D* creates copies of this configuration (as many as needed)
- On each copy, *D* simulates one of the nondeterministic moves of *N*.
- *D* places the resulting configurations to the back of the queue.

Structure of the Simulating DTM

- *N* is simulated with 2-tape DTM, *D*
  - Note that this is different from the construction in the book!

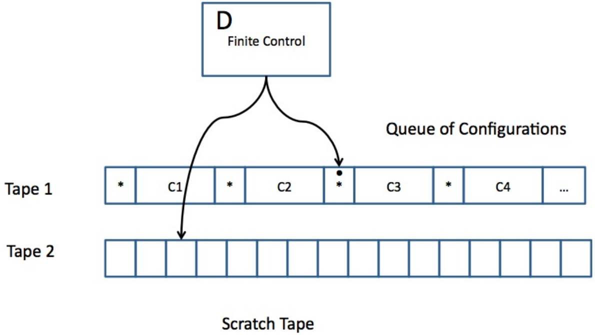


How D Simulates N



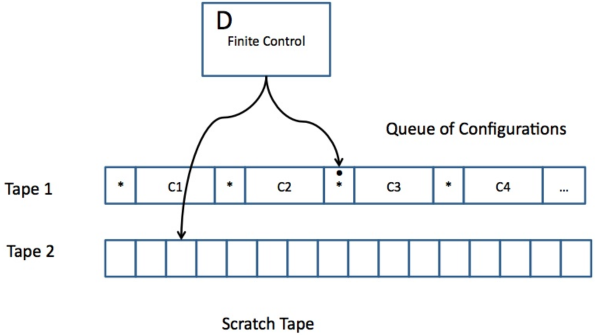
- Built into the finite control of D is the knowledge of what choices of moves N has for each state and input.

How *D* Simulates *N*



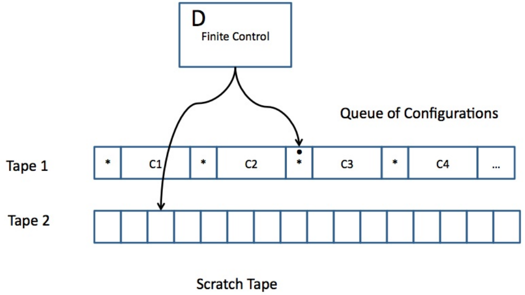
1. *D* examines the state and the input symbol of the current configuration (right after the dotted separator)
2. If the state of the current configuration is the accept state of *N*, then *D* accepts the input and stops simulating *N*.

How *D* Simulates *N*



1. *D* copies *k* copies of the current configuration to the scratch tape.
2. *D* then applies one nondeterministic move of *N* to each copy.

How *D* Simulates *N*



3. *D* then copies the new configurations from the scratch tape, back to the end of tape 1 (so they go to the back of the queue), and then clears the scratch tape.
4. *D* then returns to the marked current configuration, and “erases” the mark, and “marks” the next configuration.
5. *D* returns to step 1), if there is a next configuration. Otherwise rejects.

HOW D SIMULATES N

- Let m be the maximum number of choices N has for any of its states.
- Then, after n steps, N can reach at most $1 + m + m^2 + \cdots + m^n$ configurations (which is at most nm^n)
- Thus D has to process at most this many configurations to simulate n steps of N.
- Thus the simulation can take exponentially more time than the nondeterministic TM.
- It is not known whether or not this exponential slowdown is necessary.

IMPLICATIONS

COROLLARY

A language is Turing-recognizable if and only if some nondeterministic TM recognizes it.

COROLLARY

A language is decidable if and only of some nondeterministic TM decides it.