Formal Languages, Automata and Computation
Lecture Slides

Carnegie Mellon University in Qatar

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Administrative Stuff

- Textbook: *Introduction to the Theory of Computation* by Michael Sipser (MIT)
- Evaluation:
  - 1 Midterm Exam
  - 1 Final Exam
  - 6-8 Homeworks

What is this course about? – Formal Languages

- An abstraction of the notion of a “problem”
- Problems are cast **either** as Languages (= sets of “Strings”)
    - ”Solutions” determine if a given “string” is in the set or not
        - e.g., Is a given integer, *n*, prime?
- **Or**, as transductions between languages
    - “Solutions” transduce/transform the input string to an output string
        - e.g., What is 3+5?

What is this course about? – Formal Languages

- So essentially all computational processes can be reduced to one of
  - Determining membership in a set (of strings)
  - Mapping between sets (of strings)
- We will formalize the concept of mechanical computation by
  - giving a precise definition of the term “algorithm”
  - characterizing problems that are or are not suitable for mechanical computation.

What is this course about? – Automata

- Automata (singular *Automaton*) are abstract mathematical devices that can
  - Determine membership in a set of strings
  - Transduce strings from one set to another
- They have all the aspects of a computer
  - input and output
  - memory
  - ability to make decisions
  - transform input to output
- Memory is crucial:
  - Finite Memory
  - Infinite Memory
    - Limited Access
    - Unlimited Access

What is this course about?– Automata

- We have different types of automata for different classes of languages.
- They differ in
  - the amount of memory then have (finite vs infinite)
  - what kind of access to the memory they allow.
- Automata can behave **non-deterministically**
  - A non-deterministic automaton can at any point, among possible next steps, pick one step and proceed
  - This gives the conceptual illusion of (infinitely) parallel computation for some classes of automata
    - All branches of a computation proceed in parallel (sort of)
  - More on this later

What is this course about?– Complexity

- How much resource does a computation consume?
  - Time and Space
- What are the implications of nondeterminism for complexity?
- How can we classify problems into classes based on their resource use?
  - Are there problems with very unreasonable resource usage (Intractable problems)?
  - How can we characterize such problems?
    - P vs. NP, PSPACE, Log Space

What is this course about?– Computability

- What is computational power?
  - Automaton 1 tells Automaton 2
    "Tell me what kinds of problems you can solve and I will tell you how powerful you are? "
- What does computational power depend on? (it turns out, not "speed")
- What does it mean for a problem to be computable ?
- Are there any uncomputable functions or unsolvable problems?
  - What does this mean?
  - Why do we care?

Applications/Relevance

- Pattern matching
  - Perl Hacking
  - Bioinformatics
  - Lexical analysis
- Design and Verification
  - Hardware
  - Software
  - Communication Protocols
- Parsing Languages
  - Compiler construction
  - XML Analysis
  - Natural language processing, Machine Translation
- Algorithm design and analysis

Decision Problems

- A decision problem is a function with a YES/NO output
- We need to specify
  - the set A of possible inputs (usually A is infinite)
  - the subset $B \subseteq A$ of YES instances (usually B is also infinite)
- The subset B should have a finite description!

Decision Problems – Examples

- *A*: integers
  - is_even?(x)
  - is_prime?(x)
- *A*: integers $\times$ integers
  - is_relatively_prime?(x,y)

Decision Problems – Examples

- A: set of all pairs (G, t)
  - G is a $\{$finite set of triples of the sort $(i, j, w)\}$,
  - i and j are integers and w is real
  - The finite set encodes the edges of a weighted directed graph G.
  - $A = \{\ldots(\{\ldots,(3,4,5.6),\ldots\},8.0),\ldots\}$
- Each pair in A, (G, t), represents a graph G and a threshold t
- Does G have a path that goes through all nodes once with total weight $< t$?
  - Travelling Salesperson Problem
- A is the set of all TSP instances.

Encoding Sets

- Sets can be
  - Finite
  - Infinite
    - Countably Infinite: can be put in one-to-one correspondence with natural numbers (e.g., rational numbers, integers)
    - Uncountably Infinite: can NOT be put in one-to-one correspondence with natural numbers (e.g., real numbers)

Encoding Sets

- In real life, we use many different types of data: integers, reals, vectors, complex numbers, graphs, programs (your program is somebody else's data).
- These can all be encoded as strings
- So we will have only one data type: strings

Strings

- An alphabet is any finite set of distinct symbols
  - $\{0, 1\}$, $\{0,1,2,\ldots,9\}$, $\{a,b,c\}$
  - We denote a generic alphabet by Σ
- A string is any finite-length sequence of elements of Σ.
- e.g., if $\Sigma = \{a, b\}$ then *a*, *aba*, *aaaa*, ..., *abababbaab* are some strings over the alphabet Σ

Strings

- The length of a string ω is the number of symbols in ω. We denote it by $|\omega|$. $|aba| = 3$.
- The symbol ϵ denotes a special string called the empty string
  - ϵ has length 0
- String concatenation
  - If $\omega = a_1, \ldots, a_n$ and $\nu = b_1, \ldots, b_m$ then $\omega \cdot \nu$ (or $\omega\nu$) $= a_1, \ldots, a_n b_1, \ldots, b_m$
  - Concatenation is associative with ϵ as the identity element.
- If $a \in \Sigma$, we use a^n to denote a string of n a's concatenated
  - $\Sigma = \{0, 1\}, 0^5 = 00000$
  - $a^0 =_{def} \epsilon$
  - $a^{n+1} =_{def} a^n a$

Strings

- The reverse of a string ω is denoted by ω^R.
  - $\omega^R = a_n, \ldots, a_1$
- A substring y of a string ω is a string such that $\omega = xyz$ with $|x|, |y|, |z| \geq 0$ and $|x| + |y| + |z| = |\omega|$
- If $\omega = xy$ with $|x|, |y| \geq 0$ and $|x| + |y| = |\omega|$, then x is prefix of ω and y is a suffix of ω.
  - For $\omega = abaab$,
    - ϵ, *a*, *aba*, and *abaab* are some prefixes
    - ϵ, *abaab*, *aab*, and *baab* are some suffixes.

Strings

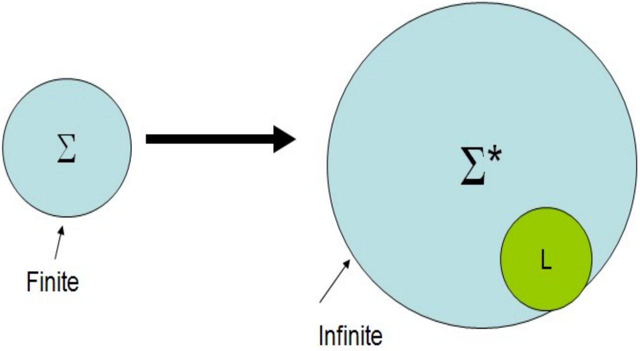
- The set of all possible strings over Σ is denoted by Σ^*.
- We define $\Sigma^0 = \{\epsilon\}$ and $\Sigma^n = \Sigma^{n-1} \cdot \Sigma$
  - with some abuse of the concatenation notation applying to sets of strings now
- So $\Sigma^n = \{\omega | \omega = xy \text{ and } x \in \Sigma^{n-1} \text{ and } y \in \Sigma\}$
- $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \cdots \Sigma^n \cup \cdots = \bigcup_0^\infty \Sigma^i$
  - Alternatively, $\Sigma^* = \{x_1, \ldots, x_n | n \geq 0 \text{ and } x_i \in \Sigma \text{ for all } i\}$
- Φ denotes the empty set of strings $\Phi = \{\}$,
  - but $\Phi^* = \{\epsilon\}$

Strings

- Σ^* is a countably infinite set of finite length strings
- If x is a string, we write x^n for the string obtained by concatenating n copies of x.
  - $(aab)^3 = aabaabaab$
  - $(aab)^0 = \epsilon$

Languages

- A language L over Σ is any subset of Σ^*



- L can be finite or (countably) infinite

Some Languages

- $L = \Sigma^*$ – The mother of all languages!
- $L = \{a, ab, aab\}$ – A fine finite language.
  - Description by enumeration
- $L = \{a^n b^n : n \geq 0\} = \{\epsilon, \text{ ab}, \text{ aabb}, \text{ aaabbb}, \ldots\}$
- $L = \{\omega | n_a(\omega) \text{ is even}\}$
  - $n_x(\omega)$ denotes the number of occurrences of x in ω
  - all strings with even number of a's.
- $L = \{\omega | \omega = \omega^R\}$
  - All strings which are the same as their reverses – palindromes.
- $L = \{\omega | \omega = xx\}$
  - All strings formed by duplicating some string once.
- $L = \{\omega | \omega$ is a syntactically correct Java program $\}$

Languages

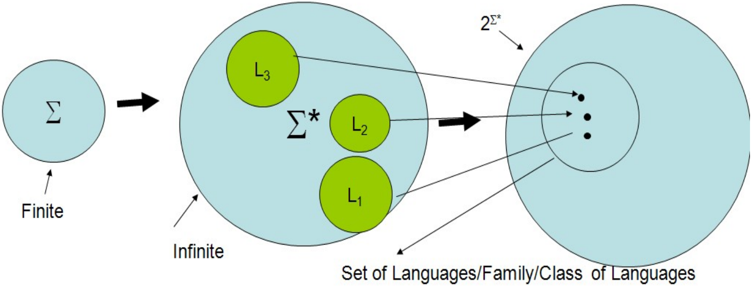
- Since languages are sets, all usual set operations such as intersection and union, etc. are defined.
- Complementation is defined with respect to the universe Σ^* : $\overline{L} = \Sigma^* - L$

Languages

- If L, L_1 and L_2 are languages:
  - $L_1 \cdot L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
  - $L^0 = \{\epsilon\}$ and $L^n = L^{n-1} \cdot L$
  - $L^* = \bigcup_0^\infty L^i$
  - $L^+ = \bigcup_1^\infty L^i = L^* - \{\epsilon\}$

Sets of Languages

- The power set of Σ^*, the set of all its subsets, is denoted as 2^{Σ^*}



Describing Languages

- Interesting languages are infinite
- We need finite descriptions of infinite sets
  - $L = \{a^n b^n : n \geq 0\}$ is fine but not terribly useful!
- We need to be able to use these descriptions in mechanizable procedures