Chapter 13

The Arithmetical Hierarchy

We may think of $\overline{K}$ as posing the problem of induction for computational devices, for it is impossible to tell for sure whether a given computation will never halt. Thus, K is effectively refutable and $\overline{K}$ is effectively verifiable. We know from the philosophy of science that universal hypotheses are refutable and existential hypotheses are verifiable. This correspondence also holds, if we think of the expressions of the sets in Kleene normal form. Kleene normal form prenex normal form with U as the only predicate. Thus, we have

$$K(x) \iff \exists z\ U(x, (z)_1, (z)_2, \langle x \rangle)$$

$$\overline{K}(x) \iff \forall z\ \neg U(x, (z)_1, (z)_2, \langle x \rangle)$$

Thus, we may think of $K(x)$ as an "existential hypothesis" and of $\overline{K}(x)$ as a "universal hypothesis" given that instances of U are "observable" (e.g., the "scientist" is treating program x as a black box and watching what it does in various numbers of steps of computation on input x.

It is also notorious in the philosophy of science that most hypotheses are neither verifiable nor refutable. Thus, Kant's antinomies of pure reason include such statements as that space is infinite, matter is infinitely divisible, and the series of efficient causes is infinite. These hypotheses all have the form

$$\forall x\ \exists y\ (\Phi(x, y))$$

But computers have their own "antinomies of pure reason", for example:

$$Tot(x) \iff \phi_x \text{ is total } \iff \forall w\ \exists z\ U(x, (z)_1, (z)_2, \langle w \rangle)$$

$$\mathit{Inf}(x) \iff W_x \text{ is infinite } \iff \forall w\ \exists z\ \exists_{y>w}\ (U(x, (z)_1, (z)_2, \langle y \rangle))$$

In both cases, things get worse as the quantifier structure of the hypothesis becomes more complex, where complexity is measured as number of blocks of quantifiers: i.e., $\exists\forall\forall\forall\exists\exists$ counts as three blocks. Also, verification and refutation depend on whether the block is $\exists$ or $\forall$ and in general we will want to keep track of the leading quantifier. We can count quantifier alternations as follows.

13.1 The Arithmetical Hierarchy

$\Sigma[A]_0(P) \iff P$ is recursive in A
$\Sigma[A]_{n+1}(P) \iff \exists R\ (\Sigma[A]_n(R)] \wedge \forall x\ (P(x) \leftrightarrow \exists y\ \neg R(x,y))$

$\Pi[A]_0(P) \iff P$ is recursive in A
$\Pi[A]_{n+1}(P) \iff \exists R\ (\Pi[A]_n(R)] \wedge \forall x\ (P(x) \leftrightarrow \forall y\ \neg R(x,y))$

$\Delta[A]_n(P) \iff \Sigma[A]_n(P) \wedge \Pi[A]_n(P)$
$Arith = \bigcup_n \Sigma[A]_n$

N.B. you can define the whole thing as a hierarchy of sets rather than relations by treating relations as sets of coded n-tuples.

Proposition 13.1 *(basic structure and closure laws)*

1. $\Delta[A]_n =$ recursive in A; $\Sigma[A]_n =$ r.e. in A; $\Pi[A]_n =$ co-r.e. in A
2. $\Pi[A]_n(R) \iff \Sigma[A]_n(\overline{R})$
3. $\Delta[A]_n \subseteq \Sigma[A]_n$, $\Pi[A]_n \subseteq \Delta[A]_{n+1}$
4. $\Delta[A]_n, S[A]n, P[A]n$ are closed downward under $\leq_M$
5. $\Delta[A]_n$ is closed under $\wedge, \vee, \neg$
6. $\Sigma[A]_n$ is closed under $\wedge, \vee, \exists$
7. $\Pi[A]_n$ is closed under $\wedge, \vee, \forall$

Exercise 13.1 *Prove it. Hint: use logical rules and some induction on n. These are all very easy so don't work too hard!*

13.2 The Arithmetical Hierarchy Theorem

We don't know yet whether the whole hierarchy collapses into some finite level. The simplest way to do it is to index the levels in the hierarchy the way we did for the r.e. sets and then diagonalize all the levels at once.

$$R \text{ is universal for } \Gamma \iff \big[\Gamma(R) \wedge \forall P\ \big(\Gamma(P) \rightarrow \exists n\ \forall \vec{x}\ (P(\vec{x}) \leftrightarrow R(n, \langle \vec{x} \rangle))\big)\big]$$

Define

$$W_{1,x}(\langle \vec{y} \rangle) \iff W_x(\langle \vec{y} \rangle)$$
$$W_{n+1,x}(\vec{y}) \iff \exists z\ (\neg W_{n,x}(\langle \vec{y}, \vec{z} \rangle))$$

Proposition 13.2 ***(universal indexing theorem)***

$$\forall_{n>0}(R(x,\vec{y}) \iff W_{n,x}(\langle\vec{y}\rangle) \text{ is universal for } \Sigma_n)$$

Proof: by induction.
The base case is immediate by the fundamental theorem for r.e. sets.
Inductively, suppose that $W_{n,x}(\langle\vec{y}\rangle)$ is universal for Σ_n.

Now suppose $\Sigma_{n+1}(R)$.

So $\exists R\ \left[\Sigma_n(R) \wedge \forall\vec{x}(P(\vec{x}) \leftrightarrow \exists y\ (\neg R(\vec{x},y)))\right]$.

By the induction hypothesis there exists k such that

$$\begin{aligned} \forall x\ P(x) &\iff \exists z\ \neg W_{n,k}(\langle\vec{x},z\rangle) \\ &\iff W_{n+1,k}(x) \end{aligned}$$

⊣

We can now form the usual Cantorian table for the Σ_n sets just as we did for the r.e. sets before:

$$T_n[x,y] = W_{n,x}(y)$$

Counterdiagonalizing, we obtain a non Σ_n set as follows

$$\begin{aligned} \overline{K}_n(x) &\iff \neg T_n[x,x] \\ &\iff \neg W_{n,x}(x) \end{aligned}$$

Proposition 13.3 ***(arithmetical hierarchy theorem)***

1. $\Sigma_n(K_n) \wedge \neg\Pi[A]_n(K_n)$.

2. $\neg\Sigma_n(\overline{K}_n) \wedge \Pi[A]_n(\overline{K}_n)$.

Proof: (1) the counter-diagonal set $\overline{K}_n$ differs from each set $W_{n,x}$. By the universal indexing theorem, $\overline{K}_n$ therefore differs from each Σ_n set. On the other hand, the definitional form of $\overline{K}_n$ witnesses its membership in Π_n.
(2) follows by duality.⊣

13.3 Jumps and the Hierarchy

We can also use the jumps to hold the hierarchy apart. An advantage of this approach is that it yields a constructive perspective on the Δ classes.

Proposition 13.4 ***(Post's theorem)***

1. $\Sigma[A]_{n+1}(R) \iff \exists P\ (\Pi[A]_n(P) \wedge R$ is r.e. in $P)$.
2. $\Sigma[A]_{n+1}(R) \iff \exists P\ (\Sigma[A]_n(P) \wedge R$ is r.e. in $P)$.
3. $\Sigma[A]_{n+1}(R) \iff R$ is r.e. in $A^{(n)}$.
4. $\Delta[A]_{n+1}(R) \iff R$ is recursive in $A^{(n)}$.
5. $n > 0 \Rightarrow A^{(n)}$ is $\Sigma[A]_n$-M-complete.

Corollary 13.5

1. $\Sigma_{n+1}(R) \iff R$ is r.e. in $\mathbf{0}^{(n)}$.
2. $\Delta_{n+1}(R) \iff R$ is recursive in $\mathbf{0}^{(n)}$.
3. $n > 0 \Rightarrow \mathbf{0}^{(n)}$ is Σ_n-M-complete.

Proof: *following Soare p. 64*

1. Suppose $\Sigma[A]_{n+1}(R)$.

So $\exists P\ \left[\Sigma[A]_n(P) \wedge \forall x\ (P(x) \leftrightarrow \exists y\ P(x,y))\right]$.

By the P-relativized projection theorem, R is r.e. in P.

Conversely, suppose $\exists P\ (\Pi[A]_n(P) \wedge R$ is r.e. in $P)$.

Then by the relativized fundamental theorem of r.e. sets:

$$\begin{aligned}
R(x) &\iff W_{P,y}(x)\\
&\iff \exists w\ U_P(y,(w)_0,(w)_1,\langle x\rangle)\\
&\iff \exists w\ U_{P|w}(y,(w)_0,(w)_1,\langle x\rangle)\\
&\iff \exists w\ \exists \text{ finite } S\ (S = P|w \wedge U_S(y,(w)_0,(w)_1,\langle x\rangle))\\
&\iff \exists w\ \exists k\ (lh(k) = w \wedge\\
&\qquad \forall_{v\le w}((P(v) \leftrightarrow \exists u \le w\ (k)_w = y) \wedge U_k(y,(w)_0,(w)_1,\langle x\rangle))
\end{aligned}$$

where we define $U_k(y,a,b,\langle x\rangle)$ just like the Kleene predicate except that when $lh(y) = 6$, we have

$$U_k(y,a,b,\langle x\rangle) = 1 \text{ if } \exists_{u\le w}\ (k)_w = y \text{ and}$$

$$U_k(y,a,b,\langle x\rangle) = 0 \text{ otherwise.}$$

This is clearly recursive in all arguments including k.

Also, $(P(v) \leftrightarrow \exists_{u \leq w}\ (k)_w = y)$ is $\Sigma[A]_{n+1}$ since $\Pi[A]_n(P)$.
Hence, $R(x)$ is $\Sigma[A]_{n+1}$.
2. The preceding construction could have been done just as well with $\overline{P}$.
5. By in induction: A' is $\Sigma[A]_1$-complete (proposition 12.4.4).
Suppose that $A^{(n)}$ is $\Sigma[A]_n$-complete.
Hence $\overline{A}^{(n)}$ is $\Sigma[A]_n$-complete.
Let $\Sigma[A]_{n+1}(B)$.
So $\exists P\ \Sigma[A]_n(P) \wedge B$ is r.e. in P (by part 2).
So B is r.e. in $A^{(n)}$, since $P{\leq_M}A^{(n)}$, (by induction hypothesis).
So $B{\leq_M}A^{(n+1)}$ (12.4.4).
3. By parts 1 and 5.
4.

$$\begin{aligned} \Delta[A]_{n+1}(R) \quad &\Longleftrightarrow \quad \Sigma[A]_{n+1}(R) \wedge \Sigma[A]_{n+1}(\overline{R}) \\ &\Longleftrightarrow \quad R, \overline{R} \text{ are r.e. in } A^{(n)} \text{ (by part 3)} \end{aligned}$$

$R \leq A^{(n)}$ (by the relativized proof that recursive sets are r.e. and co-r.e.).$\dashv$

Exercise 13.2 *How many $\Delta[A]_n$ degrees are there?*

Proposition 13.6 ***(arithmetical hierarchy theorem again)***

1. $\Sigma[A]_n(A^{(n)}) \wedge \neg\Pi[A]_n(A^{(n)})$.
2. $\neg\Sigma[A]_n(\overline{A}^{(n)}) \wedge \Pi[A]_n(\overline{A}^{(n)})$.

Proof: $A^{(n+1)}$ is $\Sigma[A]_n$-M-complete (by proposition 13.2.4).
$A^{(n)}$ is $\Delta[A]_n$ complete (by proposition 13.2.5).
Suppose $\Pi[A]_{n+1}(A^{(n+1)})$.
Then $\Delta[A]_{n+1}(A^{(n+1)})$.
Then $A^{(n+1)} \leq A^{(n)}$, contradicting proposition 12.4.2.$\dashv$