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THE COMPUTABLE TESTABILITY OF THEORIES MAKING UNCOMPUTABLE PREDICTIONS

1. INTRODUCTION

Consider a naively Popperian (1968) picture of scientific inquiry. A scientific theory may entail various observations in light of background information and what has been observed so far. To test the theory, the scientist sequentially derives predictions and checks them against the data as it comes in. When a mismatch is detected, the theory is rejected. Otherwise, the theory "passes muster" and is retained until such time as it is refuted. In general, we say that a scientist refutes a theory with certainty just in case no matter how the data comes in for eternity, the scientist eventually rejects the theory if it is false, and fails ever to reject it if it is true. According to this conception, refutation with certainty is a standard of success for scientific methods rather than a relation between theory and evidence.

Suppose that a computer refutes a given theory with certainty. Then according to the simple picture of inquiry just described, it would seem as though the computer must be able to derive each prediction made by the theory. For suppose otherwise. Then either the computer fails to derive any prediction from the theory for a given time (an error of omission) or the computer fallaciously derives a prediction that differs from the one genuinely entailed by the theory (an error of commission). If the computer is guilty of an error of omission, it must simply guess whether the (unknown) prediction of the theory would have agreed with the data, and for any guess made, the data can be arranged so that it is the wrong one. If the computer is guilty of an error of commission, the data may either agree with the erroneous prediction or agree with the theory. In the former case, the computer will forever fail to reject the theory when it is false, and in the latter case the computer will reject the theory when it is true. In each case, the computer fails to refute the theory with certainty.

Now suppose that the predictions made by a given theory are *impossible* for any computer to derive. Then each computer program for deriving predictions from the theory is guilty of *infinitely many* errors

Erkenntnis **43**: 29–66, 1995.

of omission or commission; for if it were guilty of only finitely many, its program could be "patched" with a finite lookup table correcting its mistakes, contradicting the assumption that no computer program correctly derives the theory's predictions.

A natural and important question now arises. Must it be possible for a computer to correctly derive all the predictions of a theory if the theory is to be effectively refuted with certainty? Or is there some *other* way to effectively refute a theory with certainty even though no computable method can derive all of its predictions and even though any given prediction made by the theory might be false for all we know *a priori?* The question is not merely theoretical, since there is increasing interest in the computability of prediction in physical theories.[1]

The surprising answer to our question, which will be seen to follow from a classic result of Hilbert and Bernays concerning the implicit definability of arithmetical truth, is that there is a universal hypothesis that is effectively refutable with certainty and that uniquely predicts the outcome of each observation for eternity, but whose predictions are in a precise sense *infinitely impossible* to derive by computational means. The methodological moral of this result is clear. There is a more powerful way for computable inquiry to proceed than by sequentially deriving predictions from a theory and then checking them against the data as it arrives.

Refutation with certainty is just one notion of successful inquiry. Verification with certainty is another. Or, following Peirce (1935), Reichenbach (1949), and Putnam (1963), we might demand that inquiry stabilize to the truth without ever achieving certainty (i.e. it is always possible so far as the scientist knows that his current verdict on the hypothesis might be taken back tomorrow, although after some finite time it will never again be taken back). One may now ask for each such notion of successful inquiry how uncomputable the predictions of the theory under investigation can be if computable inquiry is to be successful in that sense. In this paper we establish a complete table of the relations between different forms of computable inquiry and different forms of computable derivability of predictions. We also examine the converse questions: how computably untestable can a theory be, if its predictions are computably derivable in a given sense? One easy consequence of our results is that there exists a *computable* method whose reliability cannot be matched even by Bayesian agents of a highly idealized sort.[2] This raises the question whether Bayesian method should be viewed as an aid or as a hindrance to finding the truth.

2. EMPIRICAL HYPOTHESES AND PREDICTION

An empirical hypothesis makes claims about what will be observed, perhaps for eternity. We will suppose that there is at most a countable infinity of possible observations at a given time. We will also suppose that these possible observations can be effectively encoded by elements of some effectively decidable set $O \subseteq \omega$, where ω denotes the set of all natural numbers. For example, these numbers might be Gödel numbers of observation statements in some formalized language or code numbers representing finite vectors of dial readings. At each stage of inquiry another observation is recorded. If inquiry were to continue forever, an infinite sequence ε of observations (natural numbers) would be received. Let O^ω represent the set of all infinite sequences of code numbers in O. Each $\varepsilon \in O^\omega$ will be referred to as a ***data stream***. We will also be interested in finite sequences of observations drawn from O. Let $\varepsilon_n = \varepsilon(n)$ denote the item occurring in position n of ε. Let O^n denote the set of all sequences of length n of members of O. Let O^* denote the set of all finite sequences of objects drawn from O.

An ***empirical hypothesis*** is a proposition whose truth or falsity depends only on the actual data stream. For example, "3 will be seen by stage 5" is an empirical hypothesis, since its truth depends only on the structure of the actual data stream. Hence, an empirical hypothesis may be identified with the set of all possible, infinite data streams for which it is true. An ***empirical hypothesis*** is a subset $\mathscr{H}$ of O^ω. Of course, most such "hypotheses" cannot be expressed in a countable language. In this paper, we will focus on hypotheses expressible in a language of special interest, namely, elementary arithmetic.[3]

Hypothesis $\mathscr{H}$ is ***empirically complete*** just in case it entails a unique, unconditional prediction for each stage of inquiry. In other words, $\mathscr{H}$ is empirically complete just in case for some ε, $\mathscr{H} = \{\varepsilon\}$. More typically, a theory is empirically incomplete and entails predictions only *given* what has already been observed.[4] We will let (n, o) denote the prediction that o will be observed at stage n. Define, for each $\mathbf{e} \in O^*$, $n \in \omega$ and $o \in O$:

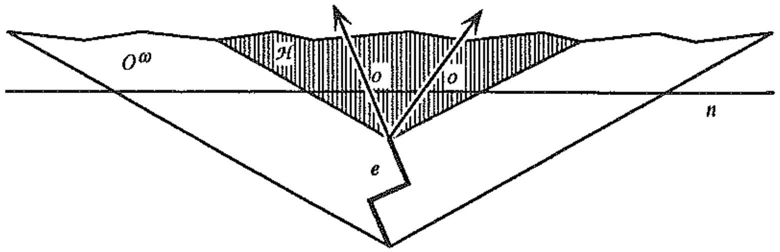

Fig. 1. Prediction entailment.

> $\boldsymbol{PRED}_{\mathscr{H}}(\mathbf{e}, n, o)$ (or, alternatively, $\mathscr{H}, \mathbf{e} \models (n, o)) \Leftrightarrow$ for each $\varepsilon \in \mathscr{H}$, if ε extends $\mathbf{e}$ then $\varepsilon(n) = o$.

Then we may say that $\mathscr{H}$, $\mathbf{e}$ ***entail*** that o will be observed at stage n. In other words, $\mathscr{H}$, $\mathbf{e} \models (n, o)$ just in case each data stream that makes both $\mathbf{e}$ and $\mathscr{H}$ true has o in position n. If no data stream in $\mathscr{H}$ extends $\mathbf{e}$, then for all (n, o), $\mathscr{H}$, $\mathbf{e} \models (n, o)$. Thus, the prediction of $\mathscr{H}$ given $\mathbf{e}$ at time n is not uniquely determined if $\mathscr{H}$ is inconsistent with $\mathbf{e}$ (no element of $\mathscr{H}$ extends $\mathbf{e}$). In all other cases, however, if $\mathscr{H}$, $\mathbf{e} \models (n, o)$, then o is the unique prediction entailed for stage n by $\mathscr{H}$ and $\mathbf{e}$.

Consider the problem of determining, for a given observation o and stage n, whether $\mathscr{H}$ predicts o at n given $\mathbf{e}$. This is a purely formal problem posed by the empirical theory $\mathscr{H}$. We are interested in how the difficulty of this formal problem relates to the difficulty of determining the truth value of $\mathscr{H}$ by empirical means. We will consider various senses of determining the truth about $\mathscr{H}$ in the next section.

3. HYPOTHESIS TEST METHODS AND RELIABILITY

Let hypothesis $\mathscr{H} \subseteq O^{\omega}$ be given. A ***hypothesis test method*** is just a function α that takes a finite data segment $\mathbf{e}$ as input and that conjectures 0, 1, or ? to indicate its guess about the truth value of $\mathscr{H}$, where ? represents refusal to draw a conclusion. We will focus on computable test methods, though some of our negative results extend to hyperarithmetical test methods.[5] One may think of the test method α as reading increasing initial segments of an infinite data stream ε. Let $\varepsilon|n$ denote the finite, initial segment of ε of length $n+1$, so that α successively sees $\varepsilon|0, \varepsilon|1, \varepsilon|2, \ldots$.

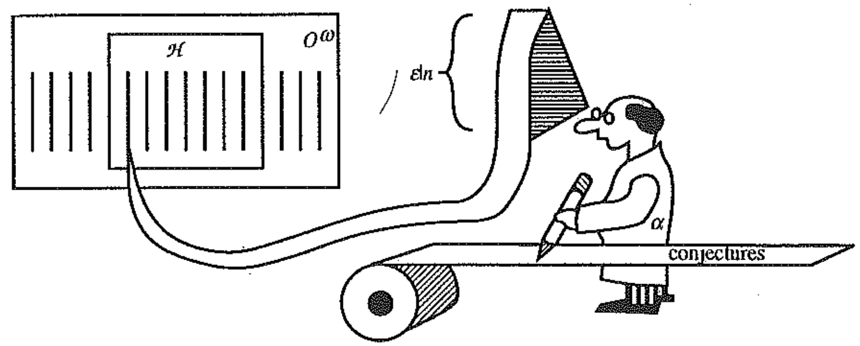

Fig. 2. The situation of a test method.

There are various senses in which a hypothesis test method might be said to be ***reliable***. We will consider the following, where it is always assumed that $\mathscr{H} \subseteq O^{\omega}$.

> α ***verifies*** $\mathscr{H}$ ***with certainty*** $\Leftrightarrow$ for every $\varepsilon \in O^{\omega}$, $\varepsilon \in \mathscr{H} \Leftrightarrow$ at some stage n, $\alpha(\varepsilon|n) = 1$ and for each stage $m < n$, $\alpha(\varepsilon|m) = ?$.
>
> α ***refutes*** $\mathscr{H}$ ***with certainty*** $\Leftrightarrow$ for every $\varepsilon \in O^{\omega}$, $\varepsilon \notin \mathscr{H} \Leftrightarrow$ at some stage n, $\alpha(\varepsilon|n) = 0$ and for each stage $m < n$, $\alpha(\epsilon|m) = ?$.
>
> α ***decides*** $\mathscr{H}$ ***with certainty*** $\Leftrightarrow$ α verifies and refutes $\mathscr{H}$ with certainty.

Refutation with certainty corresponds to the Popperian ambition discussed in the introduction. It demands that no matter how the data comes in, the method refrains from drawing any conclusion until $\mathscr{H}$ is in fact inconsistent with the data, after which the method eventually realizes this fact and concludes 0. So if α refutes $\mathscr{H}$ with certainty, then as soon as α produces its first 0 after never producing anything but ?'s, the user can be certain that $\mathscr{H}$ is false. It is important to keep in mind that refutation with certainty requires that the method succeed on *every possible* data stream, for it is trivial to refute a hypothesis with certainty on a single, fixed data stream: just output the truth value of $\mathscr{H}$ forever, without even looking at the data provided. Verification with certainty requires that the method eventually conjecture 1 after an unbroken sequence of ?'s if and only if the hypothesis is true. Decision with certainty is reminiscent of Plato's demand that inquiry eventually yield the correct truth value with certainty no matter whether the hypothesis is true or false (Kelly and Glymour 1992).

We will also entertain some limiting criteria of success:

α ***verifies*** $\mathscr{H}$ ***in the limit*** $\Leftrightarrow \forall \varepsilon \in O^{\omega}$, $\varepsilon \in \mathscr{H} \Leftrightarrow$ there is a stage n such that for all later stages m, $\alpha(\varepsilon|m) = 1$.

α ***refutes*** $\mathscr{H}$ ***in the limit*** $\Leftrightarrow \forall \varepsilon \in O^{\omega}$, $\varepsilon \notin \mathscr{H} \Leftrightarrow$ there is a stage n such that for all later stages m, $\alpha(\varepsilon|m) = 0$.

α ***decides*** $\mathscr{H}$ ***in the limit*** $\Leftrightarrow$ α verifies and refutes $\mathscr{H}$ in the limit.

Decidability in the limit requires of a method that it eventually stabilize to the truth value of the hypothesis under test. Such a method is guaranteed to stop changing its mind eventually, but there is no *a priori* bound on when this might be or on how many times the method will change its mind. This sort of "fallibilism" was proposed as an aim of science by Peirce (1935), Reichenbach (1949), and later, in a computational context, by H. Putnam (1965) and E. M. Gold (1965).[6] Verification and refutation in the limit are even weaker, "one-sided" criteria of success which serve as limiting analogues of verification and refutation with certainty, respectively (Osherson et al., 1986). Verification in the limit requires that the method stabilize to 1 when the hypothesis is true, and do anything else (i.e. stabilize to 0 or vacillate forever between 0 and 1) otherwise. Refutation in the limit requires that the method stabilize to 0 when the hypothesis is false and do anything else otherwise. Finally, define:

$$\mathscr{H} \text{ is } \textbf{\textit{computably}} \begin{bmatrix} \textbf{\textit{verifiable}} \\ \textbf{\textit{refutable}} \\ \textbf{\textit{decidable}} \end{bmatrix} \begin{bmatrix} \textbf{\textit{with certainty}} \\ \textbf{\textit{in the limit}} \end{bmatrix} \Leftrightarrow$$

there is a total, computable assessment method α such that

$$\alpha \begin{bmatrix} \textit{verifies} \\ \textit{refutes} \\ \textit{decides} \end{bmatrix} \mathscr{H} \begin{bmatrix} \textit{with certainty} \\ \textit{in the limit} \end{bmatrix}.$$

4. EFFECTIVE DEDUCTION OF PREDICTIONS

For each of the above senses of empirical testability, there is a parallel sense of effective testability for formal relations. In general, let $S \subseteq \omega$ and let M be a Turting machine.

M ***verifies*** S ***with certainty*** $\Leftrightarrow$
for each $n \in \omega$, $n \in S \Leftrightarrow M[n]$ eventually halts with output 1.

M ***refutes*** S ***with certainty*** $\Leftrightarrow$
for each $n \in \omega$, $n \notin S \Leftrightarrow M[n]$ eventually halts with output 0.

M ***decides*** S ***with certainty*** $\Leftrightarrow$
M verifies and refutes S with certainty.

M ***verifies*** S ***in the limit*** $\Leftrightarrow$ for each $n \in \omega$, $n \in S \Leftrightarrow$ $M[n]$ generates an infinite sequence stabilizing to 1.

M ***refutes*** S ***in the limit*** $\Leftrightarrow$ for each $n \in \omega$, $n \notin S \Leftrightarrow$ $M[n]$ generates an infinite sequence stabilizing to 0.

M ***decides*** S ***in the limit*** $\Leftrightarrow$ M verifies and refutes S in the limit.

Then as in the empirical case, we define:

$$S \text{ is } \begin{bmatrix} \textbf{\textit{verifiable}} \\ \textbf{\textit{refutable}} \\ \textbf{\textit{decidable}} \end{bmatrix} \textbf{\textit{with certainty}} \Leftrightarrow$$

there is a Turing machine M such that $M \begin{bmatrix} \textit{verifies} \\ \textit{refutes} \\ \textit{decides} \end{bmatrix} S$ with certainty.

$$S \text{ is } \begin{bmatrix} \textbf{\textit{verifiable}} \\ \textbf{\textit{refutable}} \\ \textbf{\textit{decidable}} \end{bmatrix} \textbf{\textit{in the limit}} \Leftrightarrow$$

$$\text{there is a Turing machine } M \text{ such that } M \begin{bmatrix} \textit{verifies} \\ \textit{refutes} \\ \textit{decides} \end{bmatrix} S \text{ in the limit.}$$

In the theory of computability, sets verifiable with certainty are said to be ***recursively enumerable*** **(*r.e.*)**, sets that are refutable with certainty are said to be ***co-r.e.*** and sets that are decidable with certainty are said to be ***recursive***. Similarly, sets that are verifiable in the limit are said to be ***limiting r.e.***, sets that are refutable in the limit are said to be ***co-limiting r.e.*** and sets that are decidable in the limit are said to be ***limiting recursive***.

5. DERIVABILITY GIVEN TESTABILITY

We may now ask a systematic set of precise questions about the relationship between the inductive testability of hypotheses and formally testing whether a given prediction follows from $\mathscr{H}$ and **e**. For example, if $\mathscr{H}$ is computably refutable with certainty, must $\text{PRED}_{\mathscr{H}}$ be verifiable with certainty (r.e.)? Or if $\mathscr{H}$ is verifiable in the limit, must $\text{PRED}_{\mathscr{H}}$ also be verifiable in the limit? Theorem 5.1 (cf. Figure 3) answers every question of this sort, both for the case in which $\mathscr{H}$ is empirically complete and for the general case, in which $\mathscr{H}$ may fail to make any prediction about what will be observed at a given time. The left-most column of the table lists the various notions of computable hypothesis testability defined in Section 3. For each such sense of testability, the table specifies a *general* upper bound on the sense in which predictions can be effectively derived from such a theory, in the sense that every problem of the specified sort has predictions *at least* as easy to derive as the table says. It may be that some problems of the specified kind have predictions even easier to derive than is indicated,[7] but the table's results are the *best possible*, in the sense that for each cell in the table, there exists a hypothesis that is effectively testable in the required sense, but whose predictions are as hard to derive as any that are derivable in the sense given by the table. The relevant sense of "as hard as" will be defined rigorously in Section 7.C below.

What does Theorem 5.1 say about the intuitive notion that science should proceed by deriving successive predictions from a theory and checking them against the data? This intuition is strongly supported

Theorem 5.1 Given sense in which $\mathscr{H}$ is computably testable as an empirical hypothesis			Best upper bound on the sense in which $\text{PRED}_{\mathscr{H}}$ is decidable: $\mathscr{H}$ is empirically complete, O is finite	$\mathscr{H}$ is empirically complete, O is infinite	General case, O is finite	General case, O is infinite
certainty case	a	decidable with certainty	Impossible unless $\lvert O \rvert = 1$		decidable with certainty	refutable with certainty
certainty case	b	verifiable with certainty	Impossible unless $\lvert O \rvert = 1$		refutable with certainty	refutable with certainty
certainty case	c	refutable with certainty	decidable with certainty	none	verifiable with certainty	none
limiting case	d	decidable in the limit	decidable with certainty	none	refutable in the limit	none
limiting case	e	verifiable in the limit	decidable with certainty	none	refutable in the limit	none
limiting case	f	refutable in the limit	none	none	none	none

Fig. 3. Theorem 5.1.

when $\mathscr{H}$ is empirically complete and O is finite, for then it must be computably decidable with certainty whether $\mathscr{H}$ predicts o at n given **e**, even if $\mathscr{H}$ is only computably verifiable in the limit. The intuition is still supported to some extent in the general case when O is finite, for in that case it must at least be verifiable with certainty whether $\mathscr{H}$ predicts o at n given **e**, if $\mathscr{H}$ is to be refutable with certainty.

But the situation changes when O is infinite. For example, even when $\mathscr{H}$ is computably decidable with certainty, it may not be possible to verify with certainty whether $\mathscr{H}$ predicts o at n given **e**. But the most curious result of all is the one alluded to in the introduction, namely, that for some empirically complete $\mathscr{H}$ that is computably refutable with certainty, it is not even refutable or verifiable in the limit whether $\mathscr{H}$ predicts o at n given **e**. This is remarkable because $\mathscr{H}$ commits itself to a unique prediction at each stage of inquiry and for all we know *a priori*, any one of these predictions might turn out wrong. Moreover, $\mathscr{H}$ is such that no Turing machine can determine or even enumerate all the predictions made by $\mathscr{H}$. Nonetheless, some Turing-computable method can refute $\mathscr{H}$ with certainty, so no matter how the data comes in, the method rejects $\mathscr{H}$ *if and only if* $\mathscr{H}$ is false. Evidently, this method cannot proceed by deriving the next prediction from $\mathcal{H}$ and then comparing the result with what is observed. There is a more powerful way for computable scientific inquiry to proceed, so far as finding the truth is concerned.[8]

6. THE ARITHMETICAL HIERARCHY

It is both technically useful and conceptually revealing to have one general way to characterize the complexities of both hypotheses and prediction sets. It turns out that the recursion-theoretic ***arithmetical hierarchy*** provides just the right scale for our purposes.[9] The basic idea behind the arithmetical hierarchy is to classify computational intractability in terms of the number of alternations between universal and existential quantifiers sufficient to define a given relation in terms of some effectively decidable relation.

Let $\mathcal{R} \subseteq (O^\omega)^m \times \omega^k$. In other words, $\mathcal{R}$ is an $m + k$-ary relation with m data-stream arguments and k numeric arguments. $\mathcal{R}$ will be said to be a ***type*** (m, k) relation. Of particular interest to us is the fact that $\mathcal{H}$ is a type $(1, 0)$ relation and $\text{PRED}_{\mathcal{H}}$ is a type $(0, 3)$ relation. Thus, if we classify the computational complexity of all relations of type (m, k) at once, we can compare the complexities of $\mathcal{H}$ and $\text{PRED}_{\mathcal{H}}$ on the same scale.

Say that a type (m, k) relation $\mathcal{R}$ is ***recursive*** or ***decidable with certainty*** just in case there exists a Turing machine M such that when M is provided with m infinite, "read only" tapes listing its infinite arguments and an ordinary work tape listing its k numeric arguments, M eventually halts with 1 if $\mathcal{R}$ holds of the arguments provided and halts with 0 otherwise. Observe that even though M is provided with tapes listing infinite functions as inputs, M can scan only some finite segment of each such tape before making its decision.

Let $\mathcal{R}$ be a type (m, k) relation. To eliminate tedious repetitions and subscripts, let $\bar{x}$ denote a k-vector of natural numbers and let $\bar{\varepsilon}$ denote an m-vector of data streams, so that we may write $\mathcal{R}(\bar{\varepsilon}, \bar{x})$ instead of $\mathcal{R}(\varepsilon[1], \ldots, \varepsilon[m], x_1, \ldots, x_k)$, where each $\varepsilon[i] \in O^\omega$. Now define:

$$\mathcal{R} \in \Sigma^0_0 \Leftrightarrow \mathcal{R} \text{ is recursive}$$

$$\begin{aligned}\mathcal{R} \in \Sigma^0_{n+1} \Leftrightarrow {} & \text{there is an } \mathcal{S} \in \Sigma^0_n \text{ such that} \\ & \text{for each } \bar{\varepsilon} \in (O^\omega)^m,\ \bar{x} \in \omega^k, \\ & \mathcal{R}(\bar{\varepsilon}, \bar{x}) \Leftrightarrow \exists x_{k+1} \text{ such that } \neg \mathcal{S}(\bar{\varepsilon}, \bar{x}, x_{k+1}).\end{aligned}$$

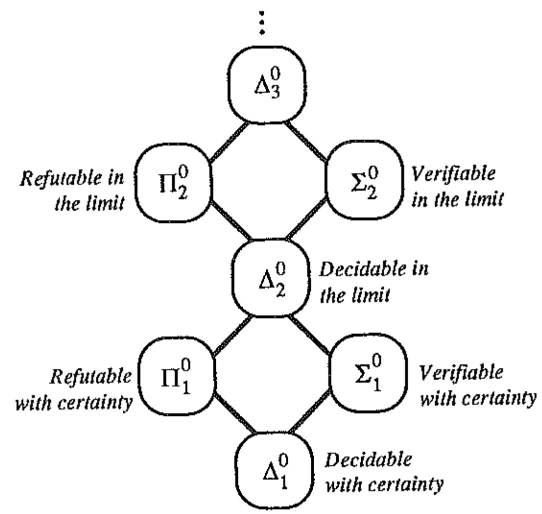

Fig. 4. The arithmetical hierarchy.

$$\mathcal{R} \in \Pi^0_n \Leftrightarrow \neg \mathcal{R} \in \Sigma^0_n.$$

$$\mathcal{R} \in \Delta^0_n \Leftrightarrow \mathcal{R} \in \Pi^0_n \cap \Sigma^0_n.$$

In other words, a relation in Σ^0_n may be defined as follows:

$$\mathcal{R}(\bar{\varepsilon}, \bar{x}) \Leftrightarrow \exists x_1 \forall x_2 \ldots \mathcal{S}(\bar{\varepsilon}, \bar{x}, x_1, \ldots, x_n).$$

where $\mathcal{S}$ is recursive and the quantifier prefix involves no more than $n - 1$ alternations between $\exists$'s and $\forall$'s. Relations in Π^0_n are similar, except that the leading quantifier is universal. Relations in Δ^0_n can be defined both ways. Evidently, $\Delta^0_n, \Sigma^0_n, \Pi^0_n \subseteq \Delta^0_{n+1}, \Sigma^0_{n+1}, \Pi^0_{n+1}$. The ***arithmetical hierarchy theorem*** says that these inclusions are all proper (cf. Section 7).

Our interest in the arithmetical hierarchy stems from the fact that it jointly characterizes the complexity of deductive and scientific inference. Regarding deductive problems, we have the following, exact correspondence (cf. Figure 4):

PROPOSITION 6.1 (E. M. Gold and H. Putnam 1965). Let $S \subseteq \omega$.

$$\text{(a)} \quad S \text{ is} \begin{bmatrix} \text{verifiable} \\ \text{refutable} \\ \text{decidable} \end{bmatrix} \text{ with certainty} \Leftrightarrow S \in \begin{bmatrix} \Sigma_1^0 \\ \Pi_1^0 \\ \Delta_1^0 \end{bmatrix}$$

$$\text{(b)} \quad S \text{ is} \begin{bmatrix} \text{verifiable} \\ \text{refutable} \\ \text{decidable} \end{bmatrix} \text{ in the limit} \Leftrightarrow S \in \begin{bmatrix} \Sigma_2^0 \\ \Pi_2^0 \\ \Delta_2^0 \end{bmatrix}.$$

■

On the empirical side, we have the exactly analogous result.

PROPOSITION 6.2 (E. M. Gold and H. Putnam 1965). Let $\mathscr{H} \subseteq O^\omega$, where O is recursive.

$$\text{(a)} \quad \mathscr{H} \text{ is computably} \begin{bmatrix} \text{verifiable} \\ \text{refutable} \\ \text{decidable} \end{bmatrix} \text{ with certainty} \Leftrightarrow \mathscr{H} \in \begin{bmatrix} \Sigma_1^0 \\ \Pi_1^0 \\ \Delta_1^0 \end{bmatrix}.$$

$$\text{(b)} \quad \mathscr{H} \text{ is computably} \begin{bmatrix} \text{verifiable} \\ \text{refutable} \\ \text{decidable} \end{bmatrix} \text{ in the limit} \Leftrightarrow \mathscr{H} \in \begin{bmatrix} \Sigma_2^0 \\ \Pi_2^0 \\ \Delta_2^0 \end{bmatrix}.$$

Proof. For a proof in an explicitly empirical setting, cf. (Kelly 1993)■

7. PROOF OF THEOREM 5.1

In light of Propositions 6.1 and 6.2, we may now restate theorem 5.1 in the form in which it will be demonstrated (cf. Fig. 7).

The table presented in Fig. 5 is more informative than the one presented in Fig. 3, for the negative results in lines (c–f) say not only that there is no Π_2^0 bound on deducing consequences from $\mathscr{H}$, but that there is no bound at any level in the arithmetical hierarchy that covers all cases. In fact, something worse will be shown; namely, that there is an empirically complete theory $\mathscr{H}$ for which there is no arithmetical bound on $\text{PRED}_{\mathscr{H}}$, and yet $\mathscr{H}$ is computably refutable with certainty.

Theorem 5.1 Given arithmetical bound on $\mathscr{H}$			Best arithmetical bound on $\text{PRED}_{\mathscr{H}}$			
			$\mathscr{H}$ is empirically complete		General case	
			1. O is finite	2. O is infinite	3. O is finite	4. O is infinite
certainty case	a	Δ_1^0	Impossible unless $\lvert O\rvert = 1$		Δ_1^0	Π_1^0
	b	Σ_1^0			Π_1^0	Π_1^0
	c	Π_1^0	Δ_1^0	none	Σ_1^0	none
limiting case	d	Δ_2^0	Δ_1^0	none	Π_2^0	none
	e	Σ_2^0	Δ_1^0	none	Π_2^0	none
	f	Π_2^0	none	none	none	none

Fig. 5. Theorem 5.1.

The proof of Theorem 5.1 will proceed in a series of lemmas. First, we show that the excluded cases 1.a, 1.b, 2.a, and 2.b cannot arise. Then we establish the general upper bounds given in the table. Finally, for each bound given in the table, we show that no lower bound holds in general. Assume in each of the following results that $\mathscr{H} \subseteq O^\omega$. Whenever we speak of an infinite data stream ε, it is to be assumed that $\varepsilon \in O^\omega$, whenever we mention a finite data sequence $\mathbf{e}$, $\mathbf{e} \in O^*$, and whenever we speak of a datum o, $o \in O$. Also, let n^k denote the sequence consisting of n repeated k times, and let n^ω denote the infinite sequence that is constantly n. Finally, let $\mathbf{e}*\mathbf{e}'$ denote the concatenation of $\mathbf{e}$ and $\mathbf{e}'$.

7.A. *Impossible Cases*

The following lemma accounts for all the impossible cases in the table. The argument is simply a formal version of the classical argument for inductive scepticism that has echoed through the works of Plato, Sextus Empiricus, William of Ockham, David Hume, Karl Popper, and many

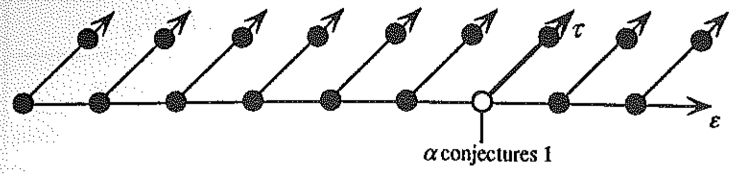

Fig. 6. Lemma 7.A.1.

others. We attribute it to Sextus because his rendition is both ancient and particularly clear.[10]

LEMMA 7.A.1 (Sextus Empiricus). If there is more than one possible observation at each stage and $\mathcal{H}$ is empirically complete, then $\mathcal{H}$ is not computably (or non-computably) verifiable with certainty.
That is, if $|O| > 1$ and $\varepsilon \in O^{\omega}$ then $\{\varepsilon\} \notin \Sigma_1^0$.

Proof. By Proposition 6.2, it suffices to show that $\{\varepsilon\}$ is not computably verifiable with certainty. Let α be an arbitrary test method. Feed successive initial segments of ε to α until α conjectures 1. If this never happens, α fails to produce 1 on ε and hence fails to verify $\{\varepsilon\}$ with certainty. If it does happen, say at stage n, then let τ be just like ε except that $\tau_{n+1} \neq \varepsilon_{n+1}$ and $\tau_{n+1} \in O$. This is possible since $|O| > 1$. α conjectures 1 on $\tau \neq \varepsilon$, and hence does not verify $\{\varepsilon\}$ with certainty. ■

7.B. *Upper Bounds*

LEMMA 7.B.1. If $\mathcal{H}$ is refutable with certainty and O is finite, then $\mathrm{PRED}_{\mathcal{H}}$ is verifiable with certainty.
That is, if O is finite and $\mathcal{H} \in \Pi_1^0$ then $\mathrm{PRED}_{\mathcal{H}} \in \Sigma_1^0$.

Proof. Suppose O is finite and $\mathcal{H} \in \Pi_1^0$. Let computable α refute $\mathcal{H}$ with certainty. We construct a procedure for verifying $\mathrm{PRED}_{\mathcal{H}}$ with certainty. Given the triple $(\mathbf{e}, x, o)$, proceed as follows: if length($\mathbf{e}$) $> x$, then return 1 if $\mathbf{e}_x = o$ and go into an infinite loop otherwise. Else, proceed in stages as follows. Begin at stage x. At stage $x + k$, construct the tree of all extensions $\mathbf{e}'$ of $\mathbf{e}$ of length $x + k$. This is possible because O is finite. Run α on each initial segment of each such $\mathbf{e}'$, labelling that "node" of the tree with α's conjecture. Say that $\mathbf{e}$ is ***dead*** $\Leftrightarrow$ there is an initial segment of $\mathbf{e}$ along which α conjectures only ?'s followed by a 0. If there is a non-dead path $\mathbf{e}'$ of length $x + k$ such that $\mathbf{e}'_x \neq o$, then go to stage $x + k + 1$. Otherwise, halt the entire process and return 1.

($\Rightarrow$) Suppose $(\mathbf{e}, x, o) \in \mathrm{PRED}_{\mathcal{H}}$. Then for each $\varepsilon \in \mathcal{H}$ such that ε extends $\mathbf{e}$, $\varepsilon_x = o$. Suppose for *reductio* that our procedure goes through infinitely many stages on input $(\mathbf{e}, x, o)$. Then at each stage k, there is a non-dead $\mathbf{e}'$ of length k extending $\mathbf{e}$. Since O is finite, the tree of all such finite paths is finitely branching, so by König's lemma, there is an infinite path ε through the tree, each initial segment of which is non-dead. Since α refutes $\mathcal{H}$ with certainty, $\varepsilon \in \mathcal{H}$. By construction, $\varepsilon_x \neq o$ and ε extends $\mathbf{e}$, so $(\mathbf{e}, x, o) \notin \mathrm{PRED}_{\mathcal{H}}$, contrary to assumption. So the procedure halts correctly with output 1. ($\Leftarrow$) Suppose that $(\mathbf{e}, x, o) \notin \mathrm{PRED}_{\mathcal{H}}$. Then for some $\varepsilon \in \mathcal{H}$ such that ε extends $\mathbf{e}$, $\varepsilon_x \neq o$. Since α refutes $\mathcal{H}$ with certainty, each initial segment of ε is non-dead, so the procedure goes through infinitely many stages and returns no output. ■

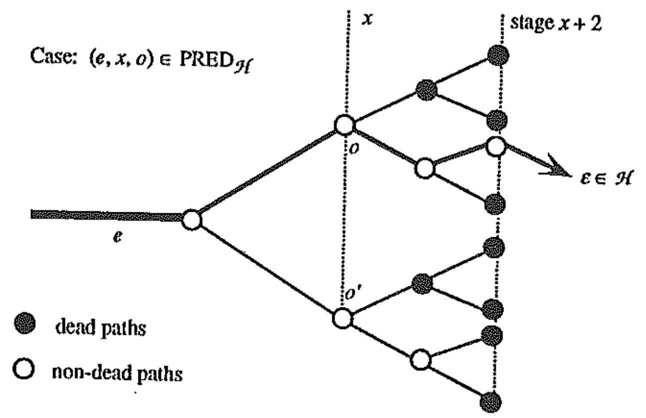

Fig. 7. Lemma 7.B.2.

LEMMA 7.B.2. If $O \subseteq \omega$ and $\mathcal{H}$ is verifiable with certainty, then $\mathrm{PRED}_{\mathcal{H}}$ is refutable with certainty.
That is, if $O \subseteq \omega$ and $\mathcal{H} \in \Sigma_1^0$ then $\mathrm{PRED}_{\mathcal{H}} \in \Pi_1^0$.

Proof. Let $O \subseteq \omega$ and let $\mathcal{H} \in \Sigma_1^0$. Let computable α verify $\mathcal{H}$ with certainty. We construct a procedure to refute $\mathrm{PRED}_{\mathcal{H}}$ with certainty. On input $(\mathbf{e}, x, o)$, the procedure simulates α sequentially on an effective enumeration $(\mathbf{e}[0], \mathbf{e}[1], \ldots, \mathbf{e}[n], \ldots)$ of the set of all finite exten-

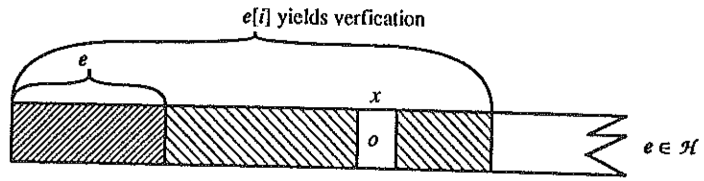

Fig. 8. Lemma 7.B.2.

sions of $\mathbf{e}$, of length $\geq x$ until some $\mathbf{e}[i]$ is found such that $\mathbf{e}[i]_x \neq o$ and α returns an unbroken sequence of ?'s followed by a 1 when applied to successive initial segments of $\mathbf{e}[i]$ (say that such a sequence ***yields verification***). Then the procedure halts and returns 0.

($\Rightarrow$) Suppose that $(\mathbf{e}, x, o) \in \text{PRED}_{\mathscr{H}}$. Then there is no $\varepsilon \in \mathscr{H}$ such that ε extends $\mathbf{e}$ and $\varepsilon_x \neq o$. Since α verifies $\mathscr{H}$ with certainty, there is no ε such that $\varepsilon_x \neq o$ and some initial segment of ε yields verification. Hence, the procedure never halts with an output.

($\Leftarrow$) So suppose that $(\mathbf{e}, x, o) \notin \text{PRED}_{\mathscr{H}}$. Then for some $\varepsilon \in \mathscr{H}$ extending $\mathbf{e}$, $\varepsilon_x \neq o$. Since α verifies $\mathscr{H}$ with certainty, some initial segment of ε of length greater than x yields verification. Hence, the procedure eventually halts with output 0. ■

LEMMA 7.B.3. If O is finite and $\mathscr{H}$ is verifiable in the limit, then $\text{PRED}_{\mathscr{H}}$ is refutable in the limit.
That is, if O is finite and $\mathscr{H} \in \Sigma^0_2$ then $\text{PRED}_{\mathscr{H}} \in \Pi^0_2$.

Proof. Let O be finite and let computable α verify $\mathscr{H}$ in the limit. We provide a Σ^0_2 definition of $\overline{\text{PRED}_{\mathscr{H}}}$, from which it follows that $\text{PRED}_{\mathscr{H}}$ is Π^0_2:

$$\begin{aligned}&(\mathbf{e}, x, o) \in \overline{PRED_{\mathscr{H}}} \Leftrightarrow\\ &\exists n \geq x,\ \exists o' \neq o,\ \exists \mathbf{e}' \in O^n \text{ such that } \mathbf{e}'_x = o' \text{ and}\\ &\forall m \geq n,\\ &\exists \mathbf{e}'' \in O^m \text{ extending } \mathbf{e}' \text{ such that}\\ &\forall k \text{ such that } n \leq k \leq m,\ \alpha(\mathbf{e}''|k) = 1.\end{aligned}$$

The quantifiers over finite data sequences are all bounded because O^n and O^m are finite (since O is finite). Hence, only the quantifiers $\exists n$ and $\forall m \geq n$ are unbounded. All other relations involved are recursive, so we have a Σ^0_2 expression. We now verify that the definition is correct (cf. Fig. 9).

($\Rightarrow$) Suppose $(\mathbf{e}, x, o) \in \overline{\text{PRED}_{\mathscr{H}}}$. Then by the definition of $\text{PRED}_{\mathscr{H}}$,

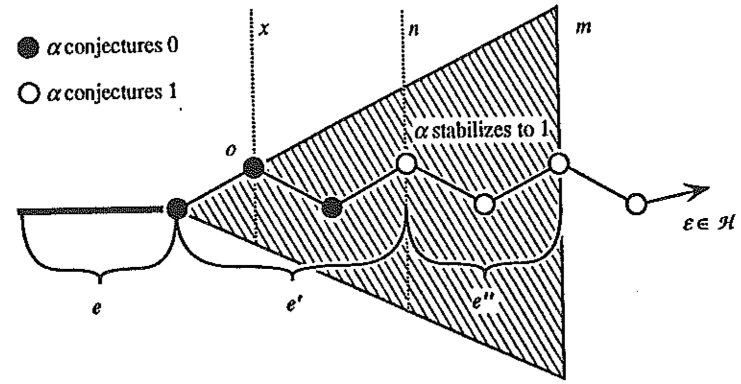

Fig. 9. Lemma 7.B.2.

there is an $\varepsilon \in \mathscr{H}$ such that ε extends $\mathbf{e}$ and $\varepsilon_x \neq o$. Since α verifies $\mathscr{H}$ in the limit, $\exists n\ \forall m \geq n\ \alpha(\varepsilon|m) = 1$. So $\exists n$ such that $\forall m \geq n\ \forall k$ such that $n \leq m \leq k$, $\alpha(\varepsilon|k) = 1$. So the right-hand side of the definition is satisfied. ($\Leftarrow$) Suppose the right-hand side of the definition is satisfied. Then for some $\mathbf{e}'$ of length $n \geq x$ such that $\mathbf{e}'_x \neq o$, we have that there is always a longer extension $\mathbf{e}''$ of $\mathbf{e}'$ along which α conjectures 1 after stage n, so there are infinitely many such $\mathbf{e}''$, of ever greater length. Since O is finite, the infinite tree of all such $\mathbf{e}''$ is finitely branching, so by König's lemma, it has an infinite path ε along which α always conjectures 1. Since α verifies $\mathscr{H}$ in the limit and stabilizes to 1 on ε, $\varepsilon \in \mathscr{H}$. But since $\varepsilon_x \neq o$, we have by the definition of $\text{PRED}_{\mathscr{H}}$ that $(\mathbf{e}, x, o) \in \overline{\text{PRED}_{\mathscr{H}}}$. ■

LEMMA 7.B.4. If O is finite and $\mathscr{H}$ is empirically complete and $\mathscr{H}$ is verifiable in the limit, then $\text{PRED}_{\mathscr{H}}$ is decidable with certainty.
That is, if O is finite and $\{\varepsilon\} \in \Sigma^0_2$ then $\text{PRED}_{\{\varepsilon\}} \in \Delta^0_1$.

Proof. Suppose O is finite and $\{\varepsilon\} \in \Sigma^0_2$. Then by Proposition 6.2.b, let computable α verify $\{\varepsilon\}$ in the limit. Then (*) $\tau = \varepsilon \Leftrightarrow \exists n$ such that for all $m \geq n$, $\alpha(\tau|m) = 1$. Let n' be the least such n along ε. In virtue of (*), ε is the unique infinite path on which α makes only finitely many non-zero conjectures. Let $m \in \omega$ and let length$(\mathbf{e}) \leq m$. Define:

$\mathbf{e}$ is ***m-dead*** $\Leftrightarrow \forall \mathbf{e}' \in O^m$, if $\mathbf{e}'$ extends $\mathbf{e}$ then $\exists k$ such that $n' \leq k \leq m$ and $\alpha(\mathbf{e}'|k) = 0$.

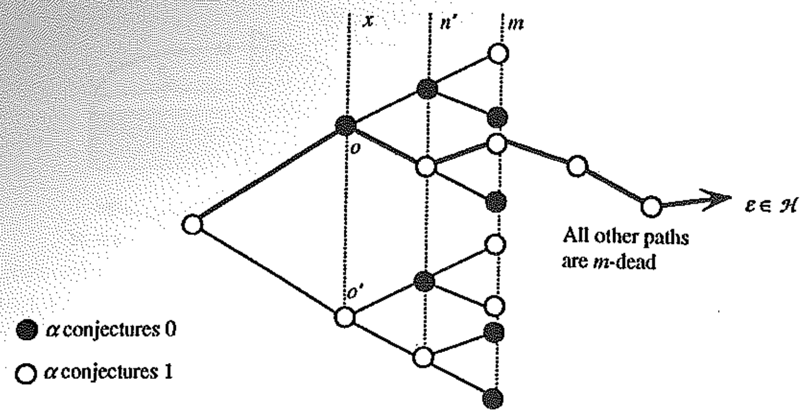

Fig. 10. Lemma 7.B.4.

By (*), it is immediate that:

(A) For each $m \geq 0$, $\forall k \varepsilon|k$ is not m-dead.

We also have:

(B) If $k \geq n'$ then there is an $m \geq k$ such that for each $\mathbf{e} \in O^k$, if $\mathbf{e} \neq \varepsilon|k$ then $\mathbf{e}$ is m-dead.

For suppose otherwise. Then for each $m \geq k$ there is *an* $\mathbf{e}[m] \in O^m$ extending $\mathbf{e}$ such that α conjectures only 1 after position k on $\mathbf{e}[m]$. Let $T = \{\mathbf{e}[m]: m \geq k\}$. Since O is finite, T is a finitely branching, infinite tree. By König's lemma, T has an infinite path τ. So for each $m \geq k$ $\alpha(\tau|m) = 1$. Since α verifies $\{\varepsilon\}$ in the limit, $\tau = \varepsilon$. But since each element of T extends $\mathbf{e}$, τ extends $\mathbf{e}$. Also, length($\mathbf{e}$) $= k$ and $\mathbf{e} \neq \varepsilon|k$, so $\tau \neq \epsilon$, which is a contradiction. Hence, we have (B).

To compute ε, use α to effectively label the finitely branching tree O^*, from bottom to top, level by level, until for some m, the tree is labelled up to level m, and it is effectively verified (by exhaustion) that there is a unique path $\mathbf{e}$ of length n that is not m-dead. (By A and B, $\mathbf{e} = \varepsilon|n$). Return $\mathbf{e}_n$ $(= \varepsilon_n)$. ■

Proof of Theorem 5.1, *Upper Bounds, General Case*. The upper bounds reported in line (a) of Theorem 5.1 follow from those reported in (b) and (c) and the upper bound reported in the finite case of (d) follows from that reported in the finite case of (e). The upper bounds reported in (b) both follow from Lemma 7.B.2. The upper bound reported in the finite case of (c) follows from Lemma 7.B.1. The upper bound reported in the finite case of (e) follows from Lemma 7.B.3.

Empirically Complete Case. The finite O cases of lines (c) and (d) follow from (e). The finite O case of (e) follows from Lemma 7.B.4. This establishes all the upper bounds given in Theorem 5.1. We now show that these upper bounds are the best general upper bounds possible.

7.C. *The Upper Bounds are Optimal*

We assume a fixed Gödel numbering of the Turing machines, and we let ϕ_i denote the (possibly partial) unary function on the natural numbers computed by machine M_i. Also, if $S \subseteq \omega^k$ and $R \subseteq \omega^n$, define

$$S \leq_m R \Rightarrow \text{there is a total recursive function } f\colon \omega^k \to \omega^n \text{ such that for each } \bar{x} \in \omega^k,\ S(\bar{x}) \Leftrightarrow R(f(\bar{x})).$$

Then it is said that S is ***many-one reducible*** to R. It is immediate that for each n, Σ^0_n, Π^0_n, and Δ^0_n are closed downward under $\leq_m$. That is, if R is in one of these classes and $S \leq_m R$, then S is also in the class. A relation R is ***complete*** in a class if R is a member of the class and every member of the class is many-one reducible to R. A complete relation in a class may be thought of as a "most complex" member of the class. For example, define:

$$K = \{i\colon \phi_i(i) \text{ is defined}\}.$$
$$T = \{i\colon \exists\ \phi_i \text{ is total}\}.$$

K is called the "halting problem". K is Σ^0_1-complete and $\bar{K}$ is Π^0_1-complete. Also, T is Π^0_2-complete.[11] We will see (Lemma 7.C.5 below) that both $\Pi^0_n - \Sigma^0_n$ and $\Sigma^0_n - \Pi^0_n$ are non-empty. Hence, if a relation S is complete in $\Sigma^0_n[\Pi^0_n]$, then the relation does not belong to the dual class $\Pi^0_n[\Sigma^0_n]$.

We have seen, for example, that if $\mathscr{H}$ is Δ^0_1 then PRED$_{\mathscr{H}}$ is Π^0_1. To show that this upper bound on PRED$_{\mathscr{H}}$ is *optimal*, we must prove that there is an $\mathscr{H}$ such that $\mathscr{H}$ is Δ^0_1 but PRED$_{\mathscr{H}}$ is as complex as a Π^0_1 function can possibly be (i.e. PRED$_{\mathscr{H}}$ is *complete* in Π^0_1). For this, it suffices to show that $\bar{K} \leq_m$ PRED$_{\mathscr{H}}$.

LEMMA 7.C.1. There is an $\mathscr{H} \subseteq \omega^\omega$ that is computably decidable with

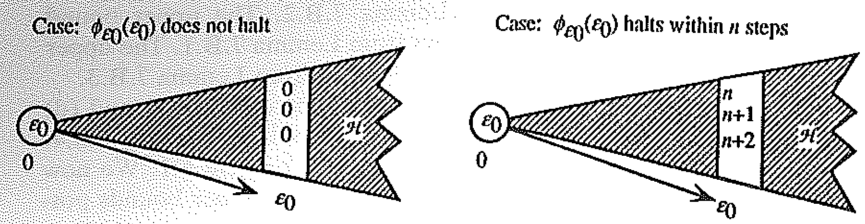

Fig. 11. Lemma 7.B.5.

certainty but whose predictions are as hard as possible to derive, given that they are computably refutable with certainty (so the predictions of $\mathcal{H}$ are not computably verifiable with certainty).

That is, there is an $\mathcal{H} \subseteq \omega^\omega$ such that

(1) $\mathcal{H} \in \Delta_1^0$ and
(2) $\text{PRED}_{\mathcal{H}}$ is complete in Π_1^0 (and hence $\text{PRED}_{\mathcal{H}} \notin \Sigma_1^0$).

Proof. Define:

$$\varepsilon \in \mathcal{H} \Leftrightarrow (\varepsilon_{\varepsilon_0} \neq 0 \Rightarrow \phi_{\varepsilon_0}(\varepsilon_0) \text{ halts within } \varepsilon_{\varepsilon_0} \text{ steps}).$$

(1) is evident. (2) Let $f(x) = ((x), x, 0)$. Clearly, f is computable and total. Let K denote the halting problem. We show that $x \in \bar{K} \Leftrightarrow f(x) \in \text{PRED}_{\mathcal{H}}$, which yields $\bar{K} \leqslant_m \text{PRED}_{\mathcal{H}}$. ($\Rightarrow$) Suppose $x \in \bar{K}$. Then for each $k \in \omega$, $\phi_x(x)$ does not halt in k steps. Suppose $\varepsilon \in \mathcal{H}$ and $\varepsilon_0 = x$. Then by the contrapositive of the definition of $\mathcal{H}$, $\varepsilon_x = 0$. Hence, $((x), x, 0) \in \text{PRED}_{\mathcal{H}}$. ($\Leftarrow$) Suppose $x \in K$. Then for some k, $\phi_x(x)$ halts within k steps. Let τ be the data stream that starts with x and that has k in each successive position. $\tau \in \mathcal{H}$ because $\phi_x(x)$ halts in k steps, but $\tau_x \neq 0$. So $((x), x, 0) \notin \text{PRED}_{\mathcal{H}}$. ■

LEMMA 7.C.2. There is an $\mathcal{H} \subseteq 2^\omega$ that is computably verifiable with certainty but whose predictions are as hard as possible to derive, given that they are computably refutable with certainty (so the predictions of $\mathcal{H}$ are not computably verifiable with certainty).

That is, there is an $\mathcal{H} \subseteq 2^\omega$ such that

(1) $\mathcal{H} \in \Sigma_1^0$ and
(2) $\text{PRED}_{\mathcal{H}}$ is complete in Π_1^0 (and hence $\text{PRED}_{\mathcal{H}} \notin \Sigma_1^0$).

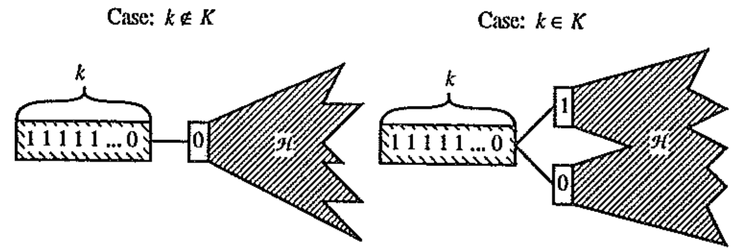

Fig. 12. Lemma 7.C.2.

Proof. Define $\varepsilon \in \mathcal{H} \Leftrightarrow \exists k$ such that $[(\forall k' < k, \varepsilon_{k'} = 1)$ and $\varepsilon_k = 0$ and $(\varepsilon_{k+1} \neq 0 \Rightarrow k \in K)]$.

(1) Since the universal quantifier is bounded, $K \in \Sigma_1^0$ and the leading quantifier is $\exists$, we have that $\mathcal{H} \in \Sigma_1^0$. (2) Recall that 1^n denotes the everywhere 1 sequence of length n, and $\mathbf{e}*\mathbf{e}'$ denotes the concatenation of $\mathbf{e}'$ onto the end of $\mathbf{e}$. Define: $f(x) = (1^x*0,\ x+1,\ 0)$. As in the preceding proof, we show that $x \in \bar{K} \Leftrightarrow f(x) \in \text{PRED}_{\mathcal{H}}$, so that $\bar{K} \leqslant_m \text{PRED}_{\mathcal{H}}$. ($\Rightarrow$) Suppose $x \in \bar{K}$.

Let $\varepsilon \in \mathcal{H}$ and let ε extend 1^x*0. By the definition of $\mathcal{H}$, $\varepsilon_{x+1} = 0$. Hence, $(1^x*0, x+1, 0) \in \text{PRED}_{\mathcal{H}}$. ($\Leftarrow$) Suppose $x \in K$. Let $\tau = 1^x*0*1^\omega$, where it should be recalled that 1^ω denotes the infinite, everywhere 1 sequence. $\tau \in \mathcal{H}$ since $x \in K$. But $\tau_{x+1} = 1$, so $(1^{x-1}*0, x+1, 0) \notin \text{PRED}_{\mathcal{H}}$. ■

LEMMA 7.C.3. There is an $\mathcal{H} \subseteq 2^\omega$ that is computably refutable with certainty but whose predictions are as hard as possible to derive, given that they are computably verifiable with certainty (so the predictions of $\mathcal{H}$ are not computably refutable with certainty).

That is, there is an $\mathcal{H} \subseteq 2^\omega$ such that

(1) $\mathcal{H} \in \Pi_1^0$ and
(2) $\text{PRED}_{\mathcal{H}}$ is complete in Σ_1^0 (and hence $\text{PRED}_{\mathcal{H}} \notin \Pi_1^0$).

Proof. Define $\varepsilon \in \mathcal{H} \Leftrightarrow \forall x[x \in \bar{K}$ or $\varepsilon_x = 0]$. (1) $\mathcal{H} \in \Pi_1^0$, since $\bar{K} \in \Pi_1^0$ and there is just one universal quantifier. (2) Let $\mathbf{0}$ denote the empty data sequence. Define $f(x) = (\mathbf{0}, x, 0)$. We show $x \in K \Leftrightarrow f(x) \in \text{PRED}_{\mathcal{H}}$, so $K \leqslant_m \text{PRED}_{\mathcal{H}}$, ($\Rightarrow$) Suppose $x \in K$. Let $\varepsilon \in \mathcal{H}$. Then $\varepsilon_x = 0$. So $(\mathbf{0}, x, 0) \in \text{PRED}_{\mathcal{H}}$. ($\Leftarrow$) Suppose $x \in \bar{K}$. Then $\tau = 0^x*1*0^\omega \in \mathcal{H}$ but $\tau_x = 1$ so $(\mathbf{0}, x, 0) \notin \text{PRED}_{\mathcal{H}}$. ■

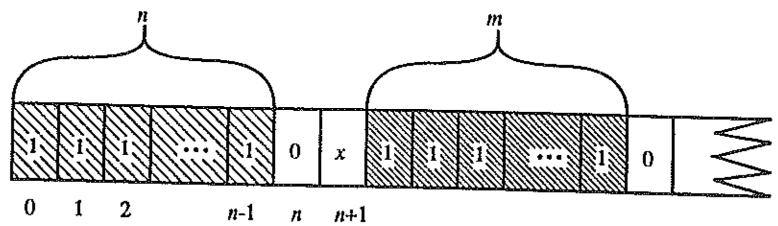

Fig. 13. Lemma 7.C.4.

LEMMA 7.C.4. There is an $\mathcal{H} \subseteq 2^\omega$ that is computably decidable in the limit but whose predictions are as hard as possible to derive, given that they are computably refutable in the limit (so the predictions of $\mathcal{H}$ are not computably verifiable in the limit).

That is there is an $\mathcal{H} \subseteq 2^\omega$ such that

(1) $\mathcal{H} \in \Delta^0_2$ and
(2) $\text{PRED}_{\mathcal{H}}$ is complete in Π^0_2 (and hence $\text{PRED}_{\mathcal{H}} \notin \Sigma^0_2$).

Proof. Let P be some arbitrary set complete in Π^0_2 (e.g. the set T defined above). Then for some $R \in \Sigma^0_1$, we have that $\forall x \in \omega$, $P(x) \Leftrightarrow \forall y R(x, y)$. Define:

$$\varepsilon \in \mathcal{H} \Leftrightarrow$$
$$\text{if } \exists n \text{ such that } (1^n{*}0{*}1) \text{ is extended by } \varepsilon$$
$$\text{then } \exists n \exists m\, (1^n{*}0{*}1{*}1^m{*}0) \text{ is extended by } \varepsilon \text{ and } \neg R(n, m).$$

(1) The definition of $\mathcal{H}$ is of the form "if $\exists\Phi$ then $\exists\Psi$", which is equivalent to the form "either $\forall\neg\Phi$ or $\exists\Psi$", which is in turn equivalent both to "$\forall\exists\neg\Phi$ or Ψ" and to "$\exists\forall\neg\Phi$ or Ψ". Hence, $\mathcal{H} \in \Delta^0_2$.

(2) We show that $P \leqslant_m \text{PRED}_{\mathcal{H}}$. In particular, we show that for each $n \in \omega$, $P(n) \Leftrightarrow ((1^n){*}0,\, n+1,\, 0) \in \text{PRED}_{\mathcal{H}}$. Evidently, the function $f(n) = ((1^n){*}0,\, n+1,\, 0)$ is computable. ($\Rightarrow$) Suppose $P(n)$. Then $\forall y\, R(n, y)$. Let $\varepsilon \in \mathcal{H}$ and let ε extend $(1^n){*}0$. Then since $\forall y\, R(n, y)$, we have by the contrapositive of the definition of $\mathcal{H}$ and the fact that ε extends $(1^n){*}0$ that $\varepsilon_{n+1} = 0$. Thus, PRED $\mathcal{H}$ $((1^n){*}0,\, n+1,\, 0) \in \text{PRED}_{\mathcal{H}}$. ($\Leftarrow$) Suppose $\neg P(n)$. Then for some $y \in \omega$, $\neg R(n, y)$. Let $\tau = (1^n){*}0{*}1{*}(1^y){*}(0^\omega)$. Since $\neg R(n, y)$, $\tau \in \mathcal{H}$. But $\tau_{n+1} = 1 \neq 0$ so $((1^n){*}0,\, n+1,\, 0) \notin \text{PRED}_{\mathcal{H}}$. ■

We have so far shown that for each arithmetical upper bound on $\text{PRED}_{\mathcal{H}}$ derived above, there is an example that realizes the full complexity of that bound. We will now show that for each case in which no arithmetical upper bound was derived, some $\mathcal{H}$ exists for which $\text{PRED}_{\mathcal{H}}$ is not arithmetically definable.

LEMMA 7.C.5 (Hilbert and Bernays).[12] There is an empirically complete $\mathcal{H} \subseteq 2^\omega$ that is computably refutable in the limit but whose predictions are not even arithmetically definable.

That is, there is an $\varepsilon \in 2^\omega$ such that

(1) $\{\varepsilon\} \in \Pi^0_2$ and
(2) $\forall n$, $\text{PRED}_{\{\varepsilon\}} \notin \Sigma^0_n$.

Proof. Let $\langle\ \rangle$ be a computable, 1–1 encoding of finite sequences of natural numbers by single natural numbers so that $\langle \bar{x} \rangle$ denotes a natural number that uniquely and effectively encodes the finite vector $\bar{x}$. Let $\bar{\varepsilon} = (\varepsilon[0], \ldots, \varepsilon[n])$ be a finite vector of infinite data streams. Let $\langle\langle \bar{\varepsilon} \rangle\rangle$ denote the infinite sequence $(\langle \varepsilon[0]_0, \ldots, \varepsilon[n]_0 \rangle, \langle \varepsilon[0]_1, \ldots, \varepsilon[n]_1 \rangle, \ldots, \langle \varepsilon[0]_k, \ldots, \varepsilon[n]_k \rangle, \ldots)$. Then $\langle\langle\ \rangle\rangle$ denotes the infinite sequence $(\langle\ \rangle, \langle\ \rangle, \ldots, \langle\ \rangle, \ldots)$, which is recursive. Now define

$\mathcal{U}^0_1(\langle\langle \bar{\varepsilon} \rangle\rangle, \langle \bar{x} \rangle, i) \Leftrightarrow \exists k$ such that Turing machine i halts on inputs $\bar{\varepsilon}$, $\bar{x}$ in at most k steps of computation.

$$\mathcal{U}^0_{n+1}(\langle\langle \bar{\varepsilon} \rangle\rangle, \langle \bar{x} \rangle, i) \Leftrightarrow \exists k \text{ such that } \neg\, \mathcal{U}^0_n(\langle\langle \bar{\varepsilon} \rangle\rangle, \langle \bar{x}{*}k \rangle, i).$$

Thus, $\mathcal{U}^0_n$ is a type $(1, 2)$ relation. The following result is a special case of the ***arithmetical indexing theorem***.[13]

(a) For each $n \geqslant 1$, for each $\mathcal{S} \in \Sigma^0_n$, there is an i such that $\mathcal{S}(\bar{\varepsilon}, \bar{x}) \Leftrightarrow \mathcal{U}^0_n(\langle\langle \bar{\varepsilon} \rangle\rangle, \langle \bar{x} \rangle, i)$.

Base case: $n = 1$. Suppose $\mathcal{S} \in \Sigma^0_1$. Let M_i be a positive test for $\mathcal{S}$. Then for all $\bar{\varepsilon}$, $\bar{x}$, $\mathcal{S}(\bar{\varepsilon}, \bar{x}) \Leftrightarrow \exists k$ such that $M_i[\bar{\varepsilon}, \bar{x}]$ halts in no more than k steps $\Leftrightarrow \mathcal{U}^0_1(\langle\langle \bar{\varepsilon} \rangle\rangle, \langle \bar{x} \rangle, i)$.

Inductive case. Suppose the result for all $n' \leqslant n$. Let $\mathcal{S} \in \Sigma^0_{n+1}$. Then for some $\mathcal{G} \in \Sigma^0_n$, for all $\bar{\varepsilon}$, $\bar{x}$, $\mathcal{S}(\bar{\varepsilon}, \bar{x}) \Leftrightarrow \exists k \neg\, \mathcal{G}(\bar{\varepsilon}, \bar{x}, k)$. By the induction hypothesis, there is some i such that for all $\bar{\varepsilon}$, $\bar{x}$, k, $\mathcal{G}(\bar{\varepsilon}, \bar{x}, k) \Leftrightarrow \mathcal{U}^0_n(\langle\langle \bar{\varepsilon} \rangle\rangle, \langle \bar{x}, k \rangle, i)$. Thus $\mathcal{S}(\bar{\varepsilon}, \bar{x}) \Leftrightarrow \exists k \neg \mathcal{U}^0_n(\bar{\varepsilon}, \langle \bar{x}, k \rangle, i)$. This establishes (a). Next define:

$$\mathcal{D}_n(x) \Leftrightarrow \mathcal{U}^0_n(\langle\langle\ \rangle\rangle, \langle x \rangle, x).$$

Now we show the ***arithmetical hierarchy theorem***:

(b) for each n, $\mathscr{D}_n \in \Sigma^0_n - \Pi^0_n$.

It is evident from the definition that $\mathscr{U}^0_n \in \Sigma^0_n$, and hence that $\overline{\mathscr{D}_n} \in \Sigma^0_n$. Now suppose for *reductio* that $\mathscr{D}_n \in \Pi^0_n$. Hence, $\overline{\mathscr{D}_n} \in \Sigma^0_n$. By (a) there is some b such that for all x, $\overline{\mathscr{D}_n}(x) \Leftrightarrow \mathscr{U}^0_n(\langle\langle\ \rangle\rangle, \langle x\rangle, b)$. So in particular, (*) $\overline{\mathscr{D}_n}(b) \Leftrightarrow \mathscr{U}^0_n(\langle\langle\ \rangle\rangle, \langle b\rangle, b)$. By the definition of $\mathscr{D}$, we have (**) $\mathscr{D}_n(b) \Leftrightarrow \mathscr{U}^0_n(\langle\langle\ \rangle\rangle, \langle b\rangle, b)$. But (*) and (**) yield $\overline{\mathscr{D}_n}(b) \Leftrightarrow \mathscr{D}_n(b)$ which is a contradiction, so we have (b). Now define:

$$v(\langle\langle\bar{x}\rangle, i, n\rangle) = \begin{cases} 1 & \text{if } \mathscr{U}^0_n(\langle\langle\ \rangle, \langle\bar{x}\rangle, i) \\ 0 & \text{otherwise.} \end{cases}$$

For each n, $\mathscr{D}_n \leq_m v$, so by (b) we have for each n, $v \notin \Pi^0_n$. Hence, for each n, $v \notin \Sigma^0_n$. Since $v(\langle\langle\bar{x}\rangle, i, n\rangle) = 1 \Leftrightarrow (\langle\ \rangle, \langle\langle\bar{x}\rangle, i, n\rangle, 1) \in \text{PRED}_{\{v\}}$, we have that (2) $\forall n\ \text{PRED}_{\{v\}} \notin \Sigma^0_n$. It remains to show that (1) $\{v\} \in \Pi^0_2$. A straightforward induction establishes that

$\varepsilon \in \{v\} \Leftrightarrow$
(i) $\forall k \in \omega$, $\varepsilon_k \leq 1$ and
(ii) $\forall i \in \omega$, $\bar{x} \in N^*$ $[\varepsilon(\langle\langle\bar{x}\rangle, i, 1\rangle) = 1 \Leftrightarrow \mathscr{U}^0_1(\langle\langle\ \rangle\rangle, \langle\bar{x}\rangle, i)]$ and
(iii) $\forall n, i, x \in \omega[\varepsilon(\langle\langle\bar{x}\rangle, i, n+1\rangle) = 1 \Leftrightarrow \exists k$ and that $\varepsilon(\langle\langle\bar{x} * k\rangle, i, n\rangle) = 0)]$.

Intuitively, condition (i) says that ε is a characteristic function, condition (ii) duplicates the base case of the definition of $\mathscr{U}^0_n$ and condition (iii) duplicates the inductive case of the definition of $\mathscr{U}^0_n$. Putting the definition into prenex normal form reveals that $\{v\} \in \Pi^0_2$. ■

LEMMA 7.C.6.[14] There is an empirically complete $\mathscr{H} \subseteq \omega^\omega$ that is computably refutable with certainty but whose predictions are not even arithmetically definable.

That is, there is an $\varepsilon \in 2^\omega$ such that

(1) $\{\varepsilon\} \in \Pi^0_1$ and
(2) $\forall n\ \text{PRED}_{\{\varepsilon\}} \notin \Sigma^0_n$.

Proof. Let v be as in the proof of the preceding proposition. $\{v\} \in \Pi^0_2$, so there is a recursive relation $\mathscr{G}$ such that for all ε, $\varepsilon = v \Leftrightarrow \forall x \exists y \mathscr{G}(\varepsilon, x, y)$. Define:

$$\delta(x) = \langle v_x, \mu y\ \mathscr{G}(v, x, y)\rangle.$$

A Σ^0_n definition for δ would yield a Σ^0_n definition for v, since v_x can be recovered by decoding $\delta(x)$ and returning the first coordinate. So by Lemma 7.C.5, we have that for each n, $\delta \notin \Sigma^0_n$. So $\text{PRED}_{\{\delta\}} \notin \Sigma^0_n$, which establishes (2).

It remains only to show that $\{\delta\} \in \Pi^0_1$. Let $(\langle x, y\rangle)_1 = x$ and $(\langle x, y\rangle)_2 = y$. Given data stream α, let $(\alpha)_1$ denote the unique data stream such that for each $x \in \omega$, $(\alpha)_1(x) = (\alpha(x))_1$. We now show:

$$\varepsilon \in \{\delta\}\ (\text{i.e. } \varepsilon = \delta) \Leftrightarrow \text{(a) } \forall x\ \mathscr{G}((\varepsilon)_1, x, (\varepsilon(x))_2) \text{ and}$$
$$\text{(b) } \forall x, y[y < (\varepsilon(x))_2 \Rightarrow \neg\mathscr{G}((\varepsilon)_1, x, y)],$$

so there is a Π^0_1 definition of $\{\delta\}$. ($\Rightarrow$) Recall that $\forall x \exists y \mathscr{G}(v, x, y)$. Thus $\mathscr{G}$ holds if we choose the least such y: $\forall x \mathscr{G}(v, x, \mu y\ \mathscr{G}(v, x, y))$. But by the definition of δ, $\mu y\ \mathscr{G}(v, x, y) = \delta(x)_2$ and $v = (\delta)_1$, so $\forall x\ \mathscr{G}((\delta)_1, x, (\delta(x))_2)$, which is (a). And (b) follows because $(\delta(x))_2$ is the least y such that $\mathscr{G}((\delta)_1, x, y)$. ($\Leftarrow$) Suppose that (a) $\forall x\ \mathscr{G}((\varepsilon)_1, x, (\varepsilon(x))_2)$. Then $\forall x \exists y\ \mathscr{G}((\varepsilon)_1, x, y)$. Thus $(\varepsilon)_1 = v$. Assuming (b), we have that for all x, $(\varepsilon(x))_2$ is the least y such that $(\mathscr{G}(v, x, y)$. Thus $\varepsilon = \delta$, as required. ■

Proof of Theorem 5.1, *Optimality*. The Δ^0_1 upper bounds cannot be improved in the arithmetical hierarchy. The non-existence of bounds in the infinite O case all follow from Lemma 7.C.6. The non-existence of bounds in the finite O cases all follow from Lemma 7.C.5. The infinite O, general case of (a) is best by Lemma 7.C.1. Both upper bounds in the general case of line (b) are best by Lemma 7.C.2. The upper bound in the finite O, general case of (c) is best by Lemma 7.C.3. The finite O, general case upper bounds given in (d) and (e) both follow from Lemma 7.C.4. This concludes the proof of Theorem 5.1. ■

8. AN EXPLANATION

It has been shown that there is an empirically complete hypothesis $\mathscr{H} = \{\delta\}$ such that a computable method α refutes $\mathscr{H}$ with certainty, but the prediction function δ determined by $\mathscr{H}$ is not even arithmetically definable, much less computable. So however the computable method α works, it does not proceed by deriving successive predictions from the theory on demand and checking them against the data. We can say somewhat more than this. Define[15]

$\mathcal{H}$ is ***consistent*** with $\mathbf{e} \Leftrightarrow$ there is an $\varepsilon \in \mathcal{H}$ such that ε extends $\mathbf{e}$.

α is ***consistent*** for $\mathcal{H}, O \Leftrightarrow$
for each finite data sequence $\mathbf{e} \in O^*$, if $\mathbf{e}$ is not consistent with $\mathcal{H}$, then $\alpha(\mathbf{e}) = 0$.

A method consistent for an empirically complete hypothesis rejects the hypothesis as soon as the data logically refutes it. This seems reasonable if all computability is neglected, for a refuted hypothesis must be false (absent any model of noise in the data). Bayesian methodologists who recommend updating by conditionalization as a scientific method are in fact committed to consistency, since the conditional probability of a hypothesis must be 0 on any data that logically refutes it. Some Bayesians have also recommended "keeping the door ajar" by witholding probability 0 until the hypothesis is refuted by the data.

α is ***conservative*** for $\mathcal{H}, O \Leftrightarrow$
for each finite data sequence $\mathbf{e} \in O^*$, $\alpha(e) = 0 \Rightarrow \mathcal{H}$ is inconsistent with $\mathbf{e}$.

But this is rather strong medicine. For example, verification and refutation in the limit countenance an arbitrary number of 0 conjectures when the hypothesis is true, and conservatism outlaws such behavior. A less onerous requirement implied by conservatism is the following:

α is ***weakly conservative*** for $\mathcal{H} \Leftrightarrow$
for each $\varepsilon \in \mathcal{H}$, there are infinitely many n such that, $\alpha(\varepsilon|n) \neq 0$
(i.e. α does not stabilize to 0 when $\mathcal{H}$ is true).

If α verifies, refutes, or decides $\mathcal{H}$ either with certainty or in the limit, then α is weakly conservative for $\mathcal{H}$. Even weaker notions of success imply weak conservatism. For example, a method that outputs rational numbers in the unit interval ***gradually decides*** $\mathcal{H}$ just in case its conjectures get ever *closer* to 1 when $\mathcal{H}$ is true and get ever closer to 0 otherwise. This is the sort of convergence that is often expected of statistical methods like Bayesian updating by conditionalization.[16] Now we have:

PROPOSITION 8.1. If α is arithmetically definable and α is consistent and weakly conservative for $\{\varepsilon\}$, then ε is arithmetically definable with complexity no greater than that of α.

Proof. Suppose arithmetically definable α is both consistent and weakly conservative for $\{\varepsilon\}$, O. Then define:

$$\varepsilon_x = y \Leftrightarrow \exists k \geq x\, \exists \mathbf{e} \in O^k \text{ such that } \alpha(\mathbf{e}) \neq 0 \text{ and } \mathbf{e}_x = y$$
$$\Leftrightarrow \forall k \geq x\, \forall \mathbf{e} \in O^k, \text{ if } a(\mathbf{e}) \neq 0 \text{ then } \mathbf{e}_x = y.$$

These definitions of ε are both correct, since α returns values other than 0 only along ε by consistency and α does return values other than 0 infinitely often along ε by weak conservatism. Since α is arithmetical, and since ε can be defined with one existential quantifier or with one universal quantifier over α, the arithmetical complexity of ε does not exceed that of α. ■

We have seen that a computable method α can refute $\{\delta\}$ with certainty, but that δ is not arithmetically definable. Suppose, for contradiction, that α is also consistent. We can readily alter α to make it both consistent and weakly conservative: let α' simulate α on each initial segment of $\mathbf{e}$. α' conjectures 1 at a given position in $\mathbf{e}$ unless α conjectures a 0 preceded only by ?'s on some initial segment of $\mathbf{e}$, after which α' conjectures 0 on $\mathbf{e}$. α' is clearly computable since α is. So it follows that δ is recursive. But by Lemma 7.C.6, δ is not even arithmetically definable. So α cannot be consistent. In other words, to refute $\{\delta\}$ with certainty, the computable method α *must* allow some time lag before "noticing" that $\{\delta\}$ has become inconsistent with the data.

It follows from Proposition 8.1 that no consistent, weakly conservative method for $\{\delta\}$ is arithmetically definable. So standard methodological recommendations like Bayesian updating can interfere with the prospects of science *even for highly uncomputable agents*:[17]

PROPOSITION 8.2.[18] Even though some *computable* method can refute $\{\delta\}$ *with certainty*, no *arithmetically definable* Bayesian conditionalizer can *gradually* decide $\{\delta\}$.

Proof. Let $\alpha(h, \mathbf{e}) = P(h|\mathbf{e})$, for some probability measure P such that all conditional probabilities of form $P(h|\mathbf{e})$ are defined. Suppose α gradually decides $\{\delta\}$. Then α is weakly conservative *a fortiori*, since α's conjectures approach 1 on data stream δ. Also, α is consistent, by coherence. Hence, α is not arithmetically definable, by Proposition 8.1. ■

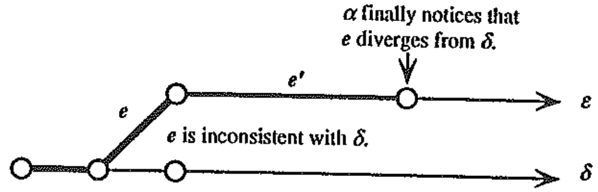

Fig. 14.

It is clear, then, that we must allow α some time delay before recognizing that δ does not agree with the data received in the past. This is not the trivial point that we must allow α some time to compute the current prediction of $\{\delta\}$ before moving on to the next prediction. The point is that the computations cannot be performed in any finite amount of time. Nonetheless, α can refute $\{\delta\}$ with certainty. Hence, if **e** diverges from δ, then for each ε extending **e**, there is some initial segment **e**$'$ of ε that α can determine to diverge from δ.

One way of viewing the situation is that α employs an *incomplete* proof system for deriving facts of the form $\delta(x) = y$ that is nonetheless *sufficiently* complete in the sense just described. A more provocative interpretation is that α requires *empirical* data about the future in order to effectively determine consistency between $\{\delta\}$ and **e**, so that the line between formal and empirical inquiry is blurred, not just for computable methods, but for all arithmetically definable methods. In fact, the provocative interpretation is supported by a closer inspection of the definition of α.

Choose the $\mathscr{G}$ in the proof of Lemma 7.C.6 as follows. $\mathscr{G}$ is the result of translating the definition of ν given in the proof of Lemma 7.C.6 into prenex normal form and then combining adjacent existential and universal quantifiers into single quantifiers over code numbers (i.e. $\forall\bar{x}\forall i\forall n\forall z\forall v$ is coded as $\forall\langle\langle\bar{x}\rangle, i, n, z, v\rangle$).

$\mathscr{G}(\varepsilon, \langle\langle\bar{x}\rangle, i, n, z, v\rangle, \langle u, w\rangle) \Leftrightarrow$
(1) $\varepsilon(\langle\langle\bar{x}\rangle, i, n\rangle) \in \{0, 1\}$ and
2(a) $\varepsilon(\langle\langle\bar{x}\rangle, i, 1\rangle) = 1$ or $\neg\phi_i(\bar{x})$ halts in v computational steps and
2(b) $\neg\varepsilon(\langle\langle\bar{x}\rangle, i, 1\rangle) = 1$ or $\phi_i(\bar{x})$ halts in w computational steps and
3(a) $\varepsilon(\langle\langle\bar{x}\rangle, i, n+1\rangle) = 1$ or $\neg\varepsilon(\langle\langle\bar{x}*u\rangle, i, n\rangle) = 0$ and
3(b) $\neg\varepsilon(\langle\langle\bar{x}\rangle, i, n+1\rangle) = 1$ or $\varepsilon(\langle\langle\bar{x}*z\rangle, i, n\rangle) = 0$

$\mathscr{G}$ is recursive. Following the definition of $\{\delta\}$ in the proof of Lemma 7.C.6, we may define the computable method α as follows:

procedure $\alpha(\mathbf{e})$:
set n = length(**e**);
for each $j < n$, do recover the pair (x_j, y_j) such that $\langle x_j, y_j\rangle = \mathbf{e}_j$.
set $\mathbf{d} = (\mathrm{x}_0, \ldots, \mathrm{x}_{n-1})$ (i.e. $\mathbf{d} = (\mathbf{e})_1$);
for each $j < n$, do
  if $\mathscr{G}(\mathbf{d}, i, y_j)$ returns 0 within n steps of computation, then halt with output 0
  else if $\mathscr{G}(\mathbf{d}, i, y_j)$ returns 1, then for each $y < y_j$ do
    if $\mathscr{G}(\mathbf{d}, i, y)$ returns 1 within n steps of computation, then halt with output 0
  else halt with output?.

Whenever $\mathscr{G}((x_1, \ldots x_n), i, y_j)$ is evaluated by α, $\mathscr{G}$ may ask to see five distinct positions on the data stream ε, namely, $\varepsilon(\langle\langle\bar{x}\rangle, i, n\rangle)$, $\varepsilon(\langle\langle\bar{x}\rangle, i, 1\rangle)$, $\varepsilon(\langle\langle\bar{x}\rangle, i, n+1\rangle)$, $\varepsilon(\langle\langle\bar{x}*u\rangle, i, n\rangle)$, and $\varepsilon(\langle\langle\bar{x}*z\rangle, i, n\rangle)$. There is no guarantee that these positions do not run off the end of $(x_1, \ldots, x_n)$ (and hence of **e**). In that case, α must stall with conjecture ? until **e** grows longer, even though **e** may already diverge from δ. So α can be viewed as waiting for future data in order to determine consistency between **e** and $\{\delta\}$, as was claimed.

9. ONWARD AND UPWARD

We have seen that there are hypotheses that are computably testable in various senses, even though there is no arithmetical bound on the problem of deriving their predictions. But the story does not end there. The ***analytical hierarchy*** is defined just like the arithmetical hierarchy, except that we start out with the arithmetical sets and build complexity by quantification over functions rather than numbers. Let $\mathscr{R}$ be a relation of type (k, m).

$\mathscr{R} \in \Sigma^1_0 \Leftrightarrow \mathscr{R}$ is arithmetically definable.
$\mathscr{R} \in \Sigma^1_{n+1} \Leftrightarrow$ there is a type $(k+1, m)$ relations $\mathscr{S} \in \Sigma^1_n$ such that for each $\bar{\varepsilon} \in (O^\omega)^k$, $\bar{x} \in \omega^m$,
  $\mathscr{R}(\bar{\varepsilon}, \bar{x}) \Leftrightarrow \exists\tau$ such that $\neg\mathscr{S}(\bar{\varepsilon}, \tau, \bar{x})$.
$\mathscr{R} \in \Pi^1_n \Leftrightarrow \neg\mathscr{R} \in \Sigma^1_n$.

Theorem 9.1 Given arithmetical bound on $\mathcal{H}$			Best arithmetical/analytical bound on $\text{PRED}_{\mathcal{H}}$: $\mathcal{H}$ is empirically complete, 1. O is finite	$\mathcal{H}$ is empirically complete, 2. O is infinite	General case, 3. O is finite	General case, 4. O is infinite
n = 1	a	Δ^0_1	Impossible unless $\lvert O\rvert = 1$		Δ^0_1	Π^0_1
	b	Σ^0_1			Π^0_1	Π^0_1
	c	Π^0_1	Δ^0_1	Δ^1_1	Σ^0_1	Π^1_1
n = 2	d	Δ^0_2	Δ^0_1	Δ^1_1	Π^0_2	Π^1_1
	e	Σ^0_2	Δ^0_1	Δ^1_1	Π^0_2	Π^1_1
	f	Π^0_2	Δ^1_1	Δ^1_1	Π^1_1	Π^1_1
n > 2	g	$\Sigma^0_n, \Pi^0_n, \Delta^0_n$	Δ^1_1	Δ^1_1	Π^1_1	Π^1_1

Fig. 15. Theorem 9.1.

$$\mathcal{R} \in \Delta^1_n \Leftrightarrow \mathcal{R} \in \Pi^1_n \cap \Sigma^1_n.$$

Now we solve for upper bounds on the complexity of deriving the predictions of an arbitrary, arithmetically definable theory. The complete table of such bounds is given in Theorem 9.1 (cf. Figure 14).

LEMMA 9.2. If $\{\varepsilon\} \in \Sigma^1_0$ then $\varepsilon \in \Delta^1_1$.

Proof. Let $\{\varepsilon\}$ be arithmetically definable, say by the arithmetical relation $\mathcal{R}(\varepsilon)$. Then

$$\varepsilon(n) = m \Leftrightarrow \forall \tau \in O^\omega, \text{ if } \mathcal{R}(\tau) \text{ then } \tau(n) = m$$
$$\Leftrightarrow \exists \tau \in O^\omega \text{ such that } \mathcal{R}(\tau) \text{ and } \tau(n) = m.$$

Hence, $\varepsilon \in \Delta^1_1$. ■

LEMMA 9.3. If $\mathcal{H} \in \Sigma^1_0$ then $\text{PRED}_{\mathcal{H}} \in \Pi^1_1$.

Proof. Let $\mathcal{H}$ be arithmetically definable. Recall that:

$$\text{PRED}_{\mathcal{H}}(\mathbf{e}, n, o) \Leftrightarrow \forall \tau \in \mathcal{H}, \text{ if } \mathbf{e} \subset \tau \text{ then } \tau_n = o.$$

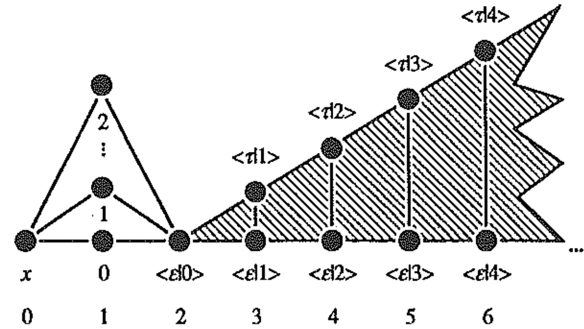

Fig. 16. Typical elements of $\mathcal{H}$.

Hence, $\text{PRED}_{\mathcal{H}} \in \Pi^1_1$. ■

Let $\langle \_ \rangle$ denote a fixed, effective bijection from ω^* to ω, and let $[\_]$ denote the inverse function from ω to ω^*.

LEMMA 9.4. $S \in \Pi^1_1 \Leftrightarrow$ there is a recursive relation G such that

$$S(x) \Leftrightarrow \forall \tau \in \omega^\omega, \exists n\, G(\langle \tau|n \rangle, x).$$

Proof (Rogers 1987), Corollary V, p. 378. ■

LEMMA 9.5. There is an $\mathcal{H} \subseteq \omega^\omega$ such that $\mathcal{H} \in \Pi^0_1$ and $\text{PRED}_{\mathcal{H}}$ is Π^1_1-complete.

Proof. Let S be a Π^1_1 complete set (e.g. the set of all indices of finite path trees (Rogers 1987)). By Lemma 9.4, there is a recursive relation G such that

$$(*) \qquad x \in S \textit{ as} \Leftrightarrow \forall \tau \in \omega^\omega, \exists n G(\langle \tau|n \rangle, x).$$

Now define:

$$\varepsilon \in \mathcal{H} \Leftrightarrow \forall n \geq 2, [\varepsilon_n] \subseteq [\varepsilon_{n+1}] \;\&\; \text{length}([\varepsilon_n]) = n - 1$$
$$\& \neg G([\varepsilon_n], \varepsilon_0).$$

Evidently, $\mathcal{H} \in \Pi^0_1$. Now we verify that $S \leq_m \text{PRED}_{\mathcal{H}}$, so that $\text{PRED}_{\mathcal{H}}$ is Π^1_1-complete. ($\Leftarrow$) Suppose $x \notin S$. Then by (*), $\exists \tau \in \omega^\omega$ such that $\forall n, \neg G(\langle \tau|n \rangle, x)$. Define $\varepsilon[y] = (x, y, \langle \tau|1 \rangle, \langle \tau|2 \rangle, \ldots, \langle \tau|k \rangle, \ldots)$. For each y, $\varepsilon[y] \in \mathcal{H}$. Hence, $\neg\text{PRED}_{\mathcal{H}}((x), 1, 1)$. ($\Rightarrow$) Suppose $x \in S$. Let $\varepsilon_0 = x$. Suppose for *reductio* that $\varepsilon \in \mathcal{H}$. Then $\forall n$, $[\varepsilon_n] \subseteq [\varepsilon_{n+1}]$, by the definition of $\mathscr{H}$. Let τ be such that for each $n \geq 2$, $\tau|n-2=[\varepsilon_n]$. Then $\forall n$, $\neg G([\varepsilon_n]\ \varepsilon_0)$, by the definition of $\mathscr{H}$. But that contradicts (*), since $x \in S$. Hence, $\varepsilon \notin \mathscr{H}$. Since ε is an arbitrary data stream such that $\varepsilon_0 = x$, x is inconsistent with $\mathscr{H}$. Hence, $\text{PRED}_{\mathscr{H}}((x), 1, 1)$. So $f(x) = ((x), 1, 1)$ reduces S to $\text{PRED}_{\mathscr{H}}$. ■

LEMMA 9.6. There is an $\mathscr{H} \subseteq 2^\omega$ such that $\mathscr{H} \in \Pi^0_2$ and $\text{PRED}_{\mathscr{H}}$ is Π^1_1-complete.

Proof. Let S be a Π^1_1 complete set. Let $\varepsilon \in \omega^\omega$. We may uniquely encode ε with a sequence $\tau \in 2^\omega$, as follows. For each n, encode ε_n as a binary numeral. Now write the successive digits of each numeral in even positions, filling in with 0 in odd positions until the numeral is entirely written down. Signal the end of the numeral by putting a 1 in the next odd position. Thus, $(0, 1, 2 \ldots)$ is encoded as $(0, 1, 1, 1, 1, 0, 1, \ldots)$. $\tau \in 2^\omega$ encodes an element of $\omega^\omega \Leftrightarrow$ 1 occurs in infinitely many distinct, odd positions of τ. Then say that τ is *significant*. If τ is significant, then define $[\tau]_n$ = the natural number denoted by the binary numeral occupying even positions between the nth occurrence of 1 in an odd position and the $n+1$th occurrence of 1 in an odd position. Now following the preceding lemma, define

$$\varepsilon \in \mathscr{H} \Leftrightarrow \varepsilon \text{ is significant and } \forall n \geq 2,\ [[\varepsilon]_n] \subseteq [[\varepsilon]_{n+1}]\ \&\ \text{length}([[\varepsilon]_n]) = n-1\ \&\ \neg G([[\varepsilon]_n], \varepsilon_0).$$

If ε is significant, then $[\varepsilon]_n$ is total and effective. Since significance is Π^0_2, so is $\mathscr{H}$. But $\text{PRED}_{\mathscr{H}}$ reduces S by the argument of Lemma 9.5. ■

Proof of Theorem 9.1. The analytical upper bounds of Theorem 9.1 follow from Lemmas 9.2 and 9.3. The optimality of the bounds in the general case follow from Lemmas 9.5 and 9.6. In the empirically complete cases, we have from Lemmas 7.C.5 and 7.C.6 that $\{\nu\} \in \Pi^0_2$ and $\{\delta\} \in \Pi^0_1$ and neither ν nor δ is in Σ^1_0. From Lemma 9.2, $\nu, \delta \in \Delta^1_1$. This shows that no tighter bound than Δ^1_1 can be given in the analytical hierarchy. But we have not shown that ν, δ are Δ^1_1-complete. In fact *no* set is Δ^1_1-complete.

LEMMA 9.7.[19] $\forall n, \forall R \subseteq \omega$, R is not Δ^1_n-complete.

Proof. Suppose, for *reductio*, that R is Δ^1_n-complete. Define $S = \{x: \exists y\ \phi_x(x) = y\ \&\ y \notin R\}$. $S \in \Delta^1_n$, since R is and Δ^1_n is closed under complementation and first-order quantification. Hence, there is a total recursive f such that $\forall x$, $x \in S \Leftrightarrow f(x) \in R$. Let $\phi_k = f$. Suppose $k \in S$. Then $\phi_k(k) = f(k) \notin R$. Suppose $k \notin S$. Since f is total, $\phi_k(k)$ is defined. Hence, $\phi_k(k) \in R$. Thus $k \in S \Leftrightarrow f(k) \notin R$. So f does not reduce S to R. Contradiction. ■

As in the empirically complete case, a computable [or arithmetically definable] method can refute $\mathscr{H}$ with certainty, but cannot notice immediately when the hypothesis is refuted. The result holds also for ***hyperarithmetically definable*** methods (i.e. methods in Δ^1_1), so Lemma 9.5 yields an even more powerful critique of Bayesian methodology than did Lemma 7.C.6.

COROLLARY 9.7. There is an $\mathscr{H}$ such that
(a) $\mathscr{H}$ is computably refutable with certainty, but
(b) no $\alpha \in \Sigma^1_1$ is consistent and weakly conservative for $\mathscr{H}$, ω, and hence
(c) no Bayesian method in Σ^1_1 can gradually decide $\mathscr{H}$.

Proof. Let $\mathscr{H}$, S be as defined in the proof of Lemma 9.5, so (a) is immediate. Then (x) is consistent with $\mathscr{H} \Leftrightarrow x \notin S$. Suppose that α is consistent and weakly conservative for $\mathscr{H}$, ω. Then (*) $x \in S \Leftrightarrow \forall \mathbf{e} \in O^*$, if $\mathbf{e}$ extends (x) then $\alpha(\mathbf{e}) = 0$. Suppose $\alpha \in \Sigma^1_1$. Then by rearrangement of quantifiers, $S \in \Sigma^1_1$, which is a contradiction, yielding (b).[20] But it was observed in Section 8 that if a Bayesian method α gradually decides $\mathscr{H}$, then α is consistent and conservative for $\mathscr{H}$, ω, so (c) follows. ■

10. CONVERSE RELATIONS

So far we have asked how complex $\text{PRED}_{\mathscr{H}}$ can be if $\mathscr{H}$ is computably testable in a given sense. We can turn the question around and ask how computably untestable $\mathscr{H}$ can be for a given complexity of $\text{PRED}_{\mathscr{H}}$. In the general case, it is easy to see that an arbitrarily untestable theory can make computationally trivial predictions, since it may refuse to entail any predictions about the future at all.

PROPOSITION 10.1. For each (arbitrarily complex) $S \subseteq \omega$, there is an $\mathscr{H} \subseteq 2^\omega$ such that $\text{PRED}_{\mathscr{H}} \in \Delta^0_1$ and S is no more complex than $\mathscr{H}$.

Proof. Let S be an arbitrary subset of ω. Define $\varepsilon \in \mathscr{H} \Leftrightarrow \varepsilon_0 \in S$. Then $(\mathbf{e}, n, o) \in \text{PRED}_{\mathscr{H}} \Leftrightarrow \mathbf{e}_n = o$, so $\text{PRED}_{\mathscr{H}}$ is Δ^0_1. Define $x \in S \Leftrightarrow x^\omega \in \mathscr{H}$. So, S is no more complex than $\mathscr{H}$. ■

Moving on to the empirically complete case, there is the following, comprehensive result:

PROPOSITION 10.2. If $\varepsilon \in \Pi^0_n$ then $\{\varepsilon\} \in \Pi^0_n$.

Proof. Define $\tau \in \{\varepsilon\} \Leftrightarrow \forall n,\ \tau_n = \varepsilon_n$. ∎

The Σ^0_n case follows from the Π^0_n case by the following lemma:

PROPOSITION 10.3. If $\varepsilon \in \Sigma^0_n$ then $\varepsilon \in \Delta^0_n$.

Proof. Since ε is total, $\varepsilon(x) \neq y \Leftrightarrow \exists y' \neq y$ such that $\varepsilon(x) = y'$. Since ε is Σ^0_n, this definition of the complement of ε is also Σ^0_n, so ε is Π^0_n and hence is Δ^0_n. ∎

Now we show that the upper bound given in Proposition 9.2 is optimal at level 1 in the hierarchy.

PROPOSITION 10.4. There is an $\varepsilon \in 2^\omega$ such that $\varepsilon \in \Delta^0_1$ but $\{\varepsilon\}$ is Π^0_1-complete (and hence is in $\Pi^0_1 - \Sigma^0_1$).

Proof. Let ζ be the 0 constant function. ζ is recursive. Let $\mathscr{H} \in \Pi^0_1$. So for some recursive $\mathscr{G}$, $\varepsilon \in \mathscr{H} \Leftrightarrow \forall x\ \mathscr{G}(\varepsilon, x)$. Define the recursive operator $\Phi(\varepsilon)_x = 0$ if $\mathscr{G}(\varepsilon, x)$ and $\Phi(\varepsilon)_x = 1$ otherwise. $\varepsilon \in \mathscr{H} \Leftrightarrow \Phi(\varepsilon) \in \{\zeta\}$, so $\mathscr{H} \leq_m \{\zeta\}$. Hence, $\{\zeta\}$ is Π^0_1-complete. ∎

At level 2, we have:

PROPOSITION 10.5. Let $O = 2$. There is an $\varepsilon \in 2^\omega$ such that $\varepsilon \in \Delta^0_2$ but $\{\varepsilon\} \in \Pi^0_2 - \Sigma^0_2$.

Proof. Let κ be the characteristic function of K. Then for each $x \in \omega$, $\kappa(x) = y \Leftrightarrow y = 1$ and $\exists k\ \phi_x(x)$ halts in k steps or $y = 0$ and $\forall k$, $\phi_x(x)$ does not halt in k steps. Hence, κ is Δ^0_2. Suppose $\{\kappa\} \in \Sigma^0_2$. Then by Lemma 7.B.4, κ is recursive, which is a contradiction, since $K \leq_m \kappa$. So $\{\kappa\} \notin \Sigma^0_2$. ∎

$\{\kappa\}$ is not Π^0_2-complete, because *no* singleton is.

PROPOSITION 10.6. If $|O| > 1$ then for each ε, $\{\varepsilon\}$ is not Π^2_0-complete.

Proof. Let $|O| > 1$, $x \in O$, and $\varepsilon \in O^\omega$. Define $\mathscr{F}(\tau) \Leftrightarrow \forall n\ \exists m > n$ such that $\tau_m = x$. $\mathscr{F} \in \Pi^0_2$. Now suppose for *reductio* that there is a recursive operator Φ such that for each $\tau \in 2^\omega$, $\tau \in \mathscr{F} \Leftrightarrow \Phi(\tau) \in \{\varepsilon\}$. Let $y \in O - \{x\}$. Hence, $y^\omega \notin \mathscr{F}$. Let n' be the least n such that $\Phi(y^\omega)_n \neq \varepsilon_n$. There is such an n, else $\Phi(y^\omega) = \varepsilon$, contrary to the reductio hypothesis. Then there is some k such that for each τ, if $\tau|k = y^k$ then $\Phi(\tau)_n = \Phi(y^\omega)_n \neq \varepsilon_n$, since Φ must proceed locally by reading increasing segments of its input (i.e. Φ is continuous). Hence, $\Phi(y^k * x^\omega)_n \neq \varepsilon_n$. But $y^k * x^\omega \in \mathscr{F}$ contradicting the reductio hypothesis. ∎

We leave optimality open for levels 3 and higher. Since $\{\varepsilon\} \in \Pi^0_2$ is consistent with ε being non-arithmetical (Lemma 7.5.C), the proof strategy of Proposition 9.5 no longer applies at level 3 or higher.

11. CONCLUSION

An important task for philosophy is to expose intriguing structure in the heart of apparent banality. It is apparently banal that science should proceed by deriving predictions from theories and checking these predictions against the data. And yet, it has been shown that this conception (as well as the increasingly popular proposal that inquiry proceed by Bayesian updating) severely underestimates the true potential of effective (and even of definable) scientific methods. In fact, the relationship between derivability of predictions, on the one hand, and scientific reliability, on the other, is a complicated matter that depends crucially on such unexpected factors as whether or not there can be infinitely many possible outcomes of an experiment. We have presented the complete table of such relations, together with proofs that the general upper bounds given cannot be improved. In pursuit of these bounds, we have been led beyond the realm of arithmetical definability.

Despite the systematic character of this study, it concerns only the outer limits of the relations between empirical testability and formal derivability. For example, it would be nice to know how pervasive the phenomenon of Lemma 7.C.6 is. Is the result closely tied to the carefully tailored structure of the hypothesis $\mathscr{H} = \{\delta\}$? Or is the existence proof merely the tip of a hidden iceberg? Is there a hypothesis of genuine interest to science that has the same formal properties? Similar questions could be asked of all our existence proofs. Recent work on computability in physics[21] may suggest less contrived examples, although that work has yet to isolate the extremely high complexities required to illustrate many of our results. One might also consider variations on our notions of computable testability, involving experimentation, probability, weaker forms of computability, theory laden data, and many other factors of interest to the philosophy of science. But even in their highly idealized form, the results presented here can serve to loosen up overly restrictive intuitions about how logic and scientific inquiry must interact.

NOTES

* Oliver Schulte was supported during the course of this work by a generous scholarship from SIEMENS AG. We also wish to thank the two anonymous reviewers for many helpful corrections to the original manuscript.

[1] E.g. (Pour-El and Richards 1980).

[2] By "highly idealized", we mean hyper-arithmetically definable.

[3] Definability in elementary arithmetic will be introduced in Section 6.

[4] In this sense, these results constitute an important generalization of those in (Kelly 1993).

[5] "Hyper-arithmetical" will be defined in Section 9 below.

[6] Also see (Kelly 1991).

[7] For example, let $O = \{0, 1\}$ and let $\varepsilon \in \mathscr{H} \Leftrightarrow$ only finitely many 0's occur in ε. Then $\text{PRED}_{\mathscr{H}}(\mathbf{e}, n, o) \Leftrightarrow \mathbf{e}_n = o$, and hence is decidable with certainty, but $\mathscr{H}$ is not refutable in the limit. For let α be an arbitrary method. A "demon" can feed data to α as follows. The demon feeds 0 so long as α conjectures something other than 0, and feeds 1 each time α conjectures 0. If α stabilizes to 0, then the data stream presented stabilizes to 1 and hence is in $\mathscr{H}$. If α conjectures a non-zero infinitely often, then infinitely many 0's occur in the data stream, so it is not in $\mathscr{H}$. Hence, no α refutes $\mathscr{H}$ in the limit.

[8] Of course, finding the truth is not the only aim of scientific inquiry. It is not our purpose in this paper to survey other aims, such as maintaining coherence or convincing others.

[9] What follows is a quick sketch intended to remind the reader of the relevant definitions. An expanded presentation may be found in any standard text on recursion theory such as (Hinman 1978) or (Rogers 1987).

[10] "[The dogmatists] claim that the universal is established from the particulars by means of induction. If this is so, they will effect it by reviewing either all the particulars or only some of them. But if they review only some, their induction will be unreliable, since it is possible that some of the particulars omitted in the induction may contradict the universal. If, on the other hand, their review is to include all the particulars, theirs will be an impossible task, because particulars are infinite and indefinite. Thus it turns out, I think, that induction, viewed from both ways, rests on a shaky foundation" (Sextus 1985), p. 105.

[11] All the definitions and results presented in this paragraph are standard. An extended presentation may be found in (Rogers 1987).

[12] Presented in (Hinman 1970), Theorem 4.3, p. 106 or (Rogers 1987) Theorem XII, p. 344.

[13] This proof follows (Hinman 1970), pp. 106–107.

[14] This proof follows (Hinman 1970), Corollary 4.5, p. 107.

[15] These definitions generalize similar notions in (Osherson et al. 1986).

[16] Gradual decidability by a computable method is equivalent to decidability in the limit by a computable method (we can convert a gradual decider α into a limiting decider by simulating α and conjecturing 1 when α produces a conjecture greater than 0.5 and conjecturing 0 otherwise). Hence, we have already obtained the bounds on $\text{PRED}_{\mathscr{H}}$ given computable gradual decidability.

[17] It can be shown that for each Bayesian conditionalizer α and for each Borel set $\mathscr{H}$ (each arithmetically definable hypothesis is a Borel set), α gradually decides $\mathscr{H}$ over some set $\mathscr{K}$ of data streams of probability 1 (in α's prior probability measure). Hence, even though no arithmetically definable α gradually decides $\{\delta\}$ over all data streams, each such method gradually decides $\{\delta\}$ measure 1. We leave it to the reader to decide whether this implies that requiring success over all data streams is too stringent (remember: a computable method can succeed in this sense concerning $\{\delta\}$!) or that requiring success with probability 1 is too lenient in infinite product spaces. Cf. (Kelly 1995), Chapter 13.

[18] This result is related to Theorem 3.7 in (Gaifman and Snir 1982).

[19] We are indebted to Aleksandar Ignjatovic for this result. It can be strengthened to the case of Turing reducibility using a similar argument based on the set $S = \{x: \exists y\ \phi^R_x(x) = 1\}$.

[20] Observe that weak conservatism could be replaced in this result with the trivial requirement that α produce a non-zero conjecture at some time later than stage 0 when $\mathscr{H}$ is true.

[21] (Pour-El and Richards 1980).

REFERENCES

Gaifman, H. and M. Snir: 1982, 'Probabilities over Rich Languages, Testing and Randomness', *Journal of Symbolic Logic* **47**, 495–548.

Gold, E. M.: 1965, 'Limiting Recursion', *Journal of Symbolic Logic* **30**, 27–48.

Hinman, P. G.: 1978, *Recursion-Theoretic Hierarchies*, New York: Springer.

Kelly, K.: 1991, 'Reichenbach, Induction, and Discovery', *Erkenntnis* **35**, 123–149.

Kelly, K.: 1993, 'Learning Theory and Descriptive Set Theory', *Journal of Logic and Computation*, **3**, 27–45.

Kelly, K.: 1995, *The Logic of Reliable Inquiry*, forthcoming, New York: Oxford University Press.

Kelly, K. and C. Glymour: 1992, 'Thoroughly Modern Meno', in J. Earman (ed.), *Inference, Explanation, and other Frustrations*, Berkeley: University of California Press, pp. 3–22.

Osherson, D., M. Stob, and S. Weinstein: 1986, *Systems that Learn*, Cambridge: MIT Press.

Peirce, C. S.: 1935, *Collected Papers of Charles Sanders Peirce*, C. Hartshorne and P. Weiss (eds).

Popper, K. (1968), *The Logic of Scientific Discovery*, New York, Harper.

Pour-El, M. and J. Richards: 1980, *Computability in Analysis and Physics*, New York, Springer.

Putnam, H.: 1963, '"Degree of Confirmation" and Inductive Logic', in A. Schilpp (ed.), *The Philosophy of Rudolph Carnap*, Lasalle, Illinois: Open Court.

Putnam, H.: 1965, 'Trial and Error Predicates and a Solution to a Problem of Mostowski', *Journal of Symbolic Logic* **30**, 49–57.

Reichenbach, H.: 1949, *The Theory of Probability*, London, Cambridge University Press.
Rogers, H.: 1987, *The Theory of Recursive Functions and Effective Computability*, Cambridge, MIT Press.
Sextus Empiricus: 1985, *Selections from the Major Writings on Scepticism, Man & God*, P. Hallie (ed.), S. Etheridge, trans. Indianapolis, Hackett.

Manuscript submitted October 18, 1993
Final version received May 9, 1994

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