II IDENTIFICATION GENERALIZED

This part is devoted to a family of learning paradigms that results from modifying the definitions proper to identification. Chapter 4 considers alternative construals of "learner" that are narrower than the class $\mathscr{F}$ of all number-theoretic functions. Chapter 5 concerns the environments in which learning takes place. Chapter 6 examines various construals of "stability" and "accuracy" in the context of alternative criteria of successful learning. Functions that learn neither too much nor too little are the topic of chapter 7.

The family of models introduced in this part may be designated *generalized identification paradigms.*

4 Strategies

4.1 Strategies as Sets of Learning Functions

To say that children implement a learning function is not to say much; a vast array of possibilities remains. Greater informativeness in this regard consists in locating human learners in proper subsets of $\mathscr{F}$.

DEFINITION 4.1A Subsets of $\mathscr{F}$ are called (*learning*) *strategies*.

Strategies can be understood as empirical hypotheses about the limitations on learning imposed by human nature. As such, the narrower a strategy, the more interesting it is as a hypothesis.

Strategies may also be conceived as alternative interpretations of the concept *learner* (see section 1.1). We leave intact for now the interpretations of *language*, *environment*, and *hypothesis* proper to the identification paradigm; similarly identification (section 1.4) is the criterion of learning relevant to the present chapter. Each strategy $\mathscr{S}$ thus constitutes a distinct learning paradigm. The identification paradigm results when $\mathscr{S} = \mathscr{F}$.

DEFINITION 4.1B Let $\mathscr{S} \subseteq \mathscr{F}$ be given.

i. The class $\{\mathscr{L} \subseteq \mathrm{RE} \mid \text{some } \varphi \in \mathscr{S} \text{ identifies } \mathscr{L}\}$ is denoted: $[\mathscr{S}]$.
ii. The class $\{\mathscr{L} \subseteq \mathrm{RE}_{\mathrm{svt}} \mid \text{some } \varphi \in \mathscr{S} \text{ identifies } \mathscr{L}\}$ is denoted: $[\mathscr{S}]_{\mathrm{svt}}$.

Thus $[\mathscr{S}]$ is the family of all collections $\mathscr{L}$ of languages such that some learning function in the strategy $\mathscr{S}$ identifies $\mathscr{L}$. $[\mathscr{S}]_{\mathrm{svt}}$ is just $[\mathscr{S}] \cap \mathscr{P}(\mathrm{RE}_{\mathrm{svt}})$, that is, the family of all collections $\mathscr{L}$ of total, single-valued languages such that some learning function in the strategy $\mathscr{S}$ identifies $\mathscr{L}$.

Example 4.1A

a. $[\mathscr{F}]$ is the family of all identifiable collections of languages. By proposition 2.2A(i), $\mathrm{RE}_{\mathrm{fin}} \cup \{N\} \notin [\mathscr{F}]$. Thus $\mathrm{RE} \notin [\mathscr{F}]$. Let $\mathscr{L}$ be any finite collection of languages. By exercise 1.4.3C, $\mathscr{L} \in [\mathscr{F}]$.
b. $[\mathscr{F}]_{\mathrm{svt}}$ is the family of all identifiable collections of total, single-valued languages. $[\mathscr{F}]_{\mathrm{svt}} = \mathscr{P}(\mathrm{RE}_{\mathrm{svt}})$ since $\mathrm{RE}_{\mathrm{svt}} \in [\mathscr{F}]_{\mathrm{svt}}$ by proposition 1.4.3C and every subset of an identifiable collection of languages is identifiable. Let h be as in the proof of proposition 1.4.3C. Then $[\mathscr{F}]_{\mathrm{svt}} = [\{h\}]$.
c. Let $f \in \mathscr{F}$ be as defined in part a of example 1.3.4B. Then $[\{f\}] = \mathscr{P}(\mathrm{RE}_{\mathrm{fin}})$.
d. The strategy $\mathscr{M} = \{\varphi \in \mathscr{F} \mid \varphi \text{ is self-monitoring}\}$ was discussed in section 1.5.2. By proposition 1.5.2A, $[\mathscr{M}] \subset [\mathscr{F}]$.

In this chapter we consider the inclusion relations between $[\mathscr{S}]$ and $[\mathscr{S}']$ as $\mathscr{S}$ and $\mathscr{S}'$ vary over learning strategies. Informally we say that $\mathscr{S}$ *restricts* $\mathscr{S}'$ just in case $[\mathscr{S} \cap \mathscr{S}'] \subset [\mathscr{S}']$. If $[\mathscr{S}] \subset [\mathscr{F}]$, then $\mathscr{S}$ is said to be *restrictive*. Similar terminology applies to $[\mathscr{S}]_{\text{svt}}$. One last notational convention will be helpful.

DEFINITION 4.1C Let P be a property of learning functions. Then the set $\{\varphi \in \mathscr{F} \mid P \text{ is true of } \varphi\}$ is denoted: $\mathscr{F}^P$.

Thus the set of recursive learning functions is denoted "$\mathscr{F}^{\text{recursive}}$," which we will continue to write as "$\mathscr{F}^{\text{rec}}$."

All the strategies to be examined may be viewed as constraints of one kind or another on the behavior of learning functions. Five kinds of constraints are considered, corresponding to the five sections that follow. Before turning to these constraints, we conclude this section with a general fact about strategies.

PROPOSITION 4.1A Let $\mathscr{S}$ be a denumerable subset of $\mathscr{F}$. Then $[\mathscr{S}] \subset [\mathscr{F}]$.

Proof For each $i \in N$ and each X a subset of N, define $L_{i,X} = \{\langle i, x\rangle \mid x \in X\}$. Now if $Q \subseteq N$, define a collection of languages $\mathscr{L}_Q$ by

$\mathscr{L}_Q = \{L_{i,N} \mid i \in Q\} \cup \{L_{i,D} \mid i \notin Q \text{ and } D \text{ finite}\}$.

Obviously for every Q, $\mathscr{L}_Q \in [\mathscr{F}]$.

Claim No $\varphi \in \mathscr{F}$ identifies both $\mathscr{L}_Q$ and $\mathscr{L}_{Q'}$ for $Q \neq Q'$.

Proof of claim Suppose that φ identifies $\mathscr{L}_Q$ and $i \in Q - Q'$. Then, since φ identifies $\mathscr{L}_Q$, φ identifies $L_{i,N}$. Let σ be a locking sequence for φ and $L_{i,N}$. Then there is a finite set D such that $\text{rng}(\sigma) \subseteq L_{i,D}$. But then σ can be extended to a text t for $L_{i,D}$. Since $L_{i,D}$ is a subset of $L_{i,N}$, φ converges to an index for $L_{i,N}$ on t. Thus φ does not identify $L_{i,D}$. Since $L_{i,D} \in \mathscr{L}_{Q'}$, φ does not identify $\mathscr{L}_{Q'}$.

It is easy to see that the claim implies the result of the proposition, since there are nondenumerably many $Q \subseteq N$ and each φ identifies at most one of the classes $\mathscr{L}_Q$. □

Exercises

4.1A Let $\mathscr{S}$ and $\mathscr{S}'$ be learning strategies such that $\mathscr{S} \subset \mathscr{S}'$.

a. Prove that $[\mathscr{S}] \subseteq [\mathscr{S}']$.
b. Show by example that $[\mathscr{S}] = [\mathscr{S}']$ is possible.

4.1B Evaluate the validity of the following claims. For learning strategies $\mathscr{S}$, $\mathscr{S}'$,

a. $[\mathscr{S} \cup \mathscr{S}'] = [\mathscr{S}] \cup [\mathscr{S}']$.
b. $[\mathscr{S} \cap \mathscr{S}'] \subseteq [\mathscr{S}] \cap [\mathscr{S}']$.
c. $[\mathscr{S}] \cap [\mathscr{S}'] \subseteq [\mathscr{S} \cap \mathscr{S}']$.
d. $[\mathscr{F} - \mathscr{S}] = [\mathscr{F}] - [\mathscr{S}]$.

4.1C Let $\varphi \in \mathscr{F}$ and $\mathscr{S} \subseteq \mathscr{F}$ be given.

a. What is the relation between $\mathscr{L}(\varphi)$ and $[\{\varphi\}]$?
b. Prove that $[\mathscr{S}] = \{\mathscr{L} \subseteq \mathscr{L}(\varphi) \mid \varphi \in \mathscr{S}\}$.

4.2 Computational Constraints

In this section we consider two attempts to specify learning strategies that approximate human computational limitations.

4.2.1 Computability

One of the most popular hypotheses in cognitive science is that human ratiocination can be simulated by computer. It is natural then to speculate that children's learning functions are effectively calculable. The corresponding strategy is $\mathscr{F}^{\text{rec}}$, the set of all partial and total recursive functions (see section 1.2.1).

Since $\mathscr{F}^{\text{rec}}$ constitutes a small fraction of $\mathscr{F}$, the computability strategy is a nontrivial hypothesis about human learners. From the fact that $\mathscr{F}^{\text{rec}} \subset \mathscr{F}$, however, we cannot immediately conclude that $\mathscr{F}^{\text{rec}}$ is restrictive (see exercise 4.1A). For this latter result it suffices to observe that $\mathscr{F}^{\text{rec}}$ is a denumerable subset of $\mathscr{F}$, from which proposition 4.1A directly yields the following.

PROPOSITION 4.2.1A $[\mathscr{F}^{\text{rec}}] \subset [\mathscr{F}]$.

It will facilitate later developments to exhibit a specific collection of languages that falls in $[\mathscr{F}] - [\mathscr{F}^{\text{rec}}]$. We proceed via a definition and three lemmata.

DEFINITION 4.2.1A The set $\{x \in N \mid \varphi_x(x)\downarrow\}$ is denoted: K.

LEMMA 4.2.1A $K \in \text{RE}$, but $\overline{K} \notin \text{RE}$.

Proof See Rogers (1967, sec. 5.2, theorem VI). □

LEMMA 4.2.1B $\{K \cup \{x\} \mid x \in N\} \in [\mathscr{F}]$.

Proof This follows from exercise 1.4.3D. □

LEMMA 4.2.1C $\{K \cup \{x\} \mid x \in N\} \notin [\mathscr{F}^{\text{rec}}]$.

Proof Suppose on the contrary, that some $\varphi \in \mathscr{F}^{\text{rec}}$ identifies $\{K \cup \{x\} \mid x \in N\}$. Fix φ, and let σ be a locking sequence for φ and K. We will show that $\overline{K}$ is r.e., contradicting lemma 4.2.1A.

Let $k_0, k_1, \ldots,$ be some fixed enumeration of K, and for every x define a text t^x for $K \cup \{x\}$ by $t^x = \sigma \wedge x \wedge k_0, k_1, \ldots$. Since σ is a locking sequence for φ and K, $\varphi(\overline{t}^x_{\text{lh}(\sigma)}) = \varphi(\sigma)$ is an index for K for every x. Now, if $x \notin K$, t^x is a text for $K \cup \{x\}$ which is not the same language as K. Thus, if $x \notin K$, there is an $n > \text{lh}(\sigma)$ such that $\varphi(\overline{t}^x_n)$ is not an index for K, and hence $\varphi(\overline{t}^x_n) \neq \varphi(\overline{t}^x_{\text{lh}(\sigma)})$. But, if $x \in K$, t^x is a text for K, and hence, since $\overline{t}^x_{\text{lh}(\sigma)}$ is a locking sequence for K, $\varphi(\overline{t}^x_n) = \varphi(\sigma)$ for all $n > \text{lh}(\sigma)$. Thus we have shown that

(*) $x \in \overline{K}$ if and only if there is $n > \text{lh}(\sigma)$ such that

$$\varphi(\overline{t}^x_n) \neq \varphi(\sigma).$$

Now it is easy to see from (*) that $\overline{K}$ is r.e. To see this, note that t^x can be constructed effectively from x and that the function

$$\psi(x) = \text{least} \quad n > \text{lh}(\sigma) \quad \text{such that} \quad \varphi(\overline{t}^x_n) \neq \varphi(\sigma)$$

is therefore partial recursive with domain $\overline{K}$. □

A fundamental result for RE_{svt} is stated in proposition 4.2.1B.

PROPOSITION 4.2.1B (Gold 1967) $\text{RE}_{\text{svt}} \notin [\mathscr{F}^{\text{rec}}]_{\text{svt}}$.

Proof (from Gold 1967) Suppose that $\varphi \in \mathscr{F}^{\text{rec}}$ identifies RE_{svt}. We will construct an $L \in \text{RE}_{\text{svt}}$ and a text t for L such that φ changes its mind infinitely often on t. This means that φ does not identify L, so the hypothesis that φ identifies RE_{svt} must be false. We will construct t in stages so that the initial segment of t constructed by the end of stage s, σ^s, is equal to $\langle 0, x_0\rangle, \langle 1, x_1\rangle, \ldots, \langle n, x_n\rangle$ for some n. (We will also have that each x_i is equal to 0 or 1.) We rely on the following claim.

Claim Given $\sigma = \langle 0, x_0\rangle, \langle 1, x_1\rangle, \ldots, \langle n, x_n\rangle$, there are numbers j and k such that if $\tau = \sigma \wedge \langle n+1, 0\rangle, \ldots, \langle n+j, 0\rangle$ and $\tau' = \tau \wedge \langle n+j+1, 1\rangle, \ldots, \langle n+j+k, 1\rangle$, then $\varphi(\tau) \neq \varphi(\tau')$.

Proof of claim The following is a text for a language $L_0 \in \text{RE}_{\text{svt}}$: $\sigma \wedge \langle n+1, 0\rangle, \ldots, \langle n+j, 0\rangle, \ldots$. Thus there is a j such that if $\tau = \sigma \wedge \langle n+1, 0\rangle, \ldots, \langle n+j, 0\rangle$, $\varphi(\tau)$ is an index for L_0. But the following is a text for another language $L_1 \in \text{RE}_{\text{svt}}$: $\tau \wedge \langle n+j+1, 1\rangle, \ldots, \langle n+j+k, 1\rangle, \ldots$. Therefore there must be a number k such that if

$\tau' = \tau \wedge \langle n+j+1, 1\rangle, \ldots, \langle n+j+k, 1\rangle$, $\varphi(\tau')$ is an index for L_1.

Since $L_0 \neq L_1$, $\varphi(\tau) \neq \varphi(\tau')$, and j and k are our desired integers.

Now we construct t in stages.

Stage 0 $\sigma^s = \langle 0, 0\rangle$.

Stage $s+1$ Given σ^s, let j and k be as in the claim using σ^s for σ. Define σ^{s+1} to be the resultant τ'.

It is clear that $t = \bigcup_s \sigma^s$ is a text for some $L \in \text{RE}_{\text{svt}}$. However, φ does not converge on t, since φ changes its value at least once for each $s \in N$. □

COROLLARY 4.2.1A $[\mathscr{F}^{\text{rec}}]_{\text{svt}} \subset [\mathscr{F}]_{\text{svt}}$.

Proof See proposition 1.4.3C. □

Exercises

4.2.1A

a. Prove that $\{K \cup \{x\} \mid x \in \overline{K}\} \in [\mathscr{F}^{\text{rec}}]$.
b. Let $L \in \text{RE}$ be recursive. Prove that $\{L \cup D \mid D \subseteq N \text{ and } D \text{ finite}\} \in [\mathscr{F}^{\text{rec}}]$.
c. Prove that $\{N\} \cup \{D \mid D \text{ finite and } D \subseteq K\} \cup \{D \mid D \text{ finite and } D \subseteq \overline{K}\} \in [\mathscr{F}^{\text{rec}}]$.
d. Prove that $\{N - \{x\} \mid x \in \overline{K}\} \cup \{N - \{x, y\} \mid x \neq y \text{ and } x, y \in K\} \in [\mathscr{F}^{\text{rec}}]$.
e. Prove that $\{N - \{x\} \mid x \in K\} \cup \{N - \{x, y\} \mid x \neq y \text{ and } x, y \in \overline{K}\} \in [\mathscr{F}^{\text{rec}}]$.

Compare exercise 2.2C.

4.2.1B For $\mathscr{L}, \mathscr{L}' \subseteq \text{RE}$, define $\mathscr{L} \times \mathscr{L}'$ as in exercise 1.4.3F. Prove that if $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$ and $\mathscr{L}' \in [\mathscr{F}^{\text{rec}}]$, then $\mathscr{L} \times \mathscr{L}' \in [\mathscr{F}^{\text{rec}}]$.

***4.2.1C** Prove: Let $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]_{\text{svt}}$. Then, there is $\varphi \in \mathscr{F}^{\text{rec}}$ such that (a) φ identifies $\mathscr{L}$, and (b) for all $L \in \mathscr{L}$, there is $i \in N$ such that for all texts t for L, φ converges on t to i. (*Hint:* Fix $\varphi' \in \mathscr{F}^{\text{rec}}$ which identifies $\mathscr{L}$. Define $\varphi \in \mathscr{F}^{\text{rec}}$ which uses φ' to compute its guesses. φ rearranges the incoming text and feeds the rearranged text to φ'.) Compare section 4.6.3.

4.2.1D Let $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]_{\text{svt}}$ be given. Show that there is $\mathscr{L}' \in [\mathscr{F}^{\text{rec}}]_{\text{svt}}$ such that $\mathscr{L} \subset \mathscr{L}'$. (Hint: Let $\varphi \in \mathscr{F}^{\text{rec}}$ identify $\mathscr{L} \subseteq \text{RE}_{\text{svt}}$. Use the proof of proposition 4.2.1A to construct $L \notin \mathscr{L}$ and $\varphi \in \mathscr{F}^{\text{rec}}$ which identifies $\mathscr{L} \cup \{L\}$.) Compare this result to exercise 2.2E.

4.2.1E Prove that $\text{RE}_{\text{sd}} \in [\mathscr{F}^{\text{rec}}]$. (For RE_{sd}, see definition 2.3B.)

4.2.1F For $\mathscr{S} \subseteq \mathscr{F}$, let $[\mathscr{S}]_{\text{rec}} = [\mathscr{S}] \cap \mathscr{P}(\text{RE}_{\text{rec}})$. Prove that $[\mathscr{F}^{\text{rec}}]_{\text{rec}} \subset [\mathscr{F}]_{\text{rec}}$. (*Hint:* Use corollary 4.2.1A and exercise 1.2.2B.)

4.2.1G Let $\text{RE}_{\text{fin}\,\bar{K}} = \{L \in \text{RE}_{\text{fin}} | L \cap \bar{K} \neq \varnothing\}$. Prove that $\{K\} \cup \text{RE}_{\text{fin}\,\bar{K}} \notin [\mathscr{F}^{\text{rec}}]$.

4.2.1H Let $\text{RE}_{\text{seg}} = \{\{0, 1, 2, \ldots, n\} | n \in N\}$. RE_{seg} thus consists of the initial segments of N. Prove:

a. Let $\mathscr{L} \in [\mathscr{F}]$ be given. Then $\mathscr{L} \cup \text{RE}_{\text{seg}} \in [\mathscr{F}]$ if and only if $N \notin \mathscr{L}$.
b. Let $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$ be given. Then $\mathscr{L} \cup \text{RE}_{\text{seg}} \in [\mathscr{F}^{\text{rec}}]$ if and only if $N \notin \mathscr{L}$.

4.2.1I Let $n \in N$ be given. A total recursive function f is called *almost everywhere n* just in case for all but finitely many $i \in N$, $f(i) = n$. Let $\mathscr{L} = \{L |$for some total recursive function f and for some $n \in N$, f is almost everywhere n and L represents $f\}$. Show that some $\varphi \in \mathscr{F}^{\text{rec}}$ identifies $\mathscr{L}$. (Compare proposition 4.5.3B.)

***4.2.1J** Prove that $\{\{\langle 0, x\rangle\} \cup \{\langle 1, y\rangle\} \cup \{\langle 2, z\rangle\} \cup \{3\} \times W_j |$at least two-thirds of $\{x, y, z\}$ are indexes for $W_j\} \in [\mathscr{F}^{\text{rec}}]$.

4.2.2 Time Bounds

Children do not effect computations of arbitrary complexity, so we are led to examine computationally limited subsets of $\mathscr{F}^{\text{rec}}$. The following definition is central to this enterprise.

DEFINITION 4.2.2A (Blum 1967) A listing $\Phi_0, \Phi_1, \ldots$, of partial recursive functions is called a *computational complexity measure* (relative to our fixed acceptable indexing of $\mathscr{F}^{\text{rec}}$) just in case it satisfies the following two conditions:

i. For all $i, x \in N$, $\varphi_i(x)\downarrow$ if and only if $\Phi_i(x)\downarrow$.
ii. The set $\{\langle i, x, y\rangle | \Phi_i(x) \leq y\}$ is recursive.

To exemplify this definition, suppose that $\mathscr{F}^{\text{rec}}$ is indexed by associated Turing machines (see section 1.2.1). Then Φ_i may be thought of as the function that counts the steps required in running the ith Turing machine; specifically, for $i, x, y \in N$, $\Phi_i(x) = y$ just in case the ith Turing machine halts in exactly y steps when started with input x. Condition i of the definition

requires that $\Phi_i(x)$ be undefined just in case the ith Turing machine never halts on x. Condition ii requires that it be possible to determine effectively whether the ith Turing machine halts on x within y steps. Both requirements are satisfied by the suggested interpretation of Φ_i. Moreover it appears that any reasonable measure of the resources required for a computation must also conform to these conditions.

As with acceptable indexings, none of our results depend on the choice of computational complexity measure. Indeed, any two computational complexity measures can be shown, in a satisfying sense, to yield similar estimates of the resources required for a computation (see Machtey and Young 1978, theorem 5.2.4). Let a fixed computational complexity measure now be selected; reference to the functions $\Phi_0, \Phi_1, \ldots$, should henceforth be understood accordingly.

These preliminaries allow us to define the following class of strategies.

DEFINITION 4.2.2B Let $h \in \mathscr{F}^{\text{rec}}$ be total. $\psi \in \mathscr{F}^{\text{rec}}$ is said to *run in h-time* just in case ψ is total and there is $i \in N$ such that (i) $\varphi_i = \psi$, and (ii) $\Phi_i(x) \leq h(x)$ for all but finitely many $x \in N$. The subset of $\mathscr{F}^{\text{rec}}$ that runs in h-time is denoted $\mathscr{F}^{h\text{-time}}$.

Note that for any total $h \in \mathscr{F}^{\text{rec}}$, $\mathscr{F}^{h\text{-time}}$ consists exclusively of total recursive functions.

Intuitively a learning function in $\mathscr{F}^{h\text{-time}}$ can be programmed to respond to finite sequences σ within $h(\sigma)$ steps of operation (recall from section 1.3.4 that "σ" in "$h(\sigma)$" denotes the number that codes σ). The strategy $\mathscr{F}^{h\text{-time}}$ corresponds to the hypothesis that children deploy limited resources in formulating grammars on the basis of finite corpora. The limitation is given by h.

Does $\mathscr{F}^{h\text{-time}}$ restrict $\mathscr{F}^{\text{rec}}$ regardless of the choice of total recursive function h? The following result suggests an affirmative answer.

LEMMA 4.2.2A (Blum 1967a) For every total $h \in \mathscr{F}^{\text{rec}}$ there is recursive $L \in \text{RE}$ such that no characteristic function for L runs in h-time.

Proof See Machtey and Young (1978, proposition 5.2.9). □

Contrary to expectation, however, the next proposition shows that for some total $h \in \mathscr{F}^{\text{rec}}$, $\mathscr{F}^{h\text{-time}}$ does not restrict $\mathscr{F}^{\text{rec}}$.

PROPOSITION 4.2.2A There is total $h \in \mathscr{F}^{\text{rec}}$ such that $[\mathscr{F}^{h\text{-time}}] = [\mathscr{F}^{\text{rec}}]$.

The proof of proposition 4.2.2A will be facilitated by a lemma and a definition. The lemma is also of independent interest.

LEMMA 4.2.2B There is total $f \in \mathscr{F}^{\text{rec}}$ such that for all $i \in N$ (i) $\varphi_{f(i)}$ is total recursive, and (ii) for all $L \in \text{RE}$, if φ_i identifies L, then $\varphi_{f(i)}$ identifies L.

Proof of the lemma Given i, we would like to define $\varphi_{f(i)}$ so that $\varphi_{f(i)}$ identifies at least as many languages as φ_i but $\varphi_{f(i)}$ is total. Thus we would like $\varphi_{f(i)}(\sigma)$, to simulate $\varphi_i(\sigma)$ but not to wait forever if $\varphi_i(\sigma)$ doesn't converge. Therefore on input σ we will only allow $\varphi_{f(i)}$ to wait $\text{lh}(\sigma)$ many steps for φ_i to converge. Now $\varphi_i(\sigma)$ may not converge in $\text{lh}(\sigma)$ many steps for any σ but, if $\varphi_i(\sigma)$ converges, there is a k such that $\varphi_i(\sigma)$ converges in k steps. Thus, in defining $\varphi_{f(i)}(\sigma)$, we will allow the simulation of φ_i to "fall back on the text", that is, to compute only $\varphi_i(\hat{\sigma})$ for some initial segment $\hat{\sigma}$ of σ. Precisely, define

$$\varphi_{f(i)}(\sigma) = \begin{cases} \varphi_i(\hat{\sigma}), & \text{where } \hat{\sigma} \text{ is the longest initial segment of } \sigma \text{ such that} \\ & \Phi_i(\hat{\sigma}) \leq \text{lh}(\sigma) \text{ if such exists,} \\ 0, & \text{otherwise.} \end{cases}$$

$\varphi_{f(i)}$ is a total recursive function for every i. The condition defining $\hat{\sigma}$ can be checked recursively, since we have bounded the waiting time by $\text{lh}(\sigma)$. To see that $\varphi_{f(i)}$ identifies any language L that φ_i identifies, let t be a text for such an L. Then there is an $n \in N$ and an index j for L such that for all $m \geq n$, $\varphi_i(\bar{t}_m) = j$. Let $s = \Phi_i(\bar{t}_n)$. Then by the definition of $\varphi_{f(i)}$, if $m > s, n$, $\varphi_{f(i)}(\bar{t}_m) = \varphi_i(\bar{t}_k)$ for some $k \geq n$. Thus $\varphi_{f(i)}$ converges on t to j. □

Proof of proposition 4.2.2A Let f be as in the statement of lemma 4.2.2B. Define

$$h(x) = \max\{\Phi_{f(i)}(j) | i, j \leq x\}.$$

h is a total recursive function, since each function $\Phi_{f(i)}$ is total.

Now suppose that $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$. Let $\varphi_i \in \mathscr{F}^{\text{rec}}$ identify $\mathscr{L}$. Then by the lemma, $\varphi_{f(i)}$ identifies $\mathscr{L}$. But by the definition of h, for all $j \geq i$, $\Phi_{f(i)}(j) \leq h(j)$. Thus $\varphi_{f(i)}$ runs in h-time. This implies that $\mathscr{L} \in [\mathscr{F}^{h\text{-time}}]$. □

Exercise

4.2.2A Let $h \in \mathscr{F}^{\text{rec}}$ be total. Let $\mathscr{L}_h = \{L \in \text{RE}_{\text{svt}} | L$ represents a function in $\mathscr{F}^{h\text{-time}}\}$. Show that for some total $g \in \mathscr{F}^{\text{rec}}$, $\mathscr{L}_h \in [\mathscr{F}^{g\text{-time}}]$.

4.2.3 On the Interest of Nonrecursive Learning Functions

Why study strategies that are not subsets of $\mathscr{F}^{\text{rec}}$? For those convinced that human intellectual capacities are computer simulable, nonrecursive learning functions might seem to be of scant empirical interest. Many of the strategies we consider are in fact subsets of $\mathscr{F}^{\text{rec}}$.

Nonrecursive learning functions will continue, however, to figure prominently in our discussion. The reason for this is not simply the lack of persuasive argumentation in favor of the view that human mentality is machine simulable. More important, consideration of nonrecursive learning functions often clarifies the respective roles of computational and information-theoretic factors in nonlearnability phenomena. To see what is at issue, compare the collections $\mathscr{L} = \{N\} \cup \text{RE}_{\text{fin}}$ and $\mathscr{L}' = \{K \cup \{x\} | x \in N\}$. By proposition 2.2A(i) and lemma 4.2.1B, respectively, no $\varphi \in \mathscr{F}^{\text{rec}}$ identifies either collection. However, the reasons for the unidentifiability differ in the two cases. On the one hand, $\mathscr{L}'$ presents a recursive learning function with an insurmountable computational problem, whereas the computational structure of $\mathscr{L}$ is trivial. On the other hand, $\mathscr{L}$ presents the learner with an insurmountable informational problem—that is, no $\sigma \in \text{SEQ}$ allows the finite and infinite cases to be distinguished (cf. proposition 2.4A). In contrast, no such informational problem exists for $\mathscr{L}'$; the available information simply cannot be put to use by a recursive learning function.

The results to be presented concerning nonrecursive learning functions may all be interpreted from this information-theoretic point of view.

4.3 Constraints on Potential Conjectures

Let $\mathscr{L} \subseteq \text{RE}$ be identifiable, and let $\sigma \in \text{SEQ}$ and $i \in N$ be given. From exercise 1.5.1A we see that some $\varphi \in \mathscr{F}$ such that $\varphi(\sigma) = i$ identifies $\mathscr{L}$. Put differently, from the premise that $\varphi \in \mathscr{F}$ identifies $\mathscr{L} \subseteq \text{RE}$, no information may be deduced about $\varphi(\sigma)$ for any $\sigma \in \text{SEQ}$, except that $\varphi(\sigma)\downarrow$ if σ is drawn from a language in $\mathscr{L}$. In this section we consider the effects on identification of constraining in various ways the learner's potential response to evidential states.

4.3.1 Totality

The most elementary constraint on a conjecture is that it exist. The corresponding strategy is the set of total learning functions, denoted $\mathscr{F}^{\text{total}}$. From part a of exercise 1.4.3G we have proposition 4.3.1A.

PROPOSITION 4.3.1A $[\mathscr{F}^{\text{total}}] = [\mathscr{F}]$.

Similarly directly from lemma 4.2.2B we obtain proposition 4.3.1B.

PROPOSITION 4.3.1B $[\mathscr{F}^{\text{rec}}] = [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{total}}]$.

Thus totality restricts neither $\mathscr{F}$ nor $\mathscr{F}^{\text{rec}}$.

4.3.2 Nontriviality

Linguists rightly emphasize the infinite quality of natural languages. No natural language, it appears, includes a longest sentence. If this universal feature of natural language corresponds to an innate constraint on children's linguistic hypotheses, then children would be barred from conjecturing a grammar for a finite language. Such a constraint on potential conjectures amounts to a strategy.

DEFINITION 4.3.2A $\varphi \in \mathscr{F}$ is called *nontrivial* just in case for all $\sigma \in \text{SEQ}$, $W_{\varphi(\sigma)}$ is infinite.

Thus the strategy of nontriviality contains just those $\varphi \in \mathscr{F}$ such that φ never conjectures an index for a finite language. Note that nontrivial learners are total. The learning function g defined in the proof of proposition 1.4.3B is nontrivial.

Obviously nontriviality is restrictive: finite languages cannot be identified without conjecturing indexes for them. Of more interest is the relation of nontriviality to the identification of infinite languages. The next proposition shows that nontriviality imposes limits on the recursive learning functions in this respect; that is, some collections of infinite languages are identifiable by recursive learning function but not by nontrivial, recursive learning function.

PROPOSITION 4.3.2A There is $\mathscr{L} \subseteq \text{RE}$ such that (i) every $L \in \mathscr{L}$ is infinite, and (ii) $\mathscr{L} \in [\mathscr{F}^{\text{rec}}] - [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{nontrivial}}]$.

To prove the proposition, a definition and lemma are helpful.

DEFINITION 4.3.2B $\mathscr{L} \subseteq \text{RE}$ is said to be *r.e. indexable* just in case there is $S \in \text{RE}$ such that $\mathscr{L} = \{W_i | i \in S\}$; in this case S is said to be an *r.e. index set for* $\mathscr{L}$.

Thus $\mathscr{L} \subseteq \text{RE}$ is r.e. indexable just in case there is an r.e. set S such that for all $L \in \text{RE}$, $L \in \mathscr{L}$ if and only if $L = W_i$ for some $i \in S$ (S is not required to contain every index for L).

LEMMA 4.3.2A $\text{RE} - \text{RE}_{\text{fin}}$ is not r.e. indexable.

Proof of the lemma Let S be an r.e. set of indexes for infinite sets. Let e_0, $e_1, \ldots,$ be a recursive enumeration of S. We show how to enumerate an infinite r.e. set A such that no index for A is in S. We enumerate A in stages.

Stage 0: Enumerate W_{e_0} until an x_0 appears in W_{e_0}. Enumerate $0, 1, \ldots, x_0 - 1$ into A.
Stage $s + 1$: Enumerate $W_{e_{s+1}}$ until an x_{s+1} appears in $W_{e_{s+1}}$ with $x_{s+1} > x_s + 1$. Such an x_{s+1} exists since $W_{e_{s+1}}$ is infinite. Enumerate $x_s + 1, \ldots, x_{s+1} - 1$ into A.

A is infinite, since at least one integer, $x_s + 1$, is enumerated in A at each stage $s + 1$. $A \neq W_{e_s}$ for each s, since $x_s \in W_{e_s}$ but $x_s \notin A$. □

Proof of proposition 4.3.2A Recall that we have fixed a recursive isomorphism between N^2 and N, the image of a pair (x, y) being denoted by $\langle x, y \rangle$. Recall also that π_1 and π_2 are the recursive component functions defined by $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$ (see section 1.2.1).

Define for each $i \in N$, $L_i = \{\langle i, x \rangle | x \in W_i\}$. Let $\mathscr{L} = \{L_i | W_i \text{ is infinite}\}$. Obviously every language in $\mathscr{L}$ is infinite. To show that $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$, define $h(\sigma) = \pi_1(\sigma_0)$ for every $\sigma \in \text{SEQ}$, and choose $f \in \mathscr{F}^{\text{rec}}$ such that for all $i \in N$, $f(i)$ is an index for $\{i\} \times W_i$. Then $f \circ h$ identifies $\mathscr{L}$; indeed, $f \circ h$ identifies L_i for every $i \in N$.

Suppose, however, that $\varphi \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{nontrivial}}$. We show that φ does not identify $\mathscr{L}$. Let

$S = \{i | \text{there is a sequence } \sigma \text{ such that } \varphi(\sigma) = i\}$.

For any recursive function φ, S defined in this way is r.e. Since φ is nontrivial, S contains only indexes for infinite sets.

Claim There is a recursive function g such that for every $i \in N$,

1. W_i infinite implies $W_{g(i)}$ infinite,
2. $W_i \in \mathscr{L}$ implies $W_{g(i)} = \{\pi_2(\langle x, y \rangle) | \langle x, y \rangle \in W_i\}$.

Proof of claim Given i, define $W_{g(i)}$ by

$$W_{g(i)} = \begin{cases} \{\pi_2(\langle x, y \rangle) | \langle x, y \rangle \in W_i\}, & \text{if } \langle x, y \rangle \in W_i \text{ and } \langle x', y' \rangle \in W_i \\ & \text{implies } x = x', \\ N, & \text{otherwise.} \end{cases}$$

Informally we enumerate in $W_{g(i)}$ the second components of elements of W_i until we have seen two elements with different first components. In this case we then switch to enumerating every integer in $W_{g(i)}$. The function g obviously has properties 1 and 2.

Now given the claim, we complete the proof of the proposition as follows. Suppose for a contradiction that φ identifies $\mathscr{L}$. Let $g(S) = \{g(i) | i \in S\}$. Since φ is nontrivial, property 1 of g implies that $g(S)$ contains only indexes for infinite sets. Since φ identifies $\mathscr{L}$, and since for every infinite W_i, $L_i \in \mathscr{L}$, for each such W_i there is a $j \in S$ such that $W_j = L_i$. Then by (2) of the claim, $g(j)$ is an index for W_i. Thus $g(S)$ is an r.e. set containing indexes for all and only the infinite r.e. sets contradicting lemma 4.3.2A. □

In section 4.3.5, proposition 4.3.2A will be exhibited as a corollary of a more general result.

Exercises

4.3.2A Let $\mathscr{L}$ be a collection of infinite languages. Prove that $\mathscr{L}$ is identifiable if and only if some nontrivial $\varphi \in \mathscr{F}$ identifies $\mathscr{L}$. Compare this result to proposition 4.3.2A.

4.3.2B Let $\mathscr{S} \subseteq \mathscr{F}$ be such that some $\mathscr{L} \in [\mathscr{S}]$ is infinite. Show that not every $\mathscr{L}' \in [\mathscr{S}]$ is r.e. indexable.

***4.3.2C** Let $\mathscr{L}$ be as defined in the proof of proposition 4.3.2A. The function $f \circ h$ defined therein is such that $\mathscr{L} \subset \mathscr{L}(f \circ h)$. Show that there is $\varphi \in \mathscr{F}^{\text{rec}}$ such that $\mathscr{L} = \mathscr{L}(\varphi)$.

***4.3.2D** $\varphi \in \mathscr{F}$ is called *nonexcessive* just in case for all $\sigma \in \text{SEQ}$, $W_{\varphi(\sigma)} \neq N$. Prove: For all $\mathscr{L} \subseteq \text{RE}$, if $N \notin \mathscr{L}$, then $\mathscr{L} \in [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{nonexcessive}}]$ if and only if $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$.

4.3.2E (John Canny) $\varphi \in \mathscr{F}$ is said to be *weakly nontrivial* just in case for all infinite $L \in \mathscr{L}(\varphi)$, $W_{\varphi(\bar{t}_n)}$ is infinite for all $n \in N$ and all texts t for L. Nontriviality implies weak nontriviality. Show that for some collection $\mathscr{L} \subseteq \text{RE}$ of infinite languages, $\mathscr{L} \in [\mathscr{F}^{\text{rec}}] - [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{weakly nontrivial}}]$.

4.3.3 Consistency

We next consider a natural constraint on conjectures.

DEFINITION 4.3.3A (Angluin 1980) $\varphi \in \mathscr{F}$ is said to be *consistent* just in case for all $\sigma \in \text{SEQ}$, $\text{rng}(\sigma) \subseteq W_{\varphi(\sigma)}$.

That is, the conjectures of a consistent learner always generate the data seen so far. Note that consistent learning functions are total.

Example 4.3.3A

a. The function f defined in part a of example 1.3.4B is consistent. Hence $\text{RE}_{\text{fin}} \in [\mathscr{F}^{\text{consistent}}]$.
b. The function g defined in the proof of proposition 1.4.3B is consistent.
c. The function h defined in part c of example 1.3.4B is consistent.
d. The function f defined in the proof of proposition 2.3A is not consistent. To see this, let i_0 be an index for $\varnothing$, and let $\sigma \in \text{SEQ}$ be such that $i_0 \in \text{rng}(\sigma)$ and i_0 is least in $\text{rng}(\sigma)$. Then $f(\sigma) = i_0$. Since $\text{rng}(\sigma) \not\subseteq \varnothing = W_{i_0}$, f is not consistent.

Consistency has the ring of rationality: Why emit a conjecture that is falsified by the data in hand? It thus comes as no surprise that consistency is not restrictive. The proof of this fact resembles the solution to exercise 4.3.2A. We now demonstrate the less evident fact that consistency restricts $\mathscr{F}^{\text{rec}}$.

PROPOSITION 4.3.3A Let consistent $\varphi \in \mathscr{F}^{\text{rec}}$ identify $\mathscr{L} \subseteq \text{RE}$. Then $\mathscr{L} \subseteq \text{RE}_{\text{rec}}$.

Proof Let $L \in \mathscr{L}$. By the locking sequence lemma, proposition 2.1A, there is a sequence σ such that $\text{rng}(\sigma) \subseteq L$, $W_{\varphi(\sigma)} = L$, and if $\tau \in \text{SEQ}$ is such that $\text{rng}(\tau) \subseteq L$, then $\varphi(\sigma \wedge \tau) = \varphi(\sigma)$.

If $x \in L$, $\varphi(\sigma \wedge x) = \varphi(\sigma)$, since σ is a locking sequence for L. On the other hand, if $x \notin L$, $\varphi(\sigma \wedge x)$ is not an index for L, since φ is consistent; hence $\varphi(\sigma \wedge x) \neq \varphi(\sigma)$. Thus $x \in L$ if and only if $\varphi(\sigma \wedge x) = \varphi(\sigma)$. This constitutes an effective test for membership in L, since φ is total. □

There are certainly $\mathscr{L} \subseteq \text{RE}$ such that (1) $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$, and (2) $\mathscr{L}$ includes nonrecursive languages. One such collection is $\{K\}$! Hence proposition 4.3.3A yields the following corollary.

COROLLARY 4.3.3A $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{consistent}}] \subset [\mathscr{F}^{\text{rec}}]$.

Proposition 4.3.3A suggests the following question. If attention is limited to the recursive languages, does consistency still restrict $\mathscr{F}^{\text{rec}}$? The next proposition provides an affirmative answer.

Proposition 4.3.3B There is $\mathscr{L} \subseteq \mathrm{RE}_{\mathrm{rec}}$ such that $\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}}] - [\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\mathrm{consistent}}]$.

The proof of proposition 4.3.3B uses the following lemma, which is interesting in its own right.

Lemma 4.3.3A Let $h(j,k)$ be a total recursive function, and let functions $f_j \in \mathscr{F}^{\mathrm{rec}}$ be defined by $f_j(k) = h(j,k)$ for all k. Then there is a recursive set S such that f_j is not the characteristic function of S for any j.

Proof Define S by $k \in S$ if and only if $h(k,k) \neq 0$. Obviously S is recursive. No f_j is the characteristic function of S, since $j \in S$ if and only if $h(j,j) \neq 0$ if and only if $f_j(j) \neq 0$. □

Recall the definition of r.e. indexable (definition 4.3.2B). Although the recursive sets are r.e. indexable (exercise 4.3.3D), lemma 4.3.3A says that the recursive sets are not r.e. indexable as recursive sets. In other words, there is no r.e. set of indexes of characteristic functions containing at least one index for a characteristic function of each recursive set.

Proof of proposition 4.3.3B As in the proof of proposition 4.3.2A, define $L_i = \{\langle i,x \rangle | x \in W_i\}$, and let $\mathscr{L} = \{L_i | W_i \text{ is recursive}\}$. $\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}}]$; in fact, as noted in the proof of proposition 4.3.2A, $\{L_i | i \in N\} \in [\mathscr{F}^{\mathrm{rec}}]$.

Suppose, however, that $g \in \mathscr{F}^{\mathrm{rec}}$ is a consistent function that identifies $\mathscr{L}$. Define a function h as follows:

$$h(\langle \sigma, i \rangle, k) = \begin{cases} 0, & \text{if } g(\sigma \wedge \langle i,k \rangle) = g(\sigma), \\ 1, & \text{otherwise.} \end{cases}$$

It is obvious that h is a total recursive function, since g must be total. Thus h satisfies the hypothesis of lemma 4.3.3A, so there is a recursive set S such that no funtion $f_{\langle \sigma, i \rangle}(k) = h(\langle \sigma, i \rangle, k)$ is a characteristic function of S. But let i' be an index for S, and let σ' be a locking sequence for $L_{i'}$ and g. Then $k \in W_{i'}$ implies that $g(\sigma' \wedge \langle i',k \rangle) = g(\sigma')$ which implies that $h(\langle \sigma', i' \rangle, k) = 0$. And if $k \notin W_{i'}$, then $g(\sigma' \wedge \langle i',k \rangle) \neq g(\sigma')$, since g is consistent so that $h(\langle \sigma', i' \rangle, k) = 1$. But this implies that $f_{\langle \sigma', i' \rangle}(k) = h(\langle \sigma', i' \rangle, k)$ is the characteristic function of S, contradicting the choice of S. □

Proposition 4.3.3B may be strengthened to the following fact about total, single-valued languages.

Proposition 4.3.3C (Wiehagen 1976) $[\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\mathrm{consistent}}]_{\mathrm{svt}} \subset [\mathscr{F}^{\mathrm{rec}}]_{\mathrm{svt}}$.

Proof See exercise 4.3.3B. □

We note that children are not thought to be consistent learners because their early grammars do not appear to generate the sentences addressed to them.

Exercises

4.3.3A $\varphi \in \mathscr{F}$ is said to be *conditionally consistent* just in case for all $\sigma \in \mathrm{SEQ}$, if $\varphi(\sigma)\downarrow$, then $\mathrm{rng}(\sigma) \subseteq W_{\varphi(\sigma)}$.

a. Refute the following variant of proposition 4.3.3A: Let conditionally consistent $\varphi \in \mathscr{F}^{\mathrm{rec}}$ identify $\mathscr{L} \subseteq \mathrm{RE}$. Then $\mathscr{L} \subseteq \mathrm{RE}_{\mathrm{rec}}$.
b. Prove the following variant of corollary 4.3.3A: $[\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\mathrm{conditionally\ consistent}}] \subset [\mathscr{F}^{\mathrm{rec}}]$.
c. Prove the following variant of proposition 4.3.3B: there is $\mathscr{L} \subseteq \mathrm{RE}_{\mathrm{rec}}$ such that $\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}}] - [\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\mathrm{conditionally\ consistent}}]$. (*Hint:* Add N to the collection $\mathscr{L}$ defined in the proof of proposition 4.3.3B.)

4.3.3B Prove proposition 4.3.3C using the proof of proposition 4.3.3B as a model.

4.3.3C Let $\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\mathrm{consistent}}]_{\mathrm{svt}}$ be given. Show that for any $L \in \mathrm{RE}_{\mathrm{svt}}$, $\mathscr{L} \cup \{L\} \in [\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\mathrm{consistent}}]_{\mathrm{svt}}$.

***4.3.3D** Show that $\mathrm{RE}_{\mathrm{rec}}$ is r.e. indexable. (*Hint:* See Rogers 1967, exercise 5–6, p. 73.)

4.3.3E (Ehud Shapiro 1981) Let $\mathscr{L} \subseteq \mathrm{RE}$ and total $h \in \mathscr{F}^{\mathrm{rec}}$ be given, and suppose that $\mathscr{L} \in [\mathscr{F}^{h\text{-time}} \cap \mathscr{F}^{\mathrm{consistent}}]$. Show that there is a total $g \in \mathscr{F}^{\mathrm{rec}}$ such that for all $L \in \mathscr{L}$, some characteristic function for L runs in g-time.

weaker than Shapiro! no bound on learner assumed

4.3.4 Prudence and r.e. Boundedness

Suppose that $\varphi \in \mathscr{F}$ is defined on $\sigma \in \mathrm{SEQ}$. Call $\varphi(\sigma)$ a "wild guess" (with respect to φ) if φ does not identify $W_{\varphi(\sigma)}$. In this section we consider learning functions that do not make wild guesses.

Definition 4.3.4A $\varphi \in \mathscr{F}$ is called *prudent* just in case for all $\sigma \in \mathrm{SEQ}$, if $\varphi(\sigma)\downarrow$ then φ identifies $W_{\varphi(\sigma)}$.

In other words, prudent learners only conjecture grammars for languages they are prepared to learn. The function f defined in part a of example 1.3.4B and the function g defined in proposition 1.4.3B are prudent.

Children acquiring language may well be prudent learners, especially if "prestorage" models of linguistic development are correct. A prestorage model posits an internal list of candidate grammars that coincides exactly with the natural languages. Language acquisition amounts to the selection of a grammar from this list in response to linguistic input. Such a prestorage learner is prudent inasmuch as his or her hypotheses are limited to grammars from the list, that is, to grammars corresponding to natural (i.e., learnable) languages. In particular, note that the prudence hypothesis implies that every incorrect grammar projected by the child in the course of language acquisition corresponds to a natural language.

It is easy to show that prudence is not restrictive. The effect of prudence on the recursive learning functions is a more difficult matter. We begin by considering an issue of a superficially different character.

The "complexity" of a learning strategy $\mathscr{S}$ can be reckoned in alternative ways, but one natural, bipartite classification may be described as follows. From exercise 4.3.2B we know that if some $\mathscr{L} \in [\mathscr{S}]$ is infinite, then not every member of $[\mathscr{S}]$ is r.e. indexable. However, even in this case it remains possible that every collection in $[\mathscr{S}]$ can be extended to an r.e. indexable collection of languages that is also in $[\mathscr{S}]$. The next definition provides a name for strategies with this property.

DEFINITION 4.3.4B $\mathscr{S} \subseteq \mathscr{F}$ is called *r.e. bounded* just in case for every $\mathscr{L} \in [\mathscr{S}]$ there is $\mathscr{L}' \in [\mathscr{S}]$ such that (i) $\mathscr{L} \subseteq \mathscr{L}'$, and (ii) $\mathscr{L}'$ is r.e. indexable.

Thus r.e. bounded strategies give rise to simple collections of languages in a satisfying sense.

We now return to the effect of prudence on $\mathscr{F}^{\text{rec}}$.

PROPOSITION 4.3.4A (Mark Fulk) $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{prudent}}] = [\mathscr{F}^{\text{rec}}]$.

Proposition 4.3.4A is a consequence of the following two lemmata, whose proofs are deferred to section 4.6.3.

LEMMA 4.3.4A If $\mathscr{F}^{\text{rec}}$ is r.e. bounded, then $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{prudent}}] = [\mathscr{F}^{\text{rec}}]$.

LEMMA 4.3.4B (Mark Fulk) $\mathscr{F}^{\text{rec}}$ is r.e. bounded.

Exercises

4.3.4A Show that the function f defined in the proof of proposition 2.3A is not prudent.

***4.3.4B** Specify prudent $\varphi \in \mathscr{F}^{\text{rec}}$ that identifies $\{K \cup \{x\} \mid x \in \overline{K}\}$.

4.3.4C Exhibit $\mathscr{S} \subseteq \mathscr{F}$ such that (a) $\mathscr{S}$ is infinite, and (b) $\mathscr{S}$ is not r.e. bounded.

4.3.4D Show that for every $\varphi \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{prudent}}$, $\mathscr{L}(\varphi)$ is r.e. indexable. Conclude that $\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{prudent}}$ is r.e. bounded.

4.3.4E Let $\mathscr{S}$ and $\mathscr{S}'$ be r.e. bounded strategies.

a. Show that $\mathscr{S} \cup \mathscr{S}'$ is r.e. bounded.
b. Show by counterexample that $\mathscr{S} \cap \mathscr{S}'$ need not be r.e. bounded.

4.3.5 Accountability

Proper scientific practice requires the testability of proposed hypotheses. In the current context this demand may be formulated in terms of the "accountability" of scientists, as suggested by the following definition.

DEFINITION 4.3.5A $\varphi \in \mathscr{F}$ is *accountable* just in case for all $\sigma \in \text{SEQ}$, $W_{\varphi(\sigma)} - \text{rng}(\sigma) \neq \varnothing$.

Thus the hypotheses of accountable learners are always subject to further confirmation.

It is easy to see that finite languages cannot be identified by accountable learners. Similarly for $\mathscr{L} \subseteq \overline{\text{RE}_{\text{fin}}}$ it is obvious that $\mathscr{L} \in [\mathscr{F}^{\text{accountable}}]$ if and only if $\mathscr{L} \in [\mathscr{F}]$. In contrast, the following proposition reveals that the interaction of $\mathscr{F}^{\text{accountable}}$ and $\mathscr{F}^{\text{rec}}$ is less intuitive.

PROPOSITION 4.3.5A There is $\mathscr{L} \subseteq \text{RE}$ such that (i) every $L \in \mathscr{L}$ is infinite, and (ii) $\mathscr{L} \in [\mathscr{F}^{\text{rec}}] - [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{accountable}}]$.

Thus the identification by machine of certain collections of infinite languages requires the occasional conjecture of hypotheses that go no further than the data at hand. The proof of the proposition relies on the following definition and lemma.

DEFINITION 4.3.5B

i. The set $\{f \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{total}} \mid \varphi_{f(0)} = f\}$ is denoted *SD*.
ii. $\text{RE}_{SD} = \{L \in \text{RE}_{\text{svt}} \mid \text{for some } f \in SD,\ L \text{ represents } f\}$.

LEMMA 4.3.5A $\text{RE}_{SD} \notin [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{accountable}}]$.

Proof Suppose $\theta \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{accountable}}$. We define, uniformly in i, a text t^i for a language $L_i \in \text{RE}_{\text{svt}}$ which θ fails to identify. An application of the recursion theorem will then suffice to yield an $L \in \mathrm{RE}_{SD}$ that θ fails to identify.

Construction of t^i We construct t^i in stages. Let k be an index for θ.

Stage 0 $\sigma^0 = \langle 0, i \rangle$.
Stage $n+1$ Let $\langle m, s \rangle$ be the least number such that $m \in W_{\theta(\sigma^n)} - \mathrm{rng}(\sigma^n)$ and $\Phi_k(\sigma^n) < s$. Such a number exists since $\theta \in \mathscr{F}^{\text{accountable}}$.
If $\pi_1(m) < \mathrm{lh}(\sigma^n)$, let $\sigma^{n+1} = \sigma^n \wedge \langle \mathrm{lh}(\sigma^n), 0 \rangle$.
If $\pi_1(m) \geq \mathrm{lh}(\sigma^n)$, let $\sigma^{n+1} = \sigma^n \wedge \langle \langle \mathrm{lh}(\sigma^n), 0 \rangle, \ldots, \langle \pi_1(m), 1 \dot{-} \pi_2(m) \rangle \rangle$.
($n \dot{-} m = \max\{0, n-m\}$.) Let $t^i = \bigcup_n \sigma^n$.

Let $L_i = \mathrm{rng}(t^i)$. It is clear that $L_i \in \mathrm{RE}_{\mathrm{svt}}$ and that θ fails to identify t^i, since for each n either $W_{\theta(\sigma^n)} \notin \mathrm{RE}_{\mathrm{svt}}$ or $W_{\theta(\sigma^n)} \not\supseteq \mathrm{rng}(\sigma^{n+1})$. Now let g be a total recursive function such that $W_{g(i)} = L_i$. By the recursion theorem, pick a j such that $W_{g(j)} = W_j$. Then, L_j is an element of RE_{SD}. □

It is plain that $\mathrm{RE}_{SD} \subseteq \mathrm{RE}_{\mathrm{svt}} \subseteq \overline{\mathrm{RE}_{\mathrm{fin}}}$ and that $\mathrm{RE}_{SD} \in [\mathscr{F}^{\mathrm{rec}}]$. Proposition 4.3.5A thus follows immediately from the lemma. It may be seen similarly that proposition 4.3.2A is a direct corollary of proposition 4.3.5A since $\mathscr{F}^{\text{nontrivial}} \subset \mathscr{F}^{\text{accountable}}$.

An analog of nontriviality relevant to $\mathrm{RE}_{\mathrm{svt}}$ may be defined as follows.

Definition 4.3.5C (Case and Ngo-Manguelle 1979) $\varphi \in \mathscr{F}$ is called *Popperian* just in case for all $\sigma \in \mathrm{SEQ}$, if $\varphi(\sigma)\downarrow$ then $W_{\varphi(\sigma)} \in \mathrm{RE}_{\mathrm{svt}}$.

Thus the conjectures of a Popperian learning function are limited to indexes for total, single-valued languages. The function h in the proof of proposition 1.4.3C is Popperian.

An index for a member S of $\mathrm{RE}_{\mathrm{svt}}$ can be mechanically converted into an index for the characteristic function of S (see part b of exercise 1.2.2B). As a consequence it is easy to test the accuracy of such an index against the data provided by a finite sequence. Such testability motivates the terminology "Popperian" since Popper (e.g., 1972) has long insisted on this aspect of scientific practice (for discussion, see Case and Ngo-Manguelle 1979).

Plainly, in the context of $\mathrm{RE}_{\mathrm{svt}}$, $\mathscr{F}^{\text{Popperian}}$ is not restrictive. In contrast, since $\mathscr{F}^{\text{Popperian}} \subset \mathscr{F}^{\text{accountable}}$, lemma 4.3.5A implies the following.

Proposition 4.3.5B (Case and Ngo-Manguelle 1979) $[\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\text{Popperian}}]_{\mathrm{svt}} \subset [\mathscr{F}^{\mathrm{rec}}]_{\mathrm{svt}}$.

Exercises

***4.3.5A** $L \in \mathrm{RE}$ is called *total* just in case for all $x \in N$ there is $y \in N$ such that $\langle x, y \rangle \in L$ (compare definition 1.2.2D). Note that a total language need not represent a function (since it need not be single valued). $\varphi \in \mathscr{F}$ is called *total minded* just in case for all $\sigma \in \mathrm{SEQ}$, if $\varphi(\sigma)\downarrow$ then $W_{\varphi(\sigma)}$ is total. Prove: There is $\mathscr{L} \subseteq \mathrm{RE}$ such that (a) every $L \in \mathscr{L}$ is total, and (b) $\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}}] - [\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\text{total-minded}}]$. (*Hint:* Rely on Rogers 1967, theorem 5-XVI: the single-valuedness theorem.)

4.3.5B (Putnam 1975) Supply a short proof that $\mathrm{RE}_{\mathrm{svt}} \notin [\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\text{Popperian}}]$.

4.3.5C Define: $L \in \mathrm{RE}_{\mathrm{char}}$ just in case $L \in \mathrm{RE}_{\mathrm{svt}}$ and for each $n \in L$ either $\pi_2(n) = 0$ or $\pi_2(n) = 1$. ($\mathrm{RE}_{\mathrm{char}}$ thus consists of the sets representing recursive characteristic functions.) Prove the following strengthening of proposition 4.5.3A: there is $\mathscr{L} \subseteq \mathrm{RE}_{\mathrm{char}}$ such that
$\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}}] - [\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\text{accountable}}]$.

*4.3.6 Simplicity

Let $L \in \mathrm{RE}$, and let $S = \{x \mid W_x = L\}$, the set of indexes for L. By lemma 1.2.1B, S is infinite. Intuitively the indexes in S correspond to grammars of increasing size and complexity. It is a plausible hypothesis that children do not conjecture grammars that are arbitrarily more complex than simpler alternatives for the same language (in view of the space requirements for storing complex grammars). In this subsection we consider learning functions that are limited to simple conjectures.

To begin, the notion of grammatical complexity must be precisely rendered. For this purpose we identify the complexity of a grammar with its size, and we formalize the notion of size as follows.

Definition 4.3.6A (Blum 1967a) Total $m \in \mathscr{F}^{\mathrm{rec}}$ is said to be a *size measure* (relative to our fixed acceptable indexing of $\mathscr{F}^{\mathrm{rec}}$) just in case m meets the following conditions.

i. For all $i \in N$, there are only finitely many $j \in N$ such that $m(j) = i$.
ii. The set $\{\langle i, j \rangle \mid \text{for all } k \geq j, m(k) \neq i\}$ is recursive.

To grasp the definition, suppose that $\mathscr{F}^{\mathrm{rec}}$ is indexed by associated Turing machines (TM) as in section 1.2.1. Then one size measure m_{TM} maps each index i into the number of symbols used to specify the ith Turing machine. This number is to be thought of as the size of i. m_{TM} can be shown to be total recursive. This size measure meets condition i of the definition, since for

$i \in N$ there are only finitely many Turing machines that can be specified using precisely i symbols. Condition ii is satisfied, since there exists an effective procedure for finding, given any $i \in N$, the largest index of a Turing machine of size i. For another example, the simplest size measure is given by the identity function $m(x) = x$. Conditions i and ii of the definition are easily seen to be satisfied. It would seem that any reasonable measure of the size of a computational agent also conforms to these conditions.

As with our choice of computational complexity measure (section 4.2.2), none of our results depend on the choice of size measure. Indeed, any two such measures can be shown, in a satisfying sense, to yield similar estimates of size (see Blum 1967a, sec. 1). Let a fixed size measure m now be selected. Reference to size should henceforth be interpreted accordingly.

DEFINITION 4.3.6B We define the function $M : \text{RE} \to N$ as follows. For all $L \in \text{RE}$, $M(L)$ is the unique $i \in N$ such that

i. there is $k \in N$ such that $W_k = L$ and $m(k) = i$,
ii. for all $j \in N$, if $W_j = L$, then $m(j) \geq i$.

Intuitively, for $L \in \text{RE}$, "$M(L)$" denotes the size of the smallest Turing machine for L. No index of size smaller than $M(L)$ is an index for L.

DEFINITION 4.3.6C Let total $f \in \mathscr{F}^{\text{rec}}$ be given. $i \in N$ is said to be *f-simple* just in case $m(i) \leq f(M(W_i))$.

In other words, i is f-simple just in case the size of i is no more than "f of" the size of the smallest possible grammar for W_i. Thus, if $f(x) = 2x$ for all $x \in N$, then i is f-simple just in case no index for W_i is less than half the size of i.

With these preliminaries in hand, we may now define strategies that limit the complexity of a learner's conjectures.

DEFINITION 4.3.6D

i. Let total $f \in \mathscr{F}^{\text{rec}}$ be given. $\varphi \in \mathscr{F}$ is said to be *f-simpleminded* just in case for all $\sigma \in \text{SEQ}$, if $\varphi(\sigma)\downarrow$, then $\varphi(\sigma)$ is f-simple.
ii. If $\varphi \in \mathscr{F}$ is f-simpleminded for some total $f \in \mathscr{F}^{\text{rec}}$, then φ is said to be *simpleminded.*

Put differently, an f-simpleminded learning function never conjectures indexes that are f-bigger than necessary. Thus, if $f(x) = 2x$ for all $x \in N$, then no conjecture of an f-simpleminded learner is more than twice the size of the smallest equivalent grammar.

Example 4.3.6A

a. Suppose that m is the size measure defined by $m(x) = x$ for all $x \in N$. Let total $h \in \mathscr{F}^{\text{rec}}$ be such that $h(x) \geq x$. Then both function f of part a of example 1.3.4B and function g of proposition 1.4.3B are h-simpleminded.
b. Irrespective of chosen size measure, the function g of part b of example 1.3.4B is simpleminded.

Provided that total $h \in \mathscr{F}^{\text{rec}}$ is such that $h(x) \geq x$ for all $x \in N$, $\mathscr{F}^{h\text{-simpleminded}}$ is not restrictive. However, for any total $h \in \mathscr{F}^{\text{rec}}$, $\mathscr{F}^{h\text{-simpleminded}}$ severely restricts $\mathscr{F}^{\text{rec}}$. To show this, we rely on the following remarkable result.

LEMMA 4.3.6A (Blum 1967a) Let $L \in \text{RE}$ be infinite, and let total $g \in \mathscr{F}^{\text{rec}}$ be given. Then there is $i \in L$ such that $m(i) > g(M(W_i))$.

Proof The lemma is a direct consequence of theorem 1 of Blum (1967a). □

The lemma asserts that every infinite r.e. set of indexes contains at least one index that is g-bigger than necessary, for any choice of total $g \in \mathscr{F}^{\text{rec}}$.

PROPOSITION 4.3.6A Let $\mathscr{L} \in [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{simpleminded}}]$. Then $\mathscr{L}$ contains only finitely many languages.

Proof Let $\varphi \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{simpleminded}}$ identify $\mathscr{L}$. Let $S = \text{rng}(\varphi)$, where S is r.e. because it is the range of a recursive function. Since φ is g-simpleminded for some g, lemma 4.3.6A implies that S is finite. Otherwise, for some σ we would have that $m(\varphi(\sigma)) > g(M(W_{\varphi(\sigma)}))$ contradicting the definition of g-simpleminded. If S is finite, obviously $\mathscr{L}$ must be finite because φ cannot learn a language for which it does not produce a conjecture. □

Thus, if children implement recursive, simpleminded learning functions, and if they can only learn languages for which they can produce grammars, then there are only finitely many natural languages.

COROLLARY 4.3.6A $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{simpleminded}}] \subset [\mathscr{F}^{\text{rec}}]$.

Exercises

4.3.6A Prove the following strengthening of proposition 4.3.6A: $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{simpleminded}}]$ is the class of all finite collections of languages.

4.3.6B $\varphi \in \mathscr{F}$ is called *loquacious* just in case $\{\varphi(\sigma) | \sigma \in \text{SEQ}\}$ is infinite. Prove: There exists total $h \in \mathscr{F}^{\text{rec}}$ such that for all total $f \in \mathscr{F}^{\text{rec}}$ and loquacious $\varphi \in \mathscr{F}^{\text{rec}}$ there exists $\sigma \in \text{SEQ}$ and $i \in N$ such that

a. $\varphi_{\varphi(\sigma)} = \varphi_i$,
b. $f(m(i)) < m(\varphi(\sigma))$,
c. for all but finitely many $\langle x, s \rangle \in N$, if $\Phi_{\varphi(\sigma)}(x) \leq s$, then $\Phi_i(x) \leq h(\langle x, s \rangle)$.

(In other words, the longer program $\varphi(\sigma)$ is not much faster than the shorter program i). (*Hint:* Use theorem 2 of Blum 1967.) This result extends proposition 4.3.6A.

4.4 Constraints on the Information Available to a Learning Function

Each initial segment $\bar{t}_n$ of a text t provides partial information about the identity of rng(t). The information embodied in $\bar{t}_n$ may be factored into two components: (1) rng($\bar{t}_n$), that is, the subset of rng(t) available to the learner by the nth moment, and (2) the order in which rng($\bar{t}_n$) occurs in $\bar{t}_n$. Human learners operate under processing constraints that limit their access to both kinds of information. In this section we examine two strategies that reflect this limitation.

4.4.1 Memory Limitation

It seems evident that children have limited memory for the sentences presented to them. Once processed, sentences are likely to be quickly erased from the child's memory. Here we shall consider learning functions that undergo similar information loss.

Definition 4.4.1A Let $\sigma \in \text{SEQ}$ be given.

i. The result of removing the last member of σ is denoted: σ^-. If $\text{lh}(\sigma) = 0$, then $\sigma^- = \sigma$.
ii. For $n \in N$ the result of removing all but the last n members of σ is denoted: $\sigma^- n$. If $\text{lh}(\sigma) < n$, then $\sigma^- n = \sigma$.

Thus, if $\sigma = 3, 3, 8, 1, 9$, then $\sigma^- = 3, 3, 8, 1$ and $\sigma^- 2 = 1, 9$.

Definition 4.4.1B (Wexler and Culicover 1980, sect. 3.2) For all $n \in N$, $\varphi \in \mathscr{F}$ is said to be *n-memory limited* just in case for all σ, $\tau \in \text{SEQ}$, if $\sigma^- n = \tau^- n$ and $\varphi(\sigma^-) = \varphi(\tau^-)$, then $\varphi(\sigma) = \varphi(\tau)$. If $\varphi \in \mathscr{F}$ is n-memory limited for some $n \in N$, then φ is said to be *memory limited.*

In other words, φ is n-memory limited just in case $\varphi(\sigma)$ depends on no more than $\varphi(\sigma^-)$ (φ's last conjecture) and $\sigma^- n$ (the n latest members of σ). Intuitively a child is memory limited if his or her conjectures arise from the interaction of recent input sentences with the latest grammar that he or she has formulated and stored. This latter grammar of course provides partial information about all the data seen to date.

Example 4.4.1A

a. The function h defined in the proof of proposition 4.3.2A is 1-memory limited.
b. Neither the function f defined in part a of example 1.3.4B nor the function g defined in the proof of proposition 1.4.3B is memory limited.
c. The function g of part b of example 1.3.4B is 0-memory limited.

Does some memory-limited $\varphi \in \mathscr{F}$ identify RE_{fin}? Let $\varphi \in \mathscr{F}$ be 2-memory limited, and consider the text $t = 4, 5, 5, 5, 5, 6, 6, 6, 6, \ldots$, for the language $\{4, 5, 6\}$. It appears that by the time φ reaches the first 6 in t, the initial 4 will have been forgotten, rendering convergence to rng(t) impossible. Since a similar problem arises for any "memory-window," it appears that memory limitation excludes identification of RE_{fin}.

However, this reasoning is incorrect. Memory limitation can often be surmounted by retrieving past data from the current conjecture. The following proposition will make this clear.

Proposition 4.4.1A $\text{RE}_{\text{fin}} \in [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{1-memory limited}}]$.

Proof Let S be a recursive set of indexes of r.e. sets containing exactly one index for each finite set and such that, given a finite set D, we can effectively find $e(D) \in S$ such that $e(D)$ is an index for D. The existence of such a set and function e is an easy exercise.

Now define $f \in \mathscr{F}^{\text{rec}}$ by $f(\sigma) = e(\text{rng}(\sigma))$ for all $\sigma \in \text{SEQ}$. Informally f chooses a canonical index for the range of σ. Now, if $f(\sigma^-) = f(\tau^-)$, then $\text{rng}(\sigma^-) = \text{rng}(\tau^-)$ and if also $\sigma^- 1 = \tau^- 1$, $\text{rng}(\sigma) = \text{rng}(\tau)$ so that $f(\sigma) = f(\tau)$. Thus $f \in \mathscr{F}^{\text{1-memory limited}}$. □

This last result notwithstanding, memory limitation is restrictive.

PROPOSITION 4.4.1B $[\mathscr{F}^{\text{memory limited}}] \subset [\mathscr{F}]$.

Proof Let $\mathscr{L}$ consist of the language $L = \{\langle 0,x\rangle | x \in N\}$ along with, for each $j \in N$, the languages $L_j = \{\langle 0,x\rangle | x \in N\} \cup \{\langle 1,i\rangle\}$ and $L_j' = \{\langle 0,x\rangle | x \neq j\} \cup \{\langle 1,j\rangle\}$. It is easy to see that $\mathscr{L} \in [\mathscr{F}]$. (In fact $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$.) But suppose that $\mathscr{L} \in [\mathscr{F}^{\text{memory limited}}]$. For instance, suppose that some $\varphi \in \mathscr{F}^{\text{1-memory limited}}$ identifies $\mathscr{L}$. (The case where $\varphi \in \mathscr{F}^{n\text{-memory limited}}$ is similar.) Intuitively, when φ first sees $\langle 1,j\rangle$ for some j, φ cannot remember whether it saw $\langle 0,j\rangle$ or not and so cannot distinguish between L_j and L_j'. Formally let σ be a locking sequence for φ and L. Let $\sigma' = \sigma \wedge \langle 1,j_0\rangle$ for some j_0 such that $\langle 0,j_0\rangle \notin \text{rng}(\sigma)$. Let $\sigma'' = \sigma \wedge \langle 0,j_0\rangle \wedge \langle 1,j_0\rangle$. Now $\varphi(\sigma') = \varphi(\sigma'')$, since $\varphi(\sigma) = \varphi(\sigma \wedge \langle 0,j_0\rangle)$ and φ is 1-memory limited. But now let $t_1 = \sigma' \wedge \langle 0,0\rangle \wedge \langle 0,1\rangle \wedge \cdots \wedge \langle 0,i\rangle \wedge \cdots$, for all $i \neq j_0$, and let $t_2 = \sigma'' \wedge \langle 0,0\rangle \wedge \langle 0,1\rangle \wedge \cdots \wedge \langle 0,i\rangle \wedge \cdots$, for $i \neq j_0$. t_1 is a text for L_{j_0}', and t_2 is a text for L_{j_0}, but φ converges on t_1 and t_2 to the very same index because of memory limitation. Thus φ cannot identify both L_{j_0} and L_{j_0}'. □

The proof of proposition 4.4.1B hinges on a collection of languages all of whose members are finite variants of each other. Exercise 4.4.1F shows that this feature of its proof is not essential.

To simplify the statement of later propositions, it is useful to record here the following result.

LEMMA 4.4.1A

i. $[\mathscr{F}^{\text{1-memory limited}}] = [\mathscr{F}^{\text{memory limited}}]$.
ii. $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{1-memory limited}}] = [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{memory limited}}]$.

The proof of this lemma turns on the following technical result (cf. lemma 1.2.1B).

LEMMA 4.4.1B There is a recursive function p such that p is one to one and for every x and y, $\varphi_x = \varphi_{p(x,y)}$.

A proof of this lemma may be found in Machtey and Young (1978). Such a function p is called a *padding function*, for to produce $p(x,y)$ from x, we take the instructions for computing φ_x and "pad" them with extra instructions to produce infinitely many distinct programs for computing the same function.

Proof of lemma 4.4.1A

i. Obviously, $[\mathscr{F}^{\text{1-memory limited}}] \subseteq [\mathscr{F}^{\text{memory limited}}]$. Suppose on the other hand that $\mathscr{L} \in [\mathscr{F}^{\text{memory limited}}]$; say $\mathscr{L}$ is identified by the n-memory limited function φ. We construct ψ which is 1-memory limited and identifies $\mathscr{L}$. Let p be the padding function provided by lemma 4.4.1B. Given any $x \in N$, define $x^{(n)}$ to be the sequence of n x's. Now given $\sigma \in \text{SEQ}$, define

$$\hat{\sigma} = \sigma_0^{(n)} \wedge \sigma_1 \wedge \sigma_0^{(n)} \wedge \cdots \wedge \sigma_{\text{lh}(\sigma)} \wedge \sigma_0^{(n)}.$$

Now define $\psi(\sigma) = p(\varphi(\hat{\sigma}), \sigma_0)$. (Intuitively we simulate φ on texts for which n-memory limitation is of no advantage over 1-memory limitation due to the repetitions.) ψ evidently identifies $\mathscr{L}$, since for any text t for $L \in \mathscr{L}$, $\hat{t}$ is also a text for L. To see that ψ is 1-memory limited, suppose that $\psi(\sigma^-) = \psi(\tau^-)$ and $\sigma^- 1 = \tau^- 1$. Since $\psi(\sigma^-) = \psi(\tau^-)$, $p(\varphi(\hat{\sigma}^-), \sigma_0) = p(\varphi(\hat{\tau}^-), \tau_0)$ so $\sigma_0 = \tau_0$. Let $x = \sigma^- 1 = \tau^- 1$. We have then that $\hat{\sigma} = \hat{\sigma}^- \wedge x \wedge \sigma_0^{(n)}$ and $\hat{\tau} = \hat{\tau}^- \wedge x \wedge \sigma_0^{(n)}$. Since $\varphi(\hat{\sigma}^-) = \varphi(\hat{\tau}^-)$, $\varphi(\hat{\sigma}^- \wedge x) = \varphi(\hat{\tau}^- \wedge x)$ by the n-memory limitation of φ. Thus $\varphi(\hat{\tau}) = \varphi(\hat{\sigma})$ by the n-memory limitation of φ. Thus

$$\psi(\sigma) = p(\varphi(\hat{\sigma}), \sigma_0) = p(\varphi(\hat{\tau}), \tau_0) = \psi(\tau).$$

ii. The transformation of φ to ψ in the proof of (i) produces a recursive ψ if φ is recursive. □

Proposition 4.4.1B show that memory limitation restricts $\mathscr{F}$. We now show that memory limitation and $\mathscr{F}^{\text{rec}}$ restrict each other.

PROPOSITION 4.4.1C $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{memory limited}}] \subset [\mathscr{F}^{\text{rec}}] \cap [\mathscr{F}^{\text{memory limited}}]$.

Proof Let A be a fixed r.e. nonrecursive set, and define $L = \{\langle 0,x\rangle | x \in A\}$, $L_n = L \cup \{\langle 1,n\rangle\}$, and $L_n' = L \cup \{\langle 0,n\rangle, \langle 1,n\rangle\}$. Let $\mathscr{L} = \{L, L_n, L_n' | n \in N\}$. It is easy to see that $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$. (Informally, conjecture L until some pair $\langle 1,n\rangle$ appears in the text. Then conjecture L_n forever unless $\langle 0,n\rangle$ appears or has already appeared in the text. In that case conjecture L_n'.) Also $\mathscr{L} \in [\mathscr{F}^{\text{memory limited}}]$. (Again informally, conjecture L until either $\langle 1,n\rangle$ appears in the text for some n, in which case behave as just described, or until $\langle 0,n\rangle$ appears in the text for some $n \notin A$. In this case conjecture L_n' forever. This procedure is 1-memory limited but not effective, since it asks whether $n \in A$ for a nonrecursive set A.)

Finally, we claim that $\mathscr{L} \notin [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{memory limited}}]$. For suppose that φ is 1-memory limited, recursive, and φ identifies $\mathscr{L}$. Let σ be a locking sequence for φ and L. This implies that for every $n \in A$, $\varphi(\sigma \wedge \langle 0, n \rangle) = \varphi(\sigma)$. Therefore for some $m \in \bar{A}$, $\varphi(\sigma \wedge \langle 0, m \rangle) = \varphi(\sigma)$ else $\bar{A}$ is recursively enumerable, implying that A is recursive. Fixing such an m, let s be an enumeration of L, and define two texts, t and t', by $t = \sigma \wedge \langle 1, m \rangle \wedge s$ and $t' = \sigma \wedge \langle 0, m \rangle \wedge \langle 1, m \rangle \wedge s$. By 1-memory limitation and the property of m, $\varphi(\sigma \wedge \langle 0, m \rangle \wedge \langle 1, m \rangle) = \varphi(\sigma \wedge \langle 1, m \rangle)$ and so again by 1-memory limitation, $\varphi(\bar{t}'_{n+1}) = \varphi(\bar{t}_n)$ for all $n \geq \mathrm{lh}(\sigma) + 1$. But t' is a text for L'_n and t for L_n and $L_n \neq L'_n$. Thus φ does not identify both L_n and L'_n. □

The interaction of memory limitation and computability may be refined yet further.

PROPOSITION 4.4.1D For every total $h \in \mathscr{F}^{\text{rec}}$, $[\mathscr{F}^{h\text{-time}} \cap \mathscr{F}^{\text{memory limited}}] \subset [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{memory limited}}]$.

The proof of proposition 4.4.1D is facilitated by the following definition.

DEFINITION 4.4.1C

i. For $i, n \in N$, we define $\varphi_{i,n} \in \mathscr{F}^{\text{rec}}$ as follows. For all $x \in N$,

$$\varphi_{i,n}(x) = \begin{cases} \varphi_i(x), & \text{if } \Phi_i(x) \leq n, \\ \uparrow, & \text{otherwise.} \end{cases}$$

ii. We define $W_{i,n}$ to be the domain of $\varphi_{i,n}$.

Thus $\varphi_{i,n}(x)$ may be thought of as the result of running the ith Turing machine for n steps starting with input x. If the machine halts within n steps, then $\varphi_{i,n}(x) = \varphi_i(x)$; if the machine does not halt within n steps, then $\varphi_{i,n}(x)$ is undefined. Definition 4.2.2A implies that the set $\{\langle i, n, x \rangle \mid \varphi_{i,n}(x)\downarrow\}$ is recursive.

Proof of proposition 4.4.1D The collection $\mathscr{L}$ of languages, which we will show to be in $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{memory limited}}]$ but not in $[\mathscr{F}^{\text{memory limited}} \cap \mathscr{F}^{h\text{-time}}]$, will be of the form $\mathscr{L}_R = \{R \cup F \mid F \text{ finite}\}$, where R is a fixed recursive set to be chosen later. It is easy to see that each such class is identifiable by a recursive, 1-memory-limited function, so it remains to choose R such that $\mathscr{L}_R \notin [\mathscr{F}^{h\text{-time}} \cap \mathscr{F}^{\text{memory limited}}]$. Fix h, and define a recursive function f by

$$f(i, \sigma, x) = \begin{cases} 1, & \text{if } \varphi_{i,h(\sigma)}(\sigma) = \varphi_{i,h(\tau)}(\tau), \text{ where } \tau = \sigma \wedge x \wedge \sigma_0^{(x)}, \\ 0, & \text{otherwise.} \end{cases}$$

(Note that equality in the first clause means that both computations converge and are equal.) f is evidently total and recursive.

Fix R recursive, and suppose that $\varphi_{i'} \in \mathscr{F}^{h\text{-time}} \cap \mathscr{F}^{n\text{-memory limited}}$ is such that $\varphi_{i'}$ identifies $\mathscr{L}_R$. Let σ' be a locking sequence for R and $\varphi_{i'}$ such that, in addition, $\varphi_{i',h(\sigma')}(\sigma')$ converges.

Claim For all but finitely many x, $x \in R$ if and only if $f(i', \sigma', x) = 1$.

Proof of claim If $x \in R$, then if $\tau_x = \sigma' \wedge x \wedge \sigma_0'^{(x)}$, $\varphi_{i'}(\tau_x) = \varphi_{i'}(\sigma')$. Now for all but finitely many x, $\varphi_{i',h(\tau_x)}(\tau_x)$ converges. Thus for all but finitely many $x \in R$, $\varphi_{i',h(\tau_x)}(\tau_x) = \varphi_{i',h(\sigma')}(\sigma')$, and therefore $f(i', \sigma', x) = 1$. On the other hand, suppose that $f(i', \sigma', x) = 1$. Then $\varphi_{i'}(\sigma') = \varphi_{i'}(\sigma' \wedge x \wedge \sigma_0'^{(x)})$ and this common value is an index for R. Since σ' is a locking sequence for R, we also have that $\varphi_{i'}(\sigma' \wedge \sigma_0'^{(n)}) = \varphi_{i'}(\sigma')$. Let t be any text for R. Since $\varphi_{i'}(\sigma' \wedge \sigma_0'^{(n)}) = \varphi_{i'}(\sigma' \wedge x \wedge \sigma_0'^{(x)})$ and $\varphi_{i'}$ is n-memory limited, $\varphi_{i'}(\sigma' \wedge \sigma_0'^{(n)} \wedge \bar{t}_m) = \varphi_{i'}(\sigma' \wedge x \wedge \sigma_0'^{(x)} \wedge \bar{t}_m)$ for every m. But since the former must be an index for R, so is the latter. Thus $x \in R$ else $\varphi_{i'}$ does not identify $R \cup \{x\}$ on the text $\sigma' \wedge x \wedge \sigma_0'^{(x)} \wedge t$.

The theorem will now be proved if we can show that there is a recursive set R such that for all σ and i there are infinitely many x such that $x \in R$ if and only if $f(i, \sigma, x) = 0$. This follows easily by a direct diagonalization argument (f is a total recursive function) or by an argument that depends on lemma 4.3.3A. We leave the details to the reader. □

Proposition 4.4.1D should be compared with proposition 4.2.2A.

Finally, we show that memory limitation restricts the identification of total, single-valued languages. Indeed, the next proposition provides more information than this (and implies proposition 4.4.1B).

PROPOSITION 4.4.1E $[\mathscr{F}^{\text{rec}}]_{\text{svt}} \not\subseteq [\mathscr{F}^{\text{memory limited}}]_{\text{svt}}$.

Proof Consider the following collection of total recursive functions:

$C = \{f \mid f$ is the characteristic function of a finite set

or f is the characteristic function of $N\}$.

If $\mathscr{L}$ is the collection of languages in RE_{svt} that represents precisely the functions in C, it is easy to see that $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]_{\text{svt}}$. Suppose, however, that $\varphi \in \mathscr{F}^{\text{memory limited}}$ identifies $\mathscr{L}$; we may suppose by lemma 4.4.1B that φ is 1-memory limited. Let σ be a locking sequence for φ and (the language representing) the characteristic function of N. Let $D = \{x \mid \langle x, 0 \rangle \in \mathrm{rng}(\sigma)\}$, and let σ' be a sequence such that $\tau = \sigma \wedge \sigma_0 \wedge \sigma'$ is a locking sequence for the characteristic function of D. (The existence of such a σ' uses corollary 2.1A to the Blum and Blum locking-sequence lemma.) Let n be an integer such that neither $\langle n,0\rangle$ nor $\langle n,1\rangle$ is in σ. Now $\varphi(\sigma \wedge \langle n,0\rangle \wedge \sigma_0) = \varphi(\sigma \wedge \sigma_0)$, since σ is a locking sequence for φ and the characteristic function of N, and so $\varphi(\sigma \wedge \langle n,0\rangle \wedge \sigma_0 \wedge \sigma') = \varphi(\sigma \wedge \sigma_0 \wedge \sigma')$ by the 1-memory limitation of φ. But then if we let t be a text that begins with $\sigma \wedge \langle n,0\rangle \wedge \sigma_0 \wedge \sigma'$ and ends with an enumeration of the characteristic function of D except for the pair $\langle n,1\rangle$, then φ must converge on t to an index for the characteristic function of D by 1-memory limitedness and the locking sequence property of σ'. However, t is a text for the characteristic function of $D \cup \{n\}$ and not D contradicting the fact that φ identifies the characteristic function of $D \cup \{n\}$. □

Corollary 4.4.1A $[\mathscr{F}^{\text{memory limited}}]_{\text{svt}} \subset [\mathscr{F}]_{\text{svt}}$.

Proposition 4.4.1E implies proposition 4.4.1B. Corollary 4.4.1A should be compared to proposition 1.4.3C.

Exercises

4.4.1A Specify 1-memory-limited, recursive learning functions that identify the following collections of languages.

a. $\{N - \{x\} | x \in N\}$.
b. RE_{sd} (see definition 2.3B).
c. $\{K \cup \{x\} | x \in \bar{K}\}$.

4.4.1B Let $n \in N$ be given, and let $\varphi \in \mathscr{F}^{n\text{-memory limited}}$ identify $L \in \mathrm{RE}$. Must there be a locking sequence σ for φ and L such that $\mathrm{lh}(\sigma) \leq n$?

4.4.1C Prove that $[\mathscr{F}^{\text{memory limited}}] \cap [\mathscr{F}^{h\text{-time}}] \subsetneq [\mathscr{F}^{\text{memory limited}} \cap \mathscr{F}^{h\text{-time}}]$ for all total $h \in \mathscr{F}^{\text{rec}}$.

***4.4.1D** Let a function $F: \mathrm{SEQ} \to \mathrm{SEQ}$ be given. $\varphi \in \mathscr{F}$ is called *F-biased* just in case for all $\sigma \in \mathrm{SEQ}$, if $F(\sigma) = F(\tau)$ and $\varphi(\sigma^-) = \varphi(\tau^-)$, then $\varphi(\sigma) = \varphi(\tau)$. To illustrate, let $H: \mathrm{SEQ} \to \mathrm{SEQ}$ be such that for all $\sigma \in \mathrm{SEQ}$, $H(\sigma) = \sigma^- 5$. Then $\varphi \in \mathscr{F}$ is H-biased if and only if φ is 5-memory limited.

a. For $n \in N$, let $G_n: \mathrm{SEQ} \to \mathrm{SEQ}$ be defined as follows. For all $\sigma \in \mathrm{SEQ}$, $G_n(\sigma)$ is the sequence that results from removing from σ all numbers greater than n. Thus $G_6(3,7,8,2) = (3,2)$. Prove: Let $n \in N$ be given. If $\mathscr{L} \in [\mathscr{F}^{G_n\text{-biased}}]$, then $\mathscr{L}$ is finite.
b. (Gisela Schäfer) For $n \in N$, let $H_n: \mathrm{SEQ} \to \mathrm{SEQ}$ be defined as follows. For all $\sigma \in \mathrm{SEQ}$, $H_n(\sigma)$ is the result of deleting all but the last n different elements of σ. [Thus $H_3(8,9,4,6,6,2) = (4,6,2)$.] Prove that for all $n \geq 1$, $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{H_n\text{-biased}}] = [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{1\text{-memory limited}}]$.

***4.4.1E** (John Canny) σ is said to be a *subsequence* of τ just in case $\mathrm{rng}(\sigma) \subseteq \mathrm{rng}(\tau)$. For $i \in N$, $m: \mathrm{SEQ} \to \mathrm{SEQ}$ is said to be an *i-memory function* just in case for all $\sigma \in \mathrm{SEQ}$, (a) $m(\sigma)$ is a subsequence of σ, (b) $\mathrm{lh}(m(\sigma)) = i$, and (c) $\mathrm{rng}(m(\sigma)) - \mathrm{rng}(m(\sigma^-)) \subseteq \{\sigma_{\mathrm{lh}(\sigma)}\}$. $\varphi \in \mathscr{F}$ is said to be *i-memory bounded* just in case there is some i-memory function m such that for all $\sigma \in \mathrm{SEQ}$, $\varphi(\sigma)$ depends on no more than $\sigma_{\mathrm{lh}(\sigma)-1}$, $\varphi(\sigma^-)$, and $m(\sigma)$—that is, just in case for all $\sigma, \tau \in \mathrm{SEQ}$, if $\sigma_{\mathrm{lh}(\sigma)-1} = \sigma_{\mathrm{lh}(\tau)-1}$, $\varphi(\sigma^-) = \varphi(\tau^-)$, and $m(\sigma) = m(\tau)$, then $\varphi(\sigma) = \varphi(\tau)$. Thus a memory-bounded learner chooses his or her current conjecture in light of a short-term memory buffer of finite capacity that evolves through time. The concept of i-memory bounded generalizes that of 1-memory limited inasmuch as $\varphi \in \mathscr{F}$ is 1-memory limited if and only if φ is 0-memory bounded.
Prove: For all $i \in N$, $[\mathscr{F}^{i\text{-memory bounded}}] \subset [\mathscr{F}]$.

Open question 4.4.1A $[\mathscr{F}^{\text{memory limited}}] \subset [\mathscr{F}^{1\text{-memory bounded}}]$?

4.4.1F Exhibit $\mathscr{L} \subseteq \mathrm{RE}$ such that (a) for all $L, L' \in \mathscr{L}$, if $L \neq L'$ then L and L' are not finite variants, and (b) $\mathscr{L} \in [\mathscr{F}] - [\mathscr{F}^{\text{memory limited}}]$.

*4.4.2 Set-Driven Learning Functions

We next consider learning functions that are insensitive to the order in which data arrive.

Definition 4.4.2A (Wexler and Culicover 1980, sec. 2.2) $\varphi \in \mathscr{F}$ is said to be *set driven* just in case for all $\sigma, \tau \in \mathrm{SEQ}$, if $\mathrm{rng}(\sigma) = \mathrm{rng}(\tau)$, then $\varphi(\sigma) = \varphi(\tau)$.

Example 4.4.2A

a. The function f defined in part a of example 1.3.4B and the function g defined in the proof of proposition 1.4.3B are set driven.
b. The function h defined in part c of example 1.4.2A is not set driven.

Identification of a language L requires identification of every text for L, and these texts constitute every possible ordering of L. This consideration encourages the belief that the internal order of a finite sequence plays little role in identifiability. The conjecture is correct with respect to $\mathscr{F}$. (see exercise 4.4.2A). However, the next proposition shows that it is wrong with respect to $\mathscr{F}^{\text{rec}}$.

Proposition 4.4.2A (Gisela Schäfer) $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{set driven}}] \subset [\mathscr{F}^{\text{rec}}]$.

Proof For each j define $L_j = \{\langle j,x\rangle | x \in N\}$. Given j and n, define $\sigma^{j,n} = \langle j,0\rangle \wedge \langle j,1\rangle \wedge \cdots \wedge \langle j,n\rangle$. Now define

$$L'_j = \begin{cases} \mathrm{rng}(\sigma^{j,n}), & \text{if there are } n, s \text{ such that } \\ & \varphi_j(\sigma^{j,n}) = i \text{ and } W_{i,s} \supset \mathrm{rng}(\sigma^{j,n}) \\ & \text{and } n, s \text{ is the least such pair,} \\ \{\langle j,0\rangle\}, & \text{otherwise.} \end{cases}$$

Let $\mathscr{L} = \{L_j, L'_j | j \in N\}$. It is easy to see that $\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}}]$. Suppose, however, that $\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\mathrm{set\ driven}}]$, and suppose that φ_j is a set-driven recursive function. Now if φ_j identifies $\mathscr{L}$, φ_j identifies the text $t = \langle j,0\rangle \wedge \langle j,1\rangle \wedge \cdots \wedge \langle j,n\rangle \wedge \cdots$. Thus there must be an $n \in N$ and an index i for L_j such that $\varphi_j(\sigma^{j,n}) = i$. In particular, there must be an n and s such that $\varphi_j(\sigma^{j,n}) = i$ and $W_{i,s} \supset \mathrm{rng}(\sigma^{j,n})$. But then φ_j does not identify $\mathrm{rng}(\sigma^{j,n})$ since on the following text $t' = \sigma^{j,n} \wedge \langle j,n\rangle \wedge \langle j,n\rangle \wedge \langle j,n\rangle \wedge \cdots$, φ_j must conjecture W_i in the limit since φ_j is set driven. Thus φ_j does not identify $\mathscr{L}$. □

Thus set-drivenness restricts $\mathscr{F}^{\mathrm{rec}}$ (but see in this connection exercise 4.4.2C).

Although children are not likely to be set driven, they may well ignore certain aspects of sentence order in the corpora they analyze.

Exercises

4.4.2A Prove that $[\mathscr{F}^{\mathrm{set\ driven}}] = [\mathscr{F}]$.

4.4.2B Let $\varphi \in \mathscr{F}^{\mathrm{set\ driven}}$ identify $\mathrm{RE}_{\mathrm{fin}}$. Show that for all $\sigma \in \mathrm{SEQ}$, σ is a locking sequence for φ and $\mathrm{rng}(\sigma)$.

4.4.2C Prove that if $\mathscr{L}$ contains only infinite languages, then $\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}}]$ if and only if $\mathscr{L} \in [\mathscr{F}^{\mathrm{rec}} \cap \mathscr{F}^{\mathrm{set\ driven}}]$.

4.5 Constraints on the Relation between Conjectures

The successive conjectures emitted by an arbitrary learning function need stand in no particular relation to each other. In this section we consider five constraints on this relation.

4.5.1 Conservatism

Definition 4.5.1A (Angluin 1980) $\varphi \in \mathscr{F}$ is said to be *conservative* just in case for all $\sigma \in \mathrm{SEQ}$, if $\mathrm{rng}(\sigma) \subseteq W_{\varphi(\sigma^-)}$, then $\varphi(\sigma) = \varphi(\sigma^-)$.

Thus a conservative learner never abandons a locally successful conjecture, a conjecture that generates all the data seen to date.

Example 4.5.1A

a. The function h defined in part c of example 1.3.4B is conservative.
b. Both the function f defined in part a of example 1.3.4B and the function g defined in the proof of proposition 1.4.3B are conservative.
c. The function f defined in the proof of proposition 2.3A is not conservative.

Conservatism is not restrictive.

Proposition 4.5.1A $[\mathscr{F}] = [\mathscr{F}^{\mathrm{conservative}}]$.

Proof This proof depends on the characterization of classes $\mathscr{L} \in [\mathscr{F}]$ given in proposition 2.4A. Recall that if $\mathscr{L} \in [\mathscr{F}]$, then for every $L \in \mathscr{L}$ there is a finite set $D_L \subseteq L$ such that if $D_L \subseteq L'$ and $L' \in \mathscr{L}$; then $L' \not\subset L$. Now given such an $\mathscr{L}$, define f by

$$f(\sigma) = \begin{cases} f(\sigma^-), & \text{if } \sigma^- \neq \varnothing \text{ and } W_{f(\sigma^-)} \supseteq \mathrm{rng}(\sigma), \\ \text{least } i \text{ such that } L = W_i & \\ \text{and } L \supseteq \mathrm{rng}(\sigma) \supseteq D_L, & \text{if such exists and } W_{f(\sigma^-)} \not\supseteq \mathrm{rng}(\sigma), \\ \text{least index for } \mathrm{rng}(\sigma), & \text{otherwise.} \end{cases}$$

By the first clause of the definition, f is conservative. Note further that for all $\gamma \in \mathrm{SEQ}$, $W_{f(\gamma)} \supseteq \mathrm{rng}(\gamma)$ (f is consistent), so this fact together with the first clause of the definition implies that f never returns to a conjectured language once it abandons a conjecture of that language.

To show that f identifies $\mathscr{L}$, suppose that $L \in \mathscr{L}$ and t is a text for L. If $f(\bar{t}_n)$ is an index for L for any n, then $f(\bar{t}_n) = f(\bar{t}_m)$ for all $m \geq n$. Further there is an n' such that $D_L \subseteq \mathrm{rng}(\bar{t}_{n'}) \subseteq L$. Thus f will adopt the conjecture of the least index for L on $\bar{t}_m$ for some $m \geq n'$ unless there is an index i for a language $L' \neq L$ such that f converges on t to i. Suppose for a contradiction that such an i exists. Then $L' \supseteq \mathrm{rng}(t) = L$, since $W_{f(\gamma)} \supseteq \mathrm{rng}(\gamma)$ for all γ. Let n be least such that $f(\bar{t}_n) = i$; $f(\bar{t}_n)$ was defined by either the second or the third clause in the definition of f. If f was defined by the third clause, $L' = \text{rng}(\bar{t}_n)$ so that $L = \text{rng}(t) \supseteq \text{rng}(\bar{t}_n) = L' \supseteq L$; thus $L = L'$, contradicting the assumption that $L \neq L'$. Suppose, on the other hand, that $f(\bar{t}_n)$ is defined by the second clause of the definition of f so that $D_{L'} \subseteq \text{rng}(\bar{t}_n) \subseteq L'$. Thus $L \supseteq D_{L'}$ which by the property of $D_{L'}$ implies $L \not\subset L'$. But this contradicts $L' \supset \text{rng}(t) = L$. □

On the other hand, conservatism does restrict $\mathscr{F}^{\text{rec}}$.

Proposition 4.5.1B (Angluin 1980) $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{conservative}}] \subset [\mathscr{F}^{\text{rec}}]$.

Proof This argument is essentially the same as that of Proposition 4.4.2A. Consider the class $\mathscr{L}$ of languages defined there and suppose that $\varphi_j \in \mathscr{F}^{\text{rec}}$ is a conservative function that identifies $\mathscr{L}$. As we argued in proposition 4.4.2A, φ_j must identify the text $t = \langle j,0\rangle, \langle j,1\rangle, \langle j,2\rangle, \ldots$, for L_j; thus there is a least pair $\langle n,s\rangle$ such that $\varphi_j(\bar{t}_n) = i$ and $W_{i,s} \supset \text{rng}(\bar{t}_n)$. Then $L'_j = \text{rng}(\bar{t}_n)$ is not identified by φ_j, since on the text $\langle j,0\rangle, \langle j,1\rangle, \ldots, \langle j,n\rangle, \langle j,n\rangle, \ldots, \varphi_j$ must continue to output i by the conservativeness of φ_j. □

There is a parallelism between consistency and conservatism. Both strategies embody palpably rational policies for learning, both constitute canonical methods of learning in the sense that neither is restrictive, but both strategies restrict $\mathscr{F}^{\text{rec}}$. Mechanical learners evidently pay a price for rationality.

Evidence that children are not conservative learners may be found in Mazurkewich and White (1984).

Exercises

4.5.1A Prove:

a. $[\mathscr{F}] = [\mathscr{F}^{\text{consistent}} \cap \mathscr{F}^{\text{conservative}} \cap \mathscr{F}^{\text{prudent}}]$.
b. $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{consistent}}] \not\subseteq [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{conservative}}]$.
c. $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{conservative}}] \not\subseteq [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{consistent}}]$.

4.5.1B

a. Let $\varphi \in \mathscr{F}^{\text{consistent}} \cap \mathscr{F}^{\text{conservative}}$ be given. Show that for all $\sigma \in \text{SEQ}$, σ is a locking sequence for φ and $W_{\varphi(\sigma)}$.
b. Let $\varphi \in \mathscr{F}^{\text{conservative}}$ identify text t. Show that there is no $n \in N$ such that $W_{\varphi(\bar{t}_n)} \supset \text{rng}(t)$. (Thus conservative learners never "overgeneralize" on languages they identify.)

***4.5.1C** Prove that $[\mathscr{F}^{\text{memory limited}}] = [\mathscr{F}^{\text{memory limited}} \cap \mathscr{F}^{\text{consistent}} \cap \mathscr{F}^{\text{conservative}}]$.

4.5.2 Gradualism

A single sentence probably cannot effect a drastic change in a child's grammar. We consider a corresponding strategy here. As a special case of notation introduced in section 2.1 for $\sigma \in \text{SEQ}$ and $n \in N$, $\sigma \wedge n$ is the result of concatenating n onto the end of σ: thus $(6,2,4) \wedge 3$ is $(6,2,4,3)$.

Definition 4.5.2A $\varphi \in \mathscr{F}$ is said to be *gradualist* just in case for all $\sigma \in \text{SEQ}$, $\{\varphi(\sigma \wedge n) | n \in N\}$ is finite.

Thus, if $\varphi \in \mathscr{F}$ is gradualist, then the effect of any single input on φ's latest conjecture is bounded. An argument similar to that for lemma 4.2.2B shows that gradualism is not restrictive.

Proposition 4.5.2A $[\mathscr{F}^{\text{gradualist}}] = [\mathscr{F}]$.

Proof We will give an informal argument to show that if $\varphi \in \mathscr{F}$ identifies $\mathscr{L}$, there is a $\varphi' \in \mathscr{F}$ such that φ' identifies $\mathscr{L}$, and for all $\sigma \in \text{SEQ}$ $\{\varphi'(\sigma \wedge n) | n \in N\}$ has size at most 3 by showing that φ' can be constructed from φ so that it never changes its conjecture in response to a new input by more than 1. The argument is a fall-behind-on-the-text argument as in lemma 4.2.2B. What φ' does on a text t is to simulate φ. Whenever φ changes its conjecture, say by n, φ' then uses the next n arguments of t to change its conjecture by ones. If φ converges on t, so will φ', although φ' will start converging much later on the text. □

Since all the procedures invoked in the preceding proof can be carried out mechanically, we have the following corollary.

Corollary 4.5.2A $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{gradualist}}] = [\mathscr{F}^{\text{rec}}]$.

The next proposition shows that gradualism restricts memory limitation.

Proposition 4.5.2B $[\mathscr{F}^{\text{gradualist}} \cap \mathscr{F}^{\text{memory limited}}] \subset [\mathscr{F}^{\text{memory limited}}]$.

Proof Let L_m be the two-element language $\{1, m\}$, and let $\mathscr{L} = \{L_m | m \in N\}$. Obviously $\mathscr{L}$ can be identified by a 1-memory-limited function. Suppose, however, that $\varphi \in \mathscr{F}^{\text{gradualist}} \cap \mathscr{F}^{\text{memory limited}}$. Suppose for simplicity that φ is 1-memory limited. Consider the texts $t^m = 1, m, 1, 1, 1, \ldots$. Since φ is gradualist and $\varphi(\bar{t}_1^m) = \varphi(\bar{t}_1^n)$ for all $m, n \in N$, there are m and n, $m \neq n$, such that $\varphi(\bar{t}_2^m) = \varphi(\bar{t}_2^n)$. Then, since φ is 1-memory limited and $t_k^m = t_k^n$ for all $k > 1$, φ converges to the same index on t^m and t^n. But then φ does not identify both L_m and L_n. □

Exercise

4.5.2A $\varphi \in \mathscr{F}$ is said to be *n-gradualist* just in case for all $\sigma \in \text{SEQ}$, $|\{\varphi(\sigma \wedge x) | x \in N\}| \leq n$. Note that as a corollary to the proof of proposition 4.5.2A, $[\mathscr{F}^{\text{3-gradualist}}] = [\mathscr{F}]$ and that $[\mathscr{F}^{\text{3-gradualist}} \cap \mathscr{F}^{\text{rec}}] = [\mathscr{F}^{\text{rec}}]$.

Let $n, m \in N$ be given. Let $\mathscr{L} \subseteq \text{RE}$ be as defined in the proof of proposition 4.5.2B. Prove: If $\varphi \in \mathscr{F}^{m\text{-gradualist}} \cap \mathscr{F}^{n\text{-memory limited}}$, then φ can identify no more than $(2m)^{n+1}$ languages in $\mathscr{L}$.

4.5.3 Induction by Enumeration

One strategy for generating conjectures is to choose the first grammar in some list of grammars that is consistent with the data seen so far.

DEFINITION 4.5.3A (Gold 1967) $\varphi \in \mathscr{F}$ is said to be an *enumerator* just in case there is total $f \in \mathscr{F}$ such that for all $\sigma \in \text{SEQ}$, $\varphi(\sigma) = f(i)$, where i is the least number such that $\text{rng}(\sigma) \subseteq W_{f(i)}$; in this case f is called the *enumerating function for* φ.

The function defined in part c of example 1.3.4B uses induction by enumeration; the enumerating function for h is the identity function.

Induction by enumeration constraints the succession of hypotheses emitted by a learner. This constraint is restrictive, but not for RE_{svt}.

PROPOSITION 4.5.3A

i. $[\mathscr{F}^{\text{enumerator}}] \subset [\mathscr{F}]$.
ii. $\text{RE}_{\text{svt}} \in [\mathscr{F}^{\text{enumerator}}]_{\text{svt}}$.

Proof

i. Let $L_n = \{x | x \geq n\}$, and let $\mathscr{L} = \{L_n | n \in N\}$. $\mathscr{L}$ can certainly be identified; in fact there is a recursive function that identifies $\mathscr{L}$. Suppose, however, that φ is an enumerator with enumerating function f. Were φ to identity $\mathscr{L}$, $\text{rng}(f)$ must contain indexes for each L_n. Thus there would be $i < j$ such that $W_{f(i)} \supset W_{f(j)}$ and $f(i)$ and $f(j)$ are the least indexes in $\text{rng}(f)$ for $W_{f(i)}$ and $W_{f(j)}$. But then φ must, on any text for $W_{f(j)}$, conjecture $W_{f(k)}$ for some $k \leq i$ and so does not identify $W_{f(j)}$.
ii. The function h of proposition 1.4.3B that identifies RE_{svt} is an enumerator with enumerating function $f(x) = x$. □

COROLLARY 4.5.3A $[\mathscr{F}^{\text{enumerator}} \cap \mathscr{F}^{\text{rec}}] \subset [\mathscr{F}^{\text{rec}}]$.

Examination of the proof of proposition 4.5.3A(ii) leads naturally to the following result.

PROPOSITION 4.5.3B (Gold 1967) Let $\mathscr{L} \subseteq \text{RE}_{\text{svt}}$ be r.e. indexable. Then $\mathscr{L} \in [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{enumerator}}]_{\text{svt}}$.

Exercises

4.5.3A Prove that for some $h \in \mathscr{F}^{\text{rec}}$, $[\mathscr{F}^{h\text{-time}} \cap \mathscr{F}^{\text{consistent}} \cap \mathscr{F}^{\text{conservative}} \cap \mathscr{F}^{\text{prudent}}] \not\subseteq [\mathscr{F}^{\text{enumerator}}]$.

4.5.3B Total $f \in \mathscr{F}$ is called *strict* just in case $i \neq j$ implies $W_{f(i)} \neq W_{f(j)}$, for all $i, j \in N$. $\varphi \in \mathscr{F}^{\text{enumerator}}$ is called *strict* just in case φ's enumerating function is strict. Prove that $[\mathscr{F}^{\text{strict enumerator}}] = [\mathscr{F}^{\text{enumerator}}]$.

4.5.3C Prove proposition 4.5.3B.

*4.5.4 Caution

Conservative learners do not overgeneralize on languages they do in fact identify, since once a conservative learner overgeneralizes it is trapped in that conjecture (see part b of exercise 4.5.1B). However, a conservative learner may well overgeneralize on a language it does not identify. We now examine learning functions that behave as if they never overgeneralize.

DEFINITION 4.5.4A $\varphi \in \mathscr{F}$ is called *cautious* just in case for all $\sigma, \tau \in \text{SEQ}$, $W_{\varphi(\sigma \wedge \tau)}$ is not a proper subset of $W_{\varphi(\sigma)}$.

Thus a cautious learner never conjectures a language that will be "cut back" to a smaller language by a later conjecture. Both the function f defined in Example 1.3.4B (part a) and the function g defined in the proof of proposition 1.4.3B are cautious.

Caution is an admirable policy. A text presents no information allowing the learner to realize that it has overgeneralized; consequently the need to cut back a conjectured language could result only from a prior miscalculation. These considerations suggest that caution is not restrictive.

PROPOSITION 4.5.4A $[\mathscr{F}^{\text{cautious}}] = [\mathscr{F}]$.

Proof The function f defined in the proof of proposition 4.5.1A is cautious. For if $W_{f(\sigma \wedge \tau)} \neq W_{f(\sigma)}$, then $W_{f(\sigma \wedge \tau)} \supseteq \text{rng}(\sigma \wedge \tau)$, but $W_{f(\sigma)} \not\supseteq \text{rng}(\sigma \wedge \tau)$ since conjectures are only abandoned by f if they do not include the input. Thus $W_{f(\sigma)} \not\supset W_{f(\sigma \wedge \tau)}$. □

As in the cases of consistency and conservatism, the calculations required for a cautious learning policy sometimes exceed the capacities of computable functions.

PROPOSITION 4.5.4B $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{cautious}}] \subset [\mathscr{F}^{\text{rec}}]$.

Proof Again, the class $\mathscr{L}$ of languages in the proof of proposition 4.4.2A is the desired example of a class $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$ that cannot be identified by $\varphi \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{cautious}}$. For if φ_j identifies $t = \langle j, 0\rangle, \langle j, 1\rangle, \ldots$, as before, φ_j must conjecture some i such that $\varphi_j(\bar{t}_n) = i$, $W_{i,s} \supset \text{rng}(\bar{t}_n)$, where $L'_j = \text{rng}(\bar{t}_n) \in \mathscr{L}$. But then φ_j on text $t = \langle j, 0\rangle, \langle j, 1\rangle, \ldots, \langle j, n\rangle, \langle j, n\rangle, \ldots$, can never later conjecture any $L \subset W_i$. However, $L'_j \subset W_i$. □

Exercises

4.5.4A Prove that $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{conservative}} \cap \mathscr{F}^{\text{cautious}}] = [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{conservative}}]$.

4.5.4B Prove that $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{cautious}}] \not\subseteq [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{conservative}}]$.
(*Hint:* See the proof of proposition 4.5.1B.)

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*4.5.5 Decisiveness

Let $\varphi \in \mathscr{F}$ be a strict enumerator in the sense of exercise 4.5.3B. Then φ never returns to a conjectured language once abandoned. The next definition isolates those learning functions whose successive conjectures meet this condition.

DEFINITION 4.5.5A $\varphi \in \mathscr{F}$ is called *decisive* just in case for all $\sigma \in \text{SEQ}$, if $W_{\varphi(\sigma^-)} \neq W_{\varphi(\sigma)}$, then there is no $\tau \in \text{SEQ}$ such that $W_{\varphi(\sigma \wedge \tau)} = W_{\varphi(\sigma^-)}$.

Both the function defined in example 1.3.4B (part a) and the function g defined in the proof of proposition 1.4.3B are decisive.

Like caution, decisiveness appears to be a sensible strategy. It is not restrictive.

PROPOSITION 4.5.5A $[\mathscr{F}^{\text{decisive}}] = [\mathscr{F}]$.

Proof The function f defined in the proof of proposition 4.5.1A is decisive, as was remarked in the proof immediately following the definition of f. There is a general fact here of note: conservative, consistent learners are also decisive (see exercise 4.5.5D). □

The next result shows that decisiveness does not restrict $\mathscr{F}^{\text{rec}}$ in the context of RE_{svt}.

PROPOSITION 4.5.5B (Gisela Schäfer) $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{decisive}}]_{\text{svt}} = [\mathscr{F}^{\text{rec}}]_{\text{svt}}$.

Proof (proof due to Gisela Schäfer) Suppose that $\theta \in \mathscr{F}^{\text{rec}}$ identifies $\mathscr{L} \in \text{RE}_{\text{svt}}$. We will define $\psi \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{decisive}}$ which identifies $\mathscr{L}$. It is easy to see that we need only define ψ on sequences σ of the form $\sigma = (\langle 0, x_0\rangle, \langle 1, x_1\rangle, \ldots, \langle n, x_n\rangle)$ (cf. exercise 4.2.1C). We may also suppose that the conjectures of θ are indexes of partial recursive functions rather than indexes for r.e. sets. This is because for $j \in N$ we can effectively compute an index i such that if W_j represents a partial recursive function, W_j represents φ_i. Define for each i, $\varphi_i[n] = (\langle 0, \varphi_i(0)\rangle, \ldots, \langle n, \varphi_i(n)\rangle)$. Suppose then that $\sigma = (\langle 0, x_0\rangle, \ldots, \langle n, x_n\rangle)$ and that $\theta(\sigma) = i$. Let $k = 1 + \max\{m \mid \theta(\bar{\sigma}_m) \neq i\}$. Define a recursive h by

$$\varphi_{h(i,k)}(x) = \begin{cases} x_j, & x \leq k, \\ \varphi_i(x), & \text{if } x > k, \text{ for all } y \leq k, \varphi_i(y) = x_y, \text{ and } \theta(\varphi_i[x]) = i, \\ \text{diverges}, & \text{otherwise.} \end{cases}$$

Now define $\psi(\sigma) = h(i, k)$.

Informally, if θ appears to be converging after the first k elements of input to φ_i, then $\varphi_{h(i,k)}$ is defined to agree with the input through k elements and, provided that φ_i agrees with the first k input elements, $\varphi_{h(i,k)}$ is also defined to agree with φ_i through the longest initial segment such that φ_i is defined and θ appears to converge to i on that initial segment of φ_i.

It is clear that $\psi(\sigma^-) \neq \psi(\sigma)$ if and only if $\theta(\sigma^-) \neq \theta(\sigma)$. Further, if θ converges on an increasing text for φ_i to i and φ_i is total, then ψ converges on t to $h(i, k)$, where $k = \max\{m \mid \theta(\bar{t}_m) \neq i\}$. Also in this case $h(i, k)$ is an index for φ_i. Thus ψ identifies at least as many total functions as does θ.

To show that ψ is decisive, suppose that $\sigma = (\langle 0, x_0\rangle, \ldots, \langle 0, x_n\rangle)$ and that $\psi(\sigma) \neq \psi(\sigma^-)$. Suppose that $\psi(\sigma) = h(i, k)$ and $\psi(\sigma^-) = h(i', k')$. Note that $k' = n$. There are two cases.

Case 1. Suppose that n is not in the domain of $\varphi_{h(i,k)}$. Then $\psi(\sigma \wedge \tau)$ is not an index for $\varphi_{\psi(\sigma^-)}$ for any τ, since the domain of $\varphi_{\psi(\sigma \wedge \tau)} \supseteq \{0, 1, 2, \ldots, n\}$ for all τ by the first clause in the defintion of $h(i, k)$.

Case 2. n is in the domain of $\varphi_{h(i,k)}$. Then $\theta(\varphi_{h(i,k)}[n]) = i$; and so, in particular, $\sigma \neq \varphi_{h(i,k)}[n]$. But $\varphi_{\psi(\sigma \wedge \tau)}$ extends σ for all $\tau \in \text{SEQ}$, again by the first clause in the definition of $h(i, k)$. Thus in this case also, $\psi(\sigma \wedge \tau)$ is not an index for $\varphi_{\psi(\sigma^-)}$. □

Whether decisiveness restricts $\mathscr{F}^{\text{rec}}$ in the general case is not known.

Open question 4.5.5A $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{decisive}}] = [\mathscr{F}^{\text{rec}}]$?

Exercises

4.5.5A $\varphi \in \mathscr{F}$ is called *weakly decisive* just in case for all $\sigma \in \text{SEQ}$, if $\varphi(\sigma^-) \neq \varphi(\sigma)$, then there is no $\tau \in \text{SEQ}$ such that $\varphi(\sigma \wedge \tau) = \varphi(\sigma^-)$; that is, weakly decisive learning functions never repeat a conjecture once abandoned. Prove that $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{weakly decisive}}] = [\mathscr{F}^{\text{rec}}]$.

4.5.5B Prove that $[\mathscr{F}^{\text{enumerator}}] \subset [\mathscr{F}^{\text{decisive}}]$.

4.5.5C Prove that $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{conservative}}] \subset [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{decisive}}]$.

4.5.5D Prove that $\mathscr{F}^{\text{consistent}} \cap \mathscr{F}^{\text{conservative}} \subset \mathscr{F}^{\text{decisive}}$.

4.6 Constraints on Convergence

If $\varphi \in \mathscr{F}$ identifies $\mathscr{L} \in \text{RE}$, then φ must converge to some index for $\text{rng}(t)$ on every text t for $L \in \mathscr{L}$. φ may or may not converge to the same index on different texts for the same language in $\mathscr{L}$, and φ may or may not converge on texts for languages outside of $\mathscr{L}$. In this section we consider three constraints on convergence that limit the freedom of learning functions in these ways.

4.6.1 Reliability

A learner that occasionally converges to an incorrect language may be termed "unreliable."

DEFINITION 4.6.1A (Minicozzi, cited in Blum and Blum 1975) $\varphi \in \mathscr{F}$ is called *reliable* just in case (i) φ is total, and (ii) for all $t \in \mathscr{T}$, if φ converges on t, then φ identifies t.

Example 4.6.1A

a. The function f in example 1.3.4B (part a) is reliable, for f identifies every text for a finite language and fails to converge on any text for an infinite language.
b. The function f defined in the proof of proposition 2.3A is not reliable, for if n is an index for N, then f converges on the text $n, n, n, \ldots$, but fails to identify it.

Reliability is a useful property of learning functions. A reliable learner never fails to signal the inaccuracy of a previous conjecture. To explain, let $f \in \mathscr{F}$ be reliable, let t be a text for some language, and suppose that for some $i, n \in N$, $f(\bar{t}_n) = i$. If $W_i \neq \text{rng}(t)$, that is, if i is incorrect, then for some $m > n$, $f(\bar{t}_m) \neq i$ (otherwise, f converges on t to the incorrect index i, contradicting f's reliability). The new index $f(\bar{t}_m)$ signals the incorrectness of i. It might thus be hoped that every identifiable collection of languages is identified by a reliable learning function. It might also be conjectured that children implement reliable learning functions on the assumption that any text for a nonnatural language would lead a child to search ceaselessly for a successful grammar, ever elusive. In view of these considerations it is interesting to learn that reliability is a debilitating constraint on learning functions.

PROPOSITION 4.6.1A Let $\varphi \in \mathscr{F}^{\text{reliable}}$ identify $L \in \text{RE}$. Then L is finite.

Proof This is a straightforward locking sequence argument. Suppose that $\varphi \in \mathscr{F}^{\text{reliable}}$ identifies L; let σ be a locking sequence for φ and L. Then, if $t = \sigma \wedge \sigma_0 \wedge \sigma_0 \wedge \cdots$, φ converges on t to an index for L. Thus $L = \text{rng}(t) = \text{rng}(\sigma)$ which is finite. □

COROLLARY 4.6.1A $[\mathscr{F}^{\text{reliable}}] \subset [\mathscr{F}]$.

The next definition relativizes reliability to RE_{svt}.

DEFINITION 4.6.1B (Minicozzi, cited in Blum and Blum 1975) $\varphi \in \mathscr{F}$ is called *reliable*-svt just in case (i) φ is total, and (ii) for all texts t for any $L \in \text{RE}_{\text{svt}}$, if φ converges on t then φ identifies t.

Since $\mathrm{RE}_{\mathrm{svt}}$ is identifiable (proposition 1.4.3C), $\mathrm{RE}_{\mathrm{svt}}$ is identified by a learning function that is (somewhat vacuously) reliable-svt. The interaction of $\mathscr{F}^{\text{reliable-svt}}$ and $\mathscr{F}^{\text{rec}}$ is more interesting, as revealed by the following results.

PROPOSITION 4.6.1B (Minicozzi, cited in Blum and Blum 1975) Let $\mathscr{L}$, $\mathscr{L}' \in [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}]_{\text{svt}}$ be given. Then $\mathscr{L} \cup \mathscr{L}' \in [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}]_{\text{svt}}$.

That is, $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}]_{\text{svt}}$ is closed under finite union (cf. exercise 2.2D).

Proof of proposition 4.6.1B We give an informal proof and invite the reader to formalize it. Suppose that ψ and $\psi' \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}$ identify $\mathscr{L}$ and $\mathscr{L}'$ in $\mathrm{RE}_{\mathrm{svt}}$ respectively. Then we define $\varphi \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}$ as follows. On text t, φ outputs the conjectures of ψ until ψ changes its mind. Then φ outputs the conjectures of ψ' until it changes its mind, and so forth. If t is a text for a language $L' \in \mathscr{L}'$ which ψ does not identify, then φ will always abandon its ψ-like conjectures for the eventually stable ψ' conjectures. Similarly for $L \in \mathscr{L}$. Should t be a text for a language that is not in $\mathscr{L} \cup \mathscr{L}'$, then φ will change its mind infinitely often. □

Now recall definition 1.2.2D of "$S \subseteq N$ represents $T \subseteq N^2$."

DEFINITION 4.6.1C

i. $\varphi \in \mathscr{F}$ is said to be *almost everywhere zero* just in case $\varphi(x) = 0$ for all but finitely many $x \in N$. The collection $\{L \in \mathrm{RE}_{\mathrm{svt}} | L$ represents a function that is almost everywhere zero$\}$ is denoted: $\mathrm{RE}_{\mathrm{aez}}$.
ii. $\varphi \in \mathscr{F}^{\text{rec}}$ is called *self-indexing* just in case the smallest $x \in N$ such that $\varphi(x) = 1$ is an index for φ. The collection $\{L \in \mathrm{RE}_{\mathrm{svt}} | L$ represents a self-indexing function$\}$ is denoted: $\mathrm{RE}_{\mathrm{si}}$.

PROPOSITION 4.6.1C (Blum and Blum 1975)

i. $\mathrm{RE}_{\mathrm{si}} \in [\mathscr{F}^{\text{rec}}]_{\text{svt}}$.
ii. $\mathrm{RE}_{\mathrm{aez}} \in [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}]_{\text{svt}}$.
iii. $\mathrm{RE}_{\mathrm{si}} \cup \mathrm{RE}_{\mathrm{aez}} \notin [\mathscr{F}^{\text{rec}}]_{\text{svt}}$.

Proof For i and ii, the obvious methods for identifying $\mathrm{RE}_{\mathrm{si}}$ and $\mathrm{RE}_{\mathrm{aez}}$ work.

For iii, suppose that ψ identifies $\mathrm{RE}_{\mathrm{aez}}$. We will define a recursive function f by the recursion theorem such that f is self-indexing and if L represents f, then ψ does not identify L. To apply the recursion theorem, we define a total

recursive function h by the following algorithm:

$\varphi_{h(i)}(x) = 0, \quad \text{if } x < i,$

$\varphi_{h(i)}(i) = 1.$

If $x > i$, and $\varphi_{h(i)}(y)$ has been defined for all $y < x$, define $\varphi_{h(i)}(x)$ as follows. For every integer n define $\sigma^n = (\langle 0,0\rangle, \langle 1,0\rangle, \ldots, \langle i-1,0\rangle, \langle i,1\rangle, \langle i+1, \varphi_{h(i)}(i+1)\rangle, \ldots, \langle x-1, \varphi_{h(i)}(x-1)\rangle, \langle x,0\rangle, \ldots, \langle x+n, 0\rangle)$. Enumerate simultaneously $W_{\psi(\sigma^0)}, W_{\psi(\sigma^1)}, \ldots, W_{\psi(\sigma^n)}$ for increasing n until a pair $\langle x + M + 1, 0\rangle$ appears in $W_{\psi(\sigma^M)}$ for some M. Then define $\varphi_{h(i)}(x) = \cdots = \varphi_{h(i)}(x+M) = 0$ and $\varphi_{h(i)}(x+M+1) = 1$. Such an n will exist, since the sequences $\sigma^0, \sigma^1, \sigma^2, \ldots$, are initial segments of a text for a function in $\mathrm{RE}_{\mathrm{aez}}$. Thus $\varphi_{h(i)}$ is total for every i.

Let i' be such that $\varphi_{h(i')} = \varphi_{i'}$. By the definition of $\varphi_{h(i')}$, $\varphi_{i'} = \varphi_{h(i')}$ is self-indexing. But there are infinitely many x such that for the corresponding M and σ^M, $\psi(\sigma^M)$ is an index for a set that does not represent φ_i, since $\varphi_{h(i')}(x+M+1) = \varphi_{i'}(x+M+1) = 1$, but $\langle x+M+1, 0\rangle \in W_{\psi(\sigma^M)}$. These σ^M are initial segments of the same text t for $\varphi_{i'}$, so ψ does not converge on a text for $\varphi_{i'}$. □

Thus $[\mathscr{F}^{\text{rec}}]_{\text{svt}}$ is not closed under finite union (cf. exercise 2.2D). The following corollaries are immediate from the two preceding propositions.

COROLLARY 4.6.1B $\mathrm{RE}_{\mathrm{si}} \notin [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}]_{\text{svt}}$.

COROLLARY 4.6.1C $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}]_{\text{svt}} \subset [\mathscr{F}^{\text{rec}}]_{\text{svt}}$.

Exercises

4.6.1A $\varphi \in \mathscr{F}$ is called *weakly reliable* just in case for all texts t for any $L \in \mathrm{RE}$, if φ converges on t then φ identifies t. (Thus weakly reliable learning functions need not be total). Prove the following strengthened version of proposition 4.6.1A: let $\varphi \in \mathscr{F}^{\text{weakly reliable}}$ identify $L \in \mathrm{RE}$. Then L is finite.

4.6.1B (Blum and Blum 1975) Let total $f \in \mathscr{F}^{\text{rec}}$ be given. Suppose that for all $i \in N$, $\varphi_{f(i)} \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}$. Show that $\bigcup_{i \in N} [\{\varphi_{f(i)}\}]_{\text{svt}} \in [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{reliable-svt}}]_{\text{svt}}$. This result generalizes proposition 4.6.1B.

***4.6.1C** $\varphi \in \mathscr{F}$ is called *finite-difference reliable* just in case for all texts t for any $L \in \mathrm{RE}$, if φ converges on t, then φ converges to a finite variant of rng(t) (see definition 2.3A). Thus $\varphi \in \mathscr{F}$ is finite-difference reliable just in case φ never converges to a conjecture that is "infinitely wrong." Reliability is a special case of finite-difference reliability. Prove the following strengthened version of proposition 4.6.1A: let finite-difference reliable $\varphi \in \mathscr{F}$ identify $L \in \mathrm{RE}$. Then L is finite.

4.6.2 Confidence

A learner that converges on every text may be termed "confident."

DEFINITION 4.6.2A $\varphi \in \mathscr{F}$ is called *confident* just in case for all $t \in \mathscr{T}$, φ converges on t.

Thus confidence is the mirror image of reliability.

Example 4.6.2A

a. The function f defined in the proof of proposition 2.3A is confident.
b. Neither the function f defined in example 1.3.4B (part a) nor the function g defined in the proof of proposition 1.4.3B is confident since neither converges on any text for N.

Children are confident learners if they eventually settle for some approximation to input, nonnatural languages.

PROPOSITION 4.6.2A $[\mathscr{F}^{\text{confident}}] \subset [\mathscr{F}]$.

Proof $\mathrm{RE}_{\text{fin}} \in [\mathscr{F}]$. Suppose that $\varphi \in \mathscr{F}$ identifies RE_{fin}. We construct sequences $\sigma^0, \sigma^1, \ldots$, such that φ does not converge on $\sigma^0 \wedge \sigma^1 \wedge \cdots$, demonstrating that $\varphi \notin \mathscr{F}^{\text{confident}}$. Let σ^0 be the shortest sequence of zeros such that $\varphi(\sigma^0)$ is an index for $\{0\}$. Since φ identifies $\{0\}$, σ^0 exists. Now let σ^1 be the shortest sequence of ones such that $\varphi(\sigma^0 \wedge \sigma^1)$ is an index for $\{0, 1\}$. Given σ^{n-1}, let σ^n be the shortest sequence of n's such that $\varphi(\sigma^0 \wedge \cdots \wedge \sigma^n)$ is an index for $\{0, 1, \ldots, n\}$. Obviously φ does not converge on $\sigma^0 \wedge \cdots \wedge \sigma^n \wedge \cdots$. □

The next proposition shows that confidence and $\mathscr{F}^{\text{rec}}$ restrict each other. First, we prove a lemma.

LEMMA 4.6.2A Let $\varphi \in \mathscr{F}^{\text{confident}}$ be given. Then for every $L \in \mathrm{RE}$, there is $\sigma \in \mathrm{SEQ}$ such that (i) $\mathrm{rng}(\sigma) \subseteq L$, and (ii) for all $\tau \in \mathrm{SEQ}$ such that $\mathrm{rng}(\tau) \subseteq L$, $\varphi(\sigma \wedge \tau) = \varphi(\sigma)$.

Proof This is much like the proof of proposition 2.1A, the locking sequence lemma. If such a σ did not exist, we could construct a text t for L on which φ does not converge, contradicting its confidence. □

PROPOSITION 4.6.2B $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{confident}}] \subset [\mathscr{F}^{\text{rec}}] \cap [\mathscr{F}^{\text{confident}}]$.

Proof $\mathscr{L} = \{K \cup \{x\} \mid x \in \overline{K}\}$ is the needed collection. We have noted before that $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$ (see exercise 4.2.1A). The following defines $f \in [\mathscr{F}^{\text{confident}}]$ which identifies $\mathscr{L}$:

$$f(\sigma) = \begin{cases} \text{index for } K, & \text{if } \mathrm{rng}(\sigma) \subset K, \\ \text{index for } K \cup \{x\}, & \text{if } x \text{ is the least element of } \mathrm{rng}(\sigma) - K. \end{cases}$$

Suppose however that $\varphi \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{confident}}$ and identifies $\mathscr{L}$. By lemma 4.6.2A there is a sequence σ such that $\mathrm{rng}(\sigma) \subset K$ and if $\mathrm{rng}(\tau) \subset K$, $\varphi(\sigma \wedge \tau) = \varphi(\sigma)$. But then, much as in the proof of lemma 4.2.1C, we now have a way of enumerating $\overline{K}$, for, $x \in \overline{K}$ if and only if there is a sequence τ such that $\mathrm{rng}(\tau) \subset K$ and $\varphi(\sigma \wedge x \wedge \tau) \neq \varphi(\sigma)$. □

Exercises

4.6.2A Recall the definition of $\mathscr{L} \times \mathscr{L}'$ from exercise 1.4.3F. Prove: Let $\mathscr{L} \in [\mathscr{F}^{\text{confident}}]$ and $\mathscr{L}' \in [\mathscr{F}^{\text{confident}}]$ be given. Then, $\mathscr{L} \times \mathscr{L}' \in [\mathscr{F}^{\text{confident}}]$.

4.6.2B $\mathscr{L} \subseteq \mathrm{RE}$ is called a *w.o. chain* just in case $\mathscr{L}$ is well ordered by inclusion.

a. Exhibit an infinite, w.o. chain in $[\mathscr{F}^{\text{rec}}]$.
b. Prove: If $\mathscr{L} \subseteq \mathrm{RE}$ is an infinite w.o. chain, then $\mathscr{L} \notin [\mathscr{F}^{\text{confident}}]$.
c. $\varphi \in \mathscr{F}$ is called *conjecture bounded* (or cb) just in case for every $t \in \mathscr{T}$, $\{\varphi(\overline{t}_m) \mid m \in N\}$ is finite. Thus $\varphi \in \mathscr{F}^{\text{cb}}$ just in case no text leads φ to produce conjectures of arbitrary size. Prove: Let $\mathscr{L}$ be an infinite w.o. chain, then $\mathscr{L} \notin [\mathscr{F}^{\text{cb}}]$.
d. Prove that $[\mathscr{F}^{\text{confident}}] = [\mathscr{F}^{\text{cb}}]$.

4.6.2C

a. $\mathscr{L} \subseteq \mathrm{RE}$ is called *maximal* just in case $\mathscr{L} \in [\mathscr{F}]$, and there is $L \in \mathrm{RE}$ such that $\mathscr{L} \cup \{L\} \notin [\mathscr{F}]$. To illustrate, proposition 2.2A(ii) shows that the collection $\{N - \{x\} \mid x \in N\}$ is maximal. (Compare the definition of "saturated" given in exercise 2.2E.) Prove that if $\mathscr{L} \subseteq \mathrm{RE}$ is maximal, then $\mathscr{L} \notin [\mathscr{F}^{\text{confident}}]$. Obtain proposition 4.6.2A as a corollary to this result.
b. Prove: Let $\mathscr{L} \in [\mathscr{F}^{\text{confident}}]$ and $\mathscr{L}' \in [\mathscr{F}^{\text{confident}}]$ be given. Then $\mathscr{L} \cup \mathscr{L}' \in [\mathscr{F}^{\text{confident}}]$. Obtain part a of the present exercise as a corollary to this result.

4.6.3 Order Independence

As a final constraint on convergence we consider learning functions that converge to the same index on every text for a language they identify.

DEFINITION 4.6.3A (Blum and Blum 1975) $\varphi \in \mathscr{F}$ is called *order independent* just in case for all $L \in \text{RE}$, if φ identifies L, then there is $i \in N$ such that for all texts t for L, φ converges on t to i.

Thus an order-independent learning function is relatively insensitive to the choice of text for a language it identifies: any such text eventually leads it to the same index (even though such behavior is not required by the definition of identification). Note that different texts for the same identified language may cause an order-independent learning function to consume different amounts of input before convergence begins (just as for order-dependent learning functions).

Example 4.6.3A

a. Both the function f defined in example 1.3.4B, part b, and the function g defined in the proof of proposition 1.4.3B are order independent.
b. The function f defined in the proof of proposition 2.3A is order independent.
c. The function f in the proof of proposition 4.4.1A is order independent.

It is easy to see that order independence is not restrictive. The relation of order independence to $\mathscr{F}^{\text{rec}}$ is a more delicate matter; the following consideration suggests that it is restrictive. An order-independent learning policy seems to require the ability to determine the equivalence of distinct indexes. But the equivalence question cannot in general be answered by a computational process; indeed, the set $\{\langle i,j\rangle \mid W_i = W_j\}$ is not even r.e. (see Rogers 1967, sec. 5.2). Contrary to expectation, however, order independence turns out not to restrict $\mathscr{F}^{\text{rec}}$.

PROPOSITION 4.6.3A (Blum and Blum 1975) $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{order independent}}] = [\mathscr{F}^{\text{rec}}]$.

The proof of proposition 4.6.3A depends on a very important construction due to Blum and Blum. We will first give this construction and then derive from it a corollary concerning classes $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$. We will then use this corollary to establish proposition 4.6.3A. Following that, we will establish lemmas 4.3.4A and 4.3.4B and thereby Fulk's proposition that prudence does not restrict recursive learning functions (proposition 4.3.4A).

Construction (the locking-sequence-hunting construction) Suppose that $\varphi \in \mathscr{F}^{\text{rec}}$ identifies $\mathscr{L}$. Now by lemma 4.2.2B there is a total recursive function f such that f identifies $\mathscr{L}$. By proposition 2.1A, for every $L \in \mathscr{L}$ there is a locking sequence σ for f and L. We will construct $g \in \mathscr{F}^{\text{rec}}$ so that on any text t for L, g converges to $f(\sigma)$, where σ is the least locking sequence for f and L. (Recall that sequences are identified with natural numbers so that the terminology "least locking sequence" is appropriate.)

Given τ, let σ be the least sequence such that $\sigma \le \tau$ and

1. $\text{rng}(\sigma) \subseteq \text{rng}(\tau)$,
2. for all $\gamma \le \tau$ such that

   a. $\text{rng}(\gamma) \subseteq \text{rng}(\tau)$,

   b. $\sigma \subseteq \gamma$,

   c. $\text{lh}(\gamma) \le \text{lh}(\tau)$,

   $f(\sigma) = f(\gamma)$.

σ exists since τ itself satisfies 1 and 2. Define $g(\tau) = f(\sigma)$. g is recursive since only finitely many sequences need be checked to define $g(\tau)$.

Claim If f identifies L and t is a text for L, then g converges on t to $f(\sigma)$, where σ is the least locking sequence for f and L.

Proof of claim Let σ be the least locking sequence for f and L, and let n be such that $\text{rng}(\sigma) \subseteq \text{rng}(\bar{t}_n)$ and $\sigma \le \bar{t}_n$. Since σ is a locking sequence for L and t is a text for L, it is clear that for every $m \ge n$, σ satisfies both 1 and 2 for $\tau = \bar{t}_m$. Thus for each $m \ge n$, $g(\bar{t}_m) = f(\sigma)$ unless there is $\sigma' < \sigma$ such that σ' also satisfies 1 and 2 for $\tau = \bar{t}_m$. Since no such σ' can be a locking sequence for f and L, there must be a γ such that $\gamma \supseteq \sigma'$, $\text{rng}(\gamma) \subseteq L$, and $f(\gamma) \ne f(\sigma')$. (Otherwise, either σ' would be a locking sequence for f and L or f would converge on some text for L to an index, $f(\sigma')$, for a language other than L.) But then if m is such that $\text{rng}(\gamma) \subseteq \bar{t}_m$ and $\gamma \le \bar{t}_m$, σ' cannot satisfy 1 and 2 with $\tau = \bar{t}_m$. Thus, for almost all m, g cannot conjecture $f(\sigma')$ for any $\sigma' < \sigma$. □

COROLLARY 4.6.3A For every $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$, there is a $g \in \mathscr{F}^{\text{rec}}$ that identifies $\mathscr{L}$ such that for all $L \in \text{RE}$, g identifies L if and only if there is a locking sequence for g and L. Furthermore, if g identifies L, g converges to $g(\sigma)$, where σ is the least locking sequence for g and L.

To prove the corollary, we need to modify the construction slightly.

Proof of corollary 4.6.3A Let p be a recursive padding function as supplied by lemma 4.4.1B. In the construction $g(\tau)$ was defined to be equal to $f(\sigma)$ for some σ. Modify the construction so that $g(\tau) = p(f(\sigma), \sigma)$.

Now if g identifies $L \in \text{RE}$, there is a locking sequence for g and L; this is just proposition 2.1A. So suppose conversely that $L \in \text{RE}$ and that σ is a locking sequence for g and L. This means that there is a $\sigma' \leq \sigma$ such that $g(\sigma) = p(f(\sigma'), \sigma')$, and furthermore for all γ such that $\sigma \subseteq \gamma$ and $\text{rng}(\gamma) \subseteq L$, $g(\gamma) = p(f(\sigma'), \sigma')$. Now suppose that t is any text for L. Let n be such that $\text{rng}(\sigma) \subseteq \bar{t}_n$ and $\sigma \leq \bar{t}_n$. Then $g(\bar{t}_n) = p(f(\sigma'), \sigma')$ since σ' satisfies 1 and 2 for $\tau = \bar{t}_n$, and if $\sigma'' < \sigma'$ satisfies 1 and 2, then σ'' would satisfy 1 and 2 for $\tau = \sigma$ also, contradicting $g(\sigma) = p(f(\sigma'), \sigma')$. Thus on t, g converges to $p(f(\sigma'), \sigma')$. □

Proof of proposition 4.6.3A Let $\mathscr{L} \in \mathscr{F}^{\text{rec}}$ and g be as in the proof of the corollary. Then if g identifies L, g converges on every text t for L to $g(\sigma)$, where σ is the least locking sequence for g and L. Thus g is order independent. □

We may now return to lemmata 4.3.4A and 4.3.4B whose proofs were deferred to this section.

LEMMA 4.3.4A $[\mathscr{F}^{\text{rec}}]$ is r.e. bounded if and only if $[\mathscr{F}^{\text{rec}}] = [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{prudent}}]$.

Proof Suppose first that $[\mathscr{F}^{\text{rec}}] = [\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{prudent}}]$. Let $\mathscr{L} \in [\mathscr{F}^{\text{rec}}]$. Let $\varphi \in \mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{prudent}}$ identify $\mathscr{L}$. Then $\text{rng}(\varphi)$ is an r.e. set S since φ is recursive. Since φ is prudent, φ identifies $\mathscr{L}' = \{W_i | i \in S\}$. $\mathscr{L}'$ is r.e. indexable and so witnesses that $[\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{prudent}}]$ is r.e. bounded.

Suppose, on the other hand, that $\mathscr{F}^{\text{rec}}$ is r.e. bounded. Let $\mathscr{L} \in \mathscr{F}^{\text{rec}}$ and let $\mathscr{L}' \supseteq \mathscr{L}$ be such that some $\varphi \in \mathscr{F}^{\text{rec}}$ identifies $\mathscr{L}'$ and $\mathscr{L}'$ is r.e. indexable, say by the r.e. set S. By proposition 4.6.3A, let g be a total order-independent recursive function that identifies $\mathscr{L}'' \supseteq \mathscr{L}'$. We show how to construct a prudent f that identifies $\mathscr{L}'$ (and hence $\mathscr{L}$). Let $s_0, s_1, \ldots,$ be a recursive enumeration of S. If $\text{lh}(\sigma) = 1$, define $f(\sigma) = s_0$. If $n = \text{lh}(\sigma) > 1$, for $i \leq n$, let σ^i be a sequence constructed from the elements that have been enumerated in W_{s_i} by stage n in order of enumeration. Then define

$$f(\sigma) = \begin{cases} s_i, & \text{if } i \text{ is least such that } g(\sigma^i) = g(\sigma), \\ f(\sigma^-), & \text{if there is no such } i. \end{cases}$$

Since g is order independent and identifies $\mathscr{L}'$, f will converge on any text t for $L \in \mathscr{L}'$ to s_i for the least i such that s_i is an index for L. Since every index in S is an index for a language in $\mathscr{L}'$ and f outputs only indexes from S, f is prudent. □

LEMMA 4.3.4B (Mark Fulk) $\mathscr{F}^{\text{rec}}$ is r.e. bounded.

Proof (Mark Fulk) Let $\mathscr{L}$ be in $[\mathscr{F}^{\text{rec}}]$. By corollary 4.6.3A, $\mathscr{L}$ is identifiable by some g such that g identifies L if and only if there is a locking sequence for g and L. We now give an r.e. indexable collection $\mathscr{L}'$ such that $\mathscr{L}' \in [\mathscr{F}^{\text{rec}}]$ and $\mathscr{L}' \supseteq \mathscr{L}$ thereby exhibiting that $[\mathscr{F}^{\text{rec}}]$ is r.e. bounded. There are two cases.

Case 1. Suppose that g identifies N.

Define a recursive function f by

$$W_{f(\sigma)} = \begin{cases} \varnothing, & \text{if } \text{rng}(\sigma) \not\subseteq W_{g(\sigma)}, \\ W_{g(\sigma)}, & \text{if } \sigma \text{ is a locking sequence for } g \text{ and } W_{g(\sigma)}, \\ N, & \text{otherwise.} \end{cases}$$

To see that f defined in this way is recursive, we argue informally. Given σ, to enumerate $W_{f(\sigma)}$, compute $g(\sigma)$ and enumerate nothing in $W_{f(\sigma)}$ until a stage s such that $W_{g(\sigma),s}$ contains all of $\text{rng}(\sigma)$. Then begin enumerating all of $W_{g(\sigma)}$ into $W_{f(\sigma)}$ until there is a sequence γ such that $\gamma \supseteq \sigma$, $\text{rng}(\gamma) \subseteq W_{g(\sigma)}$ and $g(\gamma) \neq g(\sigma)$. If such a γ exists, then begin enumerating all of N into $W_{f(\sigma)}$.

Since f is recursive, $S = \{f(\sigma) | \sigma \in \text{SEQ}\}$ is r.e. On the other hand, since $\varnothing$ and N are identified by g and since g identifies every L such that there is a locking sequence for g and L, g identifies every language with an index in S.

Case 2. g does not identify N.

Define a recursive function f by

$$W_{f(\sigma)} = \begin{cases} \varnothing, & \text{if } \text{rng}(\sigma) \not\subseteq W_{g(\sigma)}, \\ W_{g(\sigma)}, & \text{if } \sigma \text{ is a locking sequence for } g \text{ and } W_{g(\sigma)}, \\ \{0, 1, \ldots, y\}, & \text{where } y \text{ is the maximum element enumerated in } \\ & W_{f(\sigma)} \text{ when it is discovered that } \sigma \text{ is not a} \\ & \text{locking sequence for } W_{g(\sigma)}. \end{cases}$$

Again, it is easy to give an informal description of the algorithm for enumerating $W_{f(\sigma)}$ given σ.

The set $S = \{f(\sigma) | \sigma \in \text{SEQ}\}$ is an r.e. index set for some collection of languages, $\mathcal{L}'$ Let r be a total recursive function such that for every $\sigma \in \text{SEQ}$, $W_{r(\sigma)} = \text{rng}(\sigma)$. To see that $\mathcal{L}' \in [\mathcal{F}^{\text{rec}}]$, define $h \in \mathcal{F}^{\text{rec}}$ as follows:

$$h(\sigma) = \begin{cases} r(\sigma), & \text{if rng}(\sigma) \text{ is an initial segment of } N, \\ g(\sigma), & \text{otherwise.} \end{cases}$$

Since g does not identify N, h identifies all that g does together with all initial segments of N (cf. exercise 4.2.1H). □

Exercises

4.6.3A Show that $[\mathcal{F}^{\text{rec}} \cap \mathcal{F}^{\text{order independent}} \cap \mathcal{S}] = [\mathcal{F}^{\text{rec}} \cap \mathcal{S}]$ as $\mathcal{S}$ varies over the following strategies:

a. Nontriviality
b. Prudence
c. Consistency
d. Memory limitation
e. Confidence

***4.6.3B** (Gisela Schäfer) $\varphi \in \mathcal{F}$ is said to be *partly set driven* just in case for all σ, $\tau \in \text{SEQ}$, if $\text{lh}(\sigma) = \text{lh}(\tau)$ and $\text{rng}(\sigma) = \text{rng}(\tau)$, then $\varphi(\sigma) = \varphi(\tau)$. Prove that $[\mathcal{F}^{\text{rec}} \cap \mathcal{F}^{\text{partly set driven}}] = [\mathcal{F}^{\text{rec}}]$.

4.6.3C $t, t' \in \mathcal{T}$ are said to be *cousins* just in case

a. $\text{rng}(t) = \text{rng}(t')$,
b. there are $n, m \in N$ such that $t_{i+m} = t'_{i+n}$ for all $i \in N$.

$\varphi \in \mathcal{F}$ is called *monotonic* just in case for all $t, t' \in \mathcal{T}$, if t and t' are cousins, and φ identifies t, then φ identifies t'.

Prove that $[\mathcal{F}^{\text{monotonic}}] = [\mathcal{F}]$. (*Hint:* See exercise 4.4.2A.)

Open question 4.6.3A $[\mathcal{F}^{\text{rec}} \cap \mathcal{F}^{\text{monotonic}}] = [\mathcal{F}^{\text{rec}}]$?

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*4.7 Local and Nonlocal Strategies

There are an overwhelming number of potential learning strategies. As a consequence classificatory schemes are needed to suggest general properties of large classes of strategies. Two classificatory principles have already been advanced in the discussions of countable strategies (section 4.1) and r.e. bounded strategies (definition 4.3.4B). In addition exercise 4.7C

defines a classification of subsets of $\mathcal{F}^{\text{rec}}$. The classification provided by the titles of sections 4.2 through 4.6 might also serve as the beginning of a classificatory scheme, if it could be rendered formally precise. In the present section we offer yet another classificatory principle.

Compare the strategies of consistency (definition 4.3.3A) and confidence (definition 4.6.2A). Intuitively membership in consistency can be determined by examining a function's behavior in many "small" situations; specifically, it is sufficient to determine whether $\text{rng}(\sigma) \subseteq W_{\varphi(\sigma)}$ for every $\sigma \in \text{SEQ}$. Since SEQ is infinite, there are infinitely many situations of this nature to check; nonetheless, each such situation is "small" because each $\sigma \in \text{SEQ}$ is finite. In contrast, this kind of checking is useless for determining membership in confidence. Rather, determination of confidence requires examining a function's behavior on entire texts in order to verify convergence. In this sense consistency, but not confidence, may be termed a "local" strategy. Looked at from another perspective, a learner can "decide" to embody a given, local strategy by pursuing a policy bearing on small situations. In contrast, to embody a nonlocal strategy, the learner must arrange his or her local behavior in such a way as to conform to a more global criterion.

We now make this precise.

Definition 4.7A

i. The set $\{\varphi \in \mathcal{F} | \text{the domain of } \varphi \text{ is finite}\}$ is denoted: $\mathcal{F}^{\text{fin}}$.
ii. Let learning strategy $\mathcal{S}$ be given. $\mathcal{S}$ is called *local* just in case there is a subset F of $\mathcal{F}^{\text{fin}}$ such that for all $\varphi \in \mathcal{F}$, $\varphi \in \mathcal{S}$ if and only if $\{\psi \in \mathcal{F}^{\text{fin}} | \psi \subseteq \varphi\} \subseteq F$.

The subset F of $\mathcal{F}^{\text{fin}}$ in definition 4.7A(ii) represents the set of local examinations that enforce membership in $\mathcal{S}$.

Lemma 4.7A

i. There are $2^{\aleph_0}$ many local strategies.
ii. There are $2^{2^{\aleph_0}}$ many strategies that are not local.

Proof

i. There are $\aleph_0$ many functions in $\mathcal{F}^{\text{fin}}$. Thus there are $2^{\aleph_0}$ many subsets F of $\mathcal{F}^{\text{fin}}$. The local strategies are in one-to-one correspondence with such subsets.

ii. There are $2^{\aleph_0}$ many functions in $\mathscr{F}$ so $2^{2^{\aleph_0}}$ many subsets of $\mathscr{F}$. Thus there are $2^{2^{\aleph_0}} - 2^{\aleph_0} = 2^{2^{\aleph_0}}$ many nonlocal strategies. □

Thus "most" learning strategies are not local.

PROPOSITION 4.7A The following learning strategies are local:

i. Nontriviality
ii. Consistency
iii. 1-memory limitation
iv. Conservativism

Proof These are very easy. We will just say how to choose the F in the definition of locality.

i. $F = \mathscr{F}^{\text{fin}} \cap \mathscr{F}^{\text{nontrivial}}$.
ii. $F = \{\varphi | \text{domain of } \varphi \text{ is finite and if } \varphi(\sigma) \text{ converges, then } \text{rng}(\sigma) \subseteq W_{\varphi(\sigma)}\}$.
iii. $F = \mathscr{F}^{\text{fin}} \cap \mathscr{F}^{\text{1-memory limited}}$.
iv. $F = \mathscr{F}^{\text{fin}} \cap \mathscr{F}^{\text{conservative}}$. □

PROPOSITION 4.7B The following learning strategies are not local:

i. Computability
ii. Prudence
iii. Reliability
iv. Confidence
v. Order independence

Proof The key property of each of these strategies $\mathscr{S}$ is that for every $\varphi \in \mathscr{F}^{\text{fin}}$, there is a $\varphi' \in \mathscr{S}$ such that $\varphi \subseteq \varphi'$.

Thus for each of the strategies listed, were it to be local, the set F would equal all of $\mathscr{F}^{\text{fin}}$. But this would imply that $\mathscr{S} = \mathscr{F}$ which we know to be false for all of the strategies listed. □

Exercises

4.7A Show that memory limitation is not a local strategy.

4.7B Classify the remaining learning strategies discussed in sections 4.2 through 4.6 in terms of locality.

4.7C Let $\mathscr{S} \subseteq \mathscr{F}^{\text{rec}}$ be given. $\mathscr{S}$ is said to be *r.e. indexable* just in case $\mathscr{S} = \{\varphi_j | j \in W_i\}$ for some $i \in N$. $\mathscr{S}$ is said to have an *r.e. core* just in case there is $\mathscr{S}' \subseteq \mathscr{S}$ such that $\mathscr{S}'$ is r.e. indexable and $[\mathscr{S}'] = [\mathscr{S}]$. A strategy without r.e. core may be considered intrinsically complex, in a certain sense.

a. Show that $\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{total}}$ has an r.e. core. (*Hint:* See proposition 4.2.2A.) Conclude that there are non-r.e. indexable strategies with r.e. cores.
b. Show that not every $\mathscr{S} \subseteq \mathscr{F}^{\text{rec}}$ has an r.e. core. (*Hint:* Consider $\mathscr{F}^{\text{rec}} \cap \mathscr{F}^{\text{nontrivial}}$; see section 4.3.2.)