learning process. Throughout this chapter we have been mainly interested in convergence to the actual world on the basis of infinite data streams. In the setting of positive, sound, and complete data streams we have exhibited that conditioning and lexicographic revision generate universal learning methods. Minimal revision fails to be universal, and the crucial property that makes it weaker is its strong conservatism. Moreover, we have shown that the full power of learning cannot be achieved when the underlying prior plausibility assignment is assumed to be well-founded. In the case of positive and negative information, both conditioning and lexicographic revision are universal. Minimal revision again is not. Finally, in the setting of fair streams (containing a finite number of errors that all get corrected later in the stream) lexicographic revision again turns out to be universal. Both conditioning and minimal revision lack the 'error-correcting' property.

Future work consists in multi-level investigation of the relationship between learning theory, belief revision, and dynamic epistemic logic. There surely are many links still to be found, with interesting results for everyone involved. What seems to be especially interesting is the multi-agent extension of our results. In terms of the efficiency of convergence it would enrich the multi-agent approach to information flow, an interesting subject for epistemic and doxastic logic. The interactive aspect would probably be appreciated in formal learning theory, where the single-agent perspective is clearly dominating. Another way to extend the framework is to allow revision with more complex formulae. This would perhaps link to the AGM approach, and to the philosophical investigation into the process of scientific inquiry, where possible realities have a more 'theoretical' character.

Chapter 5
Epistemic Characterizations of Identifiability

In this chapter we will further investigate the connection between formal learning theory and modal-temporal logics of belief change. We will again focus on the language-learning paradigm, in which languages are treated as sets of positive integers.[1] In the previous chapter we focused on the semantic analysis of identifiability in the limit. Now, we will devote more attention to the syntactic counterparts of our logical approach to identifiability, focusing on both finite identifiability and identifiability in the limit. We will show how the previously chosen semantics can be reflected in an appropriate syntax for knowledge, belief, and their changes over time. The corresponding notions of learning theory and dynamic epistemic logic are given in Chapter 2.

Our approach to inductive learning in the context of dynamic epistemic and epistemic temporal logic is as follows. As in the previous chapter, we take the initial class of sets to be possible worlds in an epistemic model, which mirrors Learner's initial uncertainty over the range of sets. The incoming pieces of information are taken to be events that modify the initial model. We will show that iterated update on epistemic models based on finitely identifiable classes of sets is bound to lead to the emergence of irrevocable knowledge. In a similar way identifiability in the limit leads to the emergence of stable belief. Next, we observe that the structure resulting from updating the model with a sequence of events can be viewed as an epistemic temporal forest. We explicitly focus on protocols that are assigned to worlds in set-learning scenarios. We give a temporal characterization of forests that are generated from learning situations of finite identifiability and identifiability in the limit. We observe that a special case of this protocol-based setting, in which only one stream of events is allowed in each state, can be used to model the function-learning paradigm. We show that the simple setting of iterated epistemic update cannot account for all possible learning situations. In

[1] In this chapter we are concerned with logical characterizations of learning, hence we will often refer to *languages* of certain logics. To avoid confusion for the time being we will replace the name *language learning* with *set learning* (see Section 2.1).

the end we conclude our considerations and present possible directions of further work.

5.1 Learning and Dynamic Epistemic Logic

Following our observations about the power of the conditioning revision method (Chapter 4) we will still be concerned with epistemic update. To recall the idea let us consider some simple examples of single-agent propositional update.

Example 5.1.1. *Let us take a single-agent epistemic model* $\mathcal{M} = \langle W, \sim, V\rangle$, *where* $W = \{w_1, w_2, w_3\}$, $\sim = W \times W$, $\text{PROP} = \{p_1, p_2, p_3, p_4\}$ *and the valuation* $V : \text{PROP} \to \mathcal{P}(W)$ *is defined in the following way* $V(p_1) = \{w_1, w_2, w_3\}$, $V(p_2) = \{w_1, w_2\}$, $V(p_3) = \{w_2, w_3\}$, $V(p_4) = \{w_3\}$, *in other words:* $w_1 \models p_1 \wedge p_2 \wedge \neg p_3 \wedge \neg p_4$, $w_2 \models p_1 \wedge p_2 \wedge p_3 \wedge \neg p_4$, *and* $w_3 \models p_1 \wedge p_3 \wedge p_4 \wedge \neg p_2$. *Let us assume that* w_2 *is the actual world, and that the agent receives propositional information that is consistent with* w_2 *in the following order:* p_1, p_2, p_3. *Receiving* p_1 *does not change anything—every world satisfies* p_1. *Then* p_2 *comes in, eliminating* w_3, *since* $w_3 \not\models p_2$. *The agent is now uncertain only between* w_1 *and* w_2. *The last information* p_3 *allows deleting* w_1 *because* $w_1 \not\models p_3$. *The uncertainty of the agent now disappears—the only possibility left is* w_2. *Moreover, whatever true (consistent with the actual world* w_2*) information comes in,* w_2 *cannot be eliminated.*

In fact, if any of the worlds is the actual one, and the agent will receive truthful and complete propositional information about it, he will be able to eventually eliminate all other worlds, and therefore gain full certainty about his situation.

Example 5.1.2. *Let us again take a similar epistemic model, this time with the following valuation* $V(p_1) = \{w_1, w_2, w_3\}$, $V(p_2) = \{w_1, w_2\}$, $V(p_3) = \{w_1\}$. *Now, only one world, namely* w_1, *can get identified by receiving truthful and complete propositional information. In case* w_2 *(or* w_3*) is the actual world, the agent will never be able to eliminate* w_1 *(or* w_1 *and* w_2*), and therefore the uncertainty will always remain.*

5.1.1 Dynamic Epistemic Learning Scenarios

In Examples 5.1.1 and 5.1.2, the uncertainty range of the agent is revised as new pieces of data (in the form of propositions) are received. The information comes from a completely trusted source, and as such causes the agents to eliminate the worlds that do not satisfy it. In learning theory it is common to assume the truthfulness of incoming data, and therefore, in principle, it is justified to use epistemic update as a way to perform the inquiry (for such interpretation of update see Van Benthem, 2006). It is important to note that public announcement is not the main notion of dynamic epistemic logic. Our update-based approach to learning gives the first connection, but dynamic epistemic logic can typically also deal with varieties of 'soft information' that is less trusted (see Section 2.2).

In this section we will present single-agent learning scenarios in the framework of doxastic epistemic logic. We base our investigations on the learning-theoretic framework defined in Section 2.1.

First, the initial learning model is a simple epistemic model whose worlds correspond to the initial class of sets.

Definition 5.1.3. *Let* $\mathcal{C} = \{S_1, S_2, \ldots\}$ *be a class of sets such that for all* $i \in \mathbb{N}$, $S_i \subseteq \mathbb{N}$. *Our* initial learning model $\mathcal{M}_\mathcal{C}$ *is a triple:*

$$\langle W_\mathcal{C}, \sim, V_\mathcal{C}\rangle,$$

where $W_\mathcal{C} = \mathcal{C}$, $\sim = W_\mathcal{C} \times W_\mathcal{C}$, $V_\mathcal{C} : \text{PROP} \cup \text{NOM} \to \mathcal{P}(W_\mathcal{C})$, *such that* $S_i \in V_\mathcal{C}(p_n)$ *iff* $n \in S_i$ *and for each set* $S_i \in \mathcal{C}$, *we take a nominal* $i \in \text{NOM}$ *and we set* $V_\mathcal{C}(i) = \{S_i\}$.

In words, we identify states of the model with sets, we also assume that our agent does not have any particular initial information or preference over the possibilities. The interpretation of the propositional letters is as follows. Let $\mathcal{C} = \{S_1, S_2, \ldots\}$ be a class of sets, and let $U = \bigcup \mathcal{C}$ be the universal set of $\mathcal{C}$. For every piece of data $n \in U$ we take a propositional letter p_n. The nominals correspond to indices of sets. They can be interpreted as finite descriptions of sets or as theories that describe possible sequences of events.

In the previous chapter we analyzed our central topic of *iterated* update. The definitions of data streams, data sequences and related notions remain the same for this chapter. We will be concerned with sound and complete data streams (see Section 4.1).

5.1.2 Finite Identification in DEL

The research in dynamic epistemic and dynamic doxastic logic often touches the subject of converging to some desired states: (common) knowledge or (joint) true belief (see, e.g., Baltag & Smets, 2009a). In this respect it is concerned with multi-agent versions of the belief-revision problem. In this section we will show how to use the notion of finite identification to characterize convergence to irrevocable knowledge. To establish the first connection we will restrict ourselves to the single-agent case.

Definition 5.1.4. *Iterated epistemic update of model* $\mathcal{M}$ *with an infinite data stream* ε stabilizes *to* $\mathcal{M}'$ *iff there is an* $n \in \mathbb{N}$, *such that for all* $m \geq n$, $\mathcal{M}^{\varepsilon|m} = \mathcal{M}'$. *In such cases we will sometimes write that the generated epistemic model* $\mathcal{M}^\varepsilon$ *stabilizes to* $\mathcal{M}'$.

In our considerations we will use the characterization of finite identifiability of sets from positive data (Mukouchi, 1992). First we recall the notion of the finite definite tell-tale set.

Definition 5.1.5 (Mukouchi 1992). *A set D_i is a* definite finite tell-tale set *of $S_i \in \mathcal{C}$ if*

1. $D_i \subseteq S_i$,
2. *D_i is finite, and*
3. *for any index j, if $D_i \subseteq S_j$ then $S_i = S_j$.*

The non-computable case of finite identifiability can be then characterized in the following way.

Theorem 5.1.6 (Mukouchi 1992). *A class $\mathcal{C}$ is finitely identifiable from positive data if and only if for every set $S_i \in \mathcal{C}$ there is a definite finite tell-tale set D_i.*

We are now ready to show that epistemic update performed on finitely identifiable class of sets leads to irrevocable knowledge.

Theorem 5.1.7. *The following are equivalent:*

1. *$\mathcal{C}$ is finitely identifiable.*
2. *For every $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\sim' = \{(S_i, S_i)\}$.*

Proof. The proof of this assertion consists mainly in understanding our earlier semantic definitions and arguments, and seeing that they conform to a simple syntactic pattern definable in epistemic logic. Nevertheless, for once, we add some explicit formal detail to show how this works.

($1 \Rightarrow 2$) Let us assume that $\mathcal{C}$ is finitely identifiable. Then, by Theorem 5.1.6, for every set $S_i \in \mathcal{C}$ there is a finite definite tell-tale set $D_i \subseteq S_i$ such that D_i is not a subset of any other set in $\mathcal{C}$. Let us then take one S_i and the corresponding finite definite tell-tale set D_i. For every data stream ε for S_i there is a finite initial segment, $\varepsilon\restriction m$, such that $D_i \subseteq \mathrm{set}(\varepsilon\restriction m)$. Then by stage m every S_j such that $i \neq j$ has been eliminated by the update.

($2 \Rightarrow 1$) Let us assume that for every $S_i \in W_{\mathcal{C}}$ and a data stream ε for S_i, the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = S_i$ and $\sim' = \{(S_i, S_i)\}$. Assume that $\mathcal{C}$ is not finitely identifiable. Therefore, by Theorem 5.1.6, there is a set $S_i \in \mathcal{C}$ such that every finite subset of S_i is included in some $S_j \in \mathcal{C}$ such that $i \neq j$. Then for all n, if $\mathcal{M}^{\varepsilon\restriction n}_{\mathcal{C}} = \langle W^{\varepsilon\restriction n}_{\mathcal{C}}, \sim^{\varepsilon\restriction n}, V^{\varepsilon\restriction n}_{\mathcal{C}}\rangle$ then $\{S_i, S_j\} \subseteq W^{\varepsilon\restriction n}_{\mathcal{C}}$, so $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ clearly does not stabilize to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\sim' = \{(S_i, S_i)\}$. Contradiction. □

With respect to the language of epistemic logic $\mathcal{L}_{\mathrm{EL}}$ given in Definition 2.2.3, the following corollary corresponds to the semantic characterization in Theorem 5.1.7.

Corollary 5.1.8. *The following are equivalent:*

1. *$\mathcal{C}$ is finitely identifiable.*
2. *For every $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\mathcal{M}'_{\mathcal{C}}, S_i \models K\,i$.*

Proof. From Theorem 5.1.7 we know that 1 is equivalent to:

\# For all $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\sim' = \{(S_i, S_i)\}$.

($\# \Rightarrow 2$) Let us take $S_i \in W_{\mathcal{C}}$ and data stream ε for S_i and assume that the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\sim' = \{(S_i, S_i)\}$. Then, by definition of the semantics of $\mathcal{L}_{EL}$, $\mathcal{M}', S_i \models K\,i$, since it is true that for all $S_j \in \mathcal{K}[S_i]$, we have that $\mathcal{M}'_{\mathcal{C}}, S_j \models i$.

($2 \Rightarrow \#$) Let us assume that for every $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\mathcal{M}'_{\mathcal{C}} = \langle W'_{\mathcal{C}}, \sim', V_{\mathcal{C}}\rangle$, where $W'_{\mathcal{C}} = \{S_i\}$ and $\mathcal{M}'_{\mathcal{C}}, S_i \models K\,i$. This means that for all $S_j \in \mathcal{K}[S_i]$ we have that $\mathcal{M}'_{\mathcal{C}}, S_j \models i$. But from definition of the valuation $V_{\mathcal{C}}$ we know that S_i is the only state in $W_{\mathcal{C}}$ that validates i. Therefore $\sim' = \{(S_i, S_i)\}$. □

The above results provide a characterization of the outcome of finite identification in the simple language of epistemic logic. To incorporate more dynamics into the syntactic counterpart of finite learning we can use the language $\mathcal{L}_{\mathrm{PAL}}$.[2]

Corollary 5.1.9. *The following are equivalent:*

1. *$\mathcal{C}$ is finitely identifiable.*
2. *For every $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i there is an $n \in \mathbb{N}$ such that for all $m \geq n$, $\mathcal{M}_{\mathcal{C}}, S_i \models [!(\bigwedge \mathrm{set}(\varepsilon\restriction m))]\,K\,i$.*

Proof. This equivalence follows directly from Corollary 5.1.8 and the semantics of $\mathcal{L}_{\mathrm{PAL}}$, Definition 2.2.8. □

[2]To bring out the future behavior more explicitly in our syntax, we could also formulate this result in terms of *repeated announcement*, in a version of public announcement logic that also allows Kleene star. We forego such extensions here.

5.1.3 Identification in the Limit and DDL

In Chapter 4 we extensively discussed the interrelation between identifiability in the limit, update (conditioning), and the notion of belief. Now, on the basis of those results we can give the following corollary.

Corollary 5.1.10. *The following are equivalent:*

1. *$\mathcal{C}$ is identifiable in the limit.*
2. *There is a plausibility preorder $\leq \subseteq W_{\mathcal{C}} \times W_{\mathcal{C}}$ such that for every $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i in the generated epistemic model $\mathcal{M}^{\varepsilon}_{\mathcal{C}}$, $\min_{\leq} W^{\varepsilon}_{\mathcal{C}}$ stabilizes to $\{S_i\}$.*
3. *There exists a plausibility preorder $\leq \subseteq W_{\mathcal{C}} \times W_{\mathcal{C}}$ such that for every $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i there is $n \in \mathbb{N}$ such that for all $m \geq n$, $(\mathcal{M}_{\mathcal{C}}, \leq), S_i \models [!(\bigwedge \text{set}(\varepsilon \restriction m))]B\, i$.*

Clause 3 gives a characterization in public announcement logic with the operator of absolute belief.[3] The plausibility order used in the above corollary is defined and discussed in Section 4.5. It is based on the characterization of identification in the limit and the concept of the finite tell-tale set (see Section 2.1).

Let us additionally note that the last clauses of Corollaries 5.1.8, 5.1.9, and 5.1.10 describe the *persistence* of the relevant doxastic-epistemic states. In fact, under the conditions of update, it is the case that as soon as the desired doxastic-epistemic state is reached it cannot be lost later in the process.

Until now we have shown how to model learning scenarios in dynamic epistemic and doxastic logic. In order to explicitly express the possibility of convergence as a temporal property, we will view the structure generated by iterated epistemic update as a temporal branching model. In this we follow the recently established bridge between dynamic epistemic and epistemic temporal logic (see Van Benthem et al., 2009).

5.2 Learning and Temporal Logic

We have shown how basic results connecting dynamic epistemic logic and learning theory can be given syntactic formulations in terms of the K and B operators. However, we are still missing a crucial dimension, the *temporal* one. Implicitly, we already considered the temporal aspects, since in fact the knowledge and beliefs stabilized only after some finite sequences of announcements or other

[3] Given the reduction axioms of dynamic doxastic logic for 'factual formulae', such as the nominal i, an equivalent formulation would be: There is a plausibility preorder $\leq \subseteq W_{\mathcal{C}} \times W_{\mathcal{C}}$ such that for every $S_i \in W_{\mathcal{C}}$ and every data stream ε for S_i there is $n \in \mathbb{N}$ such that for all $m \geq n$, $(\mathcal{M}_{\mathcal{C}}, \leq), S_i \models B^{(\bigwedge \text{set}(\varepsilon \restriction m))}\, i$.

informative events. This long-term aspect could be formalized in extensions of public announcement logic with program operations, in particular, Kleene iteration. While this seems an interesting line to pursue, we feel that this still does not do justice to another striking logical feature of learning theory: its resemblance to *temporal logics*. In what follows, we will show how to establish the connection, taking advantage of some recent developments that have linked dynamic epistemic logic to epistemic temporal logics, via the crucial notion of a *protocol*.

To make this connection, we need to turn to the more general version of DEL based on *event models* and *product update* (Baltag et al., 1998). We will just give the absolute basics here, referring mainly to the literature.

5.2.1 Event Models and Product Update

Iterated update can be placed in a more general perspective. Obviously, the incoming information does not have to be propositional. It does not even have to be purely linguistic. It can be any *event* that itself has an epistemic structure. To consider changes caused by such arbitrary events, we will now introduce the notion of event model, which represents the epistemic and informational content of what 'happens'.

Definition 5.2.1. *An* event model *is a triple:*

$$\mathcal{E} = \langle E, (\sim^{\mathcal{E}}_a)_{a \in \mathcal{A}}, \mathtt{pre} \rangle,$$

where $E \neq \emptyset$ is a set of events; for every agent $a \in \mathcal{A}$, $\sim^{\mathcal{E}}_a$ is a binary equivalence relation on E, and $\mathtt{pre} : E \to \mathcal{L}_{EL}$, is a precondition function where $\mathcal{L}_{EL}$ is a set of formulae of some epistemic language. A pair $(\mathcal{E}, e)$, where $e \in E$ is called a pointed event model.

For every agent $a \in \mathcal{A}$, the relation $\sim^{\mathcal{E}}_a$ encodes that agent's epistemic information about the event taking place. The precondition function pre maps events to epistemic formulae. An event will be executable in some state only if that state satisfies the precondition of this event.

The effect of updating an epistemic model $\mathcal{M}$ with an event model $\mathcal{E}$ can be computed according to the *product update*.

Definition 5.2.2. *Let $\mathcal{M} = \langle W, (\sim_a)_{a \in \mathcal{A}}, V \rangle$ be an epistemic model and $\mathcal{E} = \langle E, (\sim^{\mathcal{E}}_a)_{a \in \mathcal{A}}, \mathtt{pre} \rangle$ be an event model. The* product update *of $\mathcal{M}$ with $\mathcal{E}$ gives a new epistemic model $\mathcal{M} \otimes \mathcal{E} = \langle W', (\sim'_a)_{a \in \mathcal{A}}, V' \rangle\}$, where:*

1. *$W' = \{(w, e) \mid w \in W \;\&\; e \in E \;\&\; w \models \mathtt{pre}(e)\}$;*
2. *$(w, e) \sim'_a (w', e')$ iff $w \sim_a w'$ and $e \sim^{\mathcal{E}}_a e'$;*
3. *and the valuation is as follows: $(w, e) \in V'(p)$ iff $w \in V(p)$.*

Illustrations of the strength of product update can be found in (Baltag & Moss, 2004; Van Benthem, 2010; Van Benthem & Dégremont, 2010; Dégremont, 2010).

5.2.2 Dynamic Epistemic Logic Protocols

By making a step from dynamic epistemic logic into epistemic temporal logic we can analyze the temporal structure of update. Redefining the iterated epistemic update in terms of protocols (see Fagin, Halpern, Moses, & Vardi, 1995; Parikh & Ramanujam, 2003) will bring us closer to the temporal setting. A protocol specifies sequences of events that are admissible in certain epistemic situations. In this section, following Van Benthem et al. (2009), we will give the definition of local protocols and epistemic models generated with respect to a protocol. By doing this we prepare the grounds for our learning-theoretic setting.

A protocol P maps states in an epistemic model to sets of finite and infinite sequences of event models closed under taking prefixes. It defines the admissible runs of some informational process: In general, not every sequence of events may be possible at a given state.

Let $\mathbb{E}$ be the class of all event models. Every state of the epistemic model is assigned a set of sequences (infinite and finite) of event models closed under taking finite prefixes, an element of the set

$$\mathrm{Prot}(\mathbb{E}) = \{\mathrm{P} \subseteq \mathcal{P}(\mathbb{E}^* \cup \mathbb{E}^\omega) \mid \mathrm{P} \text{ is closed under finite prefixes}\}.$$

Definition 5.2.3. *Let us take an epistemic model $\mathcal{M} = \langle W, (\sim_a)_{a\in\mathcal{A}}, V\rangle$. A* local protocol *for $\mathcal{M}$ is a function $P : W \to \mathrm{Prot}(\mathbb{E})$.*

Until now we have been concerned with the $\varepsilon{\restriction}n$-generated epistemic model $\mathcal{M}$, where $\varepsilon{\restriction}n$ is some sequence of propositions. We will now provide an analogous notion of a model generated from a sequence of event models but according to some specific local protocol.

Definition 5.2.4. *Let $\mathcal{M} = \langle W, (\sim_a)_{a\in\mathcal{A}}, V\rangle$ be an epistemic model. We define the* $(P, \varepsilon{\restriction}n)$-generated epistemic model $\mathcal{M}^{P,\varepsilon\restriction n}$ *inductively, as follows:*

$$\begin{aligned}
\mathcal{M}^{P,\varepsilon\restriction 0} &= \mathcal{M}\\
\mathcal{M}^{P,\varepsilon\restriction n+1} &= \langle W^{P,\varepsilon\restriction n+1}, \sim^{P,\varepsilon\restriction n+1}, V^{P,\varepsilon\restriction n+1}\rangle, \text{ where:}\\
& W^{P,\varepsilon\restriction n+1} := \{s \mid s \in W^{P,\varepsilon\restriction n}; s \models \mathtt{pre}(\varepsilon_{n+1}) \;\&\; \varepsilon{\restriction}n+1 \in P(s)\};\\
& \sim^{P,\varepsilon\restriction n+1} := \sim^{P,\varepsilon\restriction n}{\restriction} W^{P,\varepsilon\restriction n+1};\\
& V^{P,\varepsilon\restriction n+1} := V^{P,\varepsilon\restriction n}{\restriction} W^{P,\varepsilon\restriction n+1}.
\end{aligned}$$

The protocol-based approach to update has a straightforward temporal interpretation. The question is how iterated product update can be interpreted in epistemic temporal logics, which are widely used to study the evolution of a system over time focusing on the information that agents possess. And this perspective is exactly what we need.

5.2.3 Dynamic Epistemic and Epistemic Temporal Logic

Epistemic temporal logics are interpreted on epistemic temporal forests (see, e.g., Parikh & Ramanujam, 2003).

Definition 5.2.5. An epistemic temporal model $\mathcal{H}$ *is a tuple:*

$$\langle W, \Sigma, H, (\sim_a)_{a\in\mathcal{A}}, V\rangle,$$

where $W \neq \emptyset$ is a countable set of initial states; Σ is a countable set of events; $H \subseteq W\Sigma^*$ is a set of histories (sequences of events starting at states from W) closed under non-empty finite prefixes; for each $a \in A$, $\sim_a \subseteq H \times H$ is an equivalence relation; and V : $\textsc{Prop} \to \mathcal{P}(H)$ is a valuation. We write wh to denote some finite history starting in the state w.

We sometimes refer to the $\langle W, \Sigma, H\rangle$-part of an ETL model as the *temporal protocol* this model is based on. We refer to the information of an agent a at h with $\mathcal{K}_a[wh] = \{vh' \in H \mid wh \sim_a vh'\}$.

The question is now how to make the step from dynamic epistemic logic to epistemic temporal logic. The relation between the two frameworks has already been studied (see, e.g., Van Benthem & Liu, 2004; Van Benthem & Pacuit, 2006). In particular, it has been observed that iterated epistemic update in dynamic epistemic logic generates epistemic temporal forests satisfying certain properties (see Van Benthem et al., 2009). We will refer to this construction by $\mathtt{For}(\mathcal{M}, P)$ and define it below.

We construct the forest by induction, starting with the epistemic model and and then checking which events can be executed according to the precondition function and to the protocol. Finally, the new information partition is updated at each stage according to the product update. Since product update describes purely epistemic change, the valuation stays the same as in the initial model.

Definition 5.2.6 (ETL forest generated by a DEL protocol). *Each epistemic model $\mathcal{M} = \langle W, (\sim_a^{\mathcal{M}})_{a\in\mathcal{A}}, V^{\mathcal{M}}\rangle$ and a local protocol $P : W \to \mathrm{Prot}(\mathbb{E})$ generates an ETL forest $\mathtt{For}(\mathcal{M}, P)$ of the form:*

$$\mathcal{H} = \langle W^{\mathcal{H}}, \mathbb{E}, H, (\sim_a)_{a\in\mathcal{A}}, V\rangle, \text{ where:}$$

1. *$W^{\mathcal{H}} := W$;*
2. *H is defined inductively as follows:*

   $H_0 := W^{\mathcal{H}}$;
   $H_{n+1} := \{(we_1 \ldots e_{n+1}) \mid (we_1 \ldots e_n) \in H_n; \mathcal{M}^{\varepsilon\restriction n}, w \models \mathtt{pre}(e_{n+1})$ *and* $(e_1 \ldots e_{n+1}) \in P(w)\}$;
   $H := \bigcup_{0\leq k<\omega} H_k$;

3. *If $w, v \in W^{\mathcal{H}}$, then $w \sim_a v$ iff $w \sim_a^{\mathcal{M}} v$;*
4. *$whe \sim_a vh'e'$ iff $whe, vh'e' \in H_k$, $wh \sim_a vh'$, e and e' are states in an event model $\mathcal{E}$ and $e \sim_a^{\mathcal{E}} e'$;*

5. *Finally,* $wh \in V(p)$ *iff* $w \in V^{\mathcal{M}}(p)$.

The correspondence between the iterated product update and an epistemic temporal forest relies on some properties of epistemic temporal agents. To be precise, it has been shown that the structures of iterated DEL update are in fact epistemic temporal forests that satisfy the following conditions: perfect recall, synchronicity, uniform no miracles and propositional stability. Let us introduce those epistemic multi-agent assumptions.

Definition 5.2.7. *Let us take* $\mathcal{H} = \langle W, \Sigma, H, (\sim_a)_{a\in\mathcal{A}}, V\rangle$ *to be an epistemic temporal model.*

Perfect Recall $\mathcal{H}$ *satisfies perfect recall iff*

$$\text{for all } whe, vh'f \in H \text{ if } \mathcal{K}_a[whe] = \mathcal{K}_a[vh'f], \text{ then } \mathcal{K}_a[wh] = \mathcal{K}_a[vh'].$$

The condition of perfect recall expresses that agents do not forget past information as further events take place.

Synchronicity $\mathcal{H}$ *satisfies synchronicity iff*

$$\text{for all } wh, vh' \in H \text{ if } \mathcal{K}_a[wh] = \mathcal{K}_b[vh'], \text{ then } \mathrm{length}[wh] = \mathrm{length}[vh'].$$

Synchronicity is satisfied if the agents have access to some external discrete clock and thus can keep track of the time.

Uniform-No-Miracles $\mathcal{H}$ *satisfies uniform no miracles iff*

$$\text{for all } wh, vh' \in H \text{ such that } wh \sim_a vh'$$
$$\text{and for all } e_1, e_2 \in \Sigma \text{ with } whe_1, vh'e_2 \in H$$
$$\text{if there are } sh'', th''' \in H \text{ such that } sh''e_1 \sim_a th'''e_2, \text{ then } whe_1 \sim_a vh'e_2.$$

Uniform-No-Miracles means that if an agent cannot distinguish between a history terminating with e_1 and a history whose last event is e_2, then at any time if he is unable to distinguish between two histories wh and vh' then he is still unable to distinguish between whe_1 and vhe_2. This property characterizes local 'updaters' that do not take into account the whole history but that proceed in a step-by-step manner.

Propositional Stability $\mathcal{H}$ *satisfies propositional stability iff for all* $wh, whe \in H$ *we have* $p \in V(whe)$ *iff* $p \in V(wh)$.

The following result says that the iterated product update of an epistemic model $\mathcal{M}$ according to a protocol P generates an epistemic temporal forest that validates the above-mentioned epistemic properties.

Theorem 5.2.8 (Van Benthem et al. 2009). *An ETL-model* $\mathcal{H}$ *is isomorphic to the forest generated by the sequential product update of an epistemic model according to some state-dependent DEL-protocol iff it satisfies perfect recall, synchronicity, uniform-no-miracles and propositional stability.*

5.2.4 Learning in a Temporal Perspective

Let us now see how the above construction can be used to analyze learning scenarios.

Learning Event Models In our learning setting the incoming information has a purely propositional character. A simple *event learning model* can be obviously associated with every such piece of data in the following way.

Definition 5.2.9. *Let* $\mathcal{C} = \{S_1, S_2, \ldots\}$ *be a class of sets and, as before,* $U = \bigcup \mathcal{C}$ *is the universal set of* $\mathcal{C}$. *Let* $\mathtt{E} : \mathbb{N} \to \mathbb{E}$ *be a function that transforms an integer into an event model in the following way: for each* $n \in \mathbb{N}$, $\mathtt{E}(n) = \mathcal{E}_n = \langle\{e\}, \sim^{\mathcal{E}_n}, \mathtt{pre}_{\mathcal{E}}\rangle$, *where* $\sim = \{(e,e)\}$ *and* $\mathtt{pre}_{\mathcal{E}}(e) = p_n$. *Similarly, if* $S \subseteq \mathbb{N}$, $\mathtt{E}(S) = \{\mathtt{E}(n) \mid n \in S\}$.

In other words, for every piece of data n from U we take a propositional letter p_n. Then for each p_n we take a simple public announcement event model. By making the conceptual transition from the simple propositional update to the event models we want to show that our framework conforms to the general setting described in the previous section.

Local Set-Learning Protocol Intuitively, given a state $S_i \in W_{\mathcal{C}}$, our protocol P should authorize at S_i any ω-sequence that enumerates S_i and nothing more. Our set-learning scenarios allow any enumeration of elements of a given set. Therefore, the corresponding local protocol can be defined in the following way.

Definition 5.2.10. *Let* $\mathcal{C} = \{S_1, S_2, \ldots\}$ *be a class of sets and* $U = \bigcup \mathcal{C}$ *be the universal set of* $\mathcal{C}$. *For every* $S_i \in W_{\mathcal{C}}$, *the* set-learning local protocol, $P(S_i)$, *is the smallest subset of* $(\mathtt{E}(U))^{\omega}$ *that contains:*

$$\{f : \omega \to \mathtt{E}(S_i) \mid f \text{ is surjective}\},$$

and that is closed under non-empty finite prefixes.

Set-learning local protocols restrict the admissible sequences of events only in terms of content and not in terms of ordering. It is easy to observe that such a local protocol can replace the sets in learning scenarios. In principle we can then skip the precondition check and instead decide whether an event can take place just on the basis of the protocols. We will return to this issue in the end of this chapter.

To sum up we will now complement our definition of the initial learning model (Definition 5.1.3) with the local set-learning protocol.

Definition 5.2.11. *Let* $\mathcal{C} = \{S_1, S_2, \ldots\}$ *be a class of sets such that for all* $i \in \mathbb{N}$, $S_i \subseteq \mathbb{N}$. *The* initial learning model with local protocol *consists of:*

1. *an epistemic model* $\mathcal{M}_C = \langle W_C, \sim, V_C \rangle$, *where* $W_C = \mathcal{C}$, $\sim\, = W_C \times W_C$, $V_C : \textsc{Prop} \cup \textsc{nom} \to \mathcal{P}(W_C)$, *such that* $S_i \in V_C(p_n)$ *iff* $n \in S_i$ *and for each set* $S_i \in \mathcal{C}$, *we take a nominal* i *and we set* $V_C(i) = \{S_i\}$.

2. *for each* $S_i \in W_C$, *a set-learning local protocol* $P(S_i)$.

Now we are ready to define how our initial learning model and the local set-learning protocol generate an epistemic temporal forest. We define the additional set of designated propositional letters based on the previously used set of nominals NOM, $\textsc{Prop}_{\textsc{nom}} := \{q_i \mid i \in \textsc{nom}\}$, and we assume that $\textsc{Prop}_{\textsc{nom}} \subseteq \textsc{Prop}$.

Definition 5.2.12 (Epistemic Temporal Learning Forest). *A learning model* $\mathcal{M}_C = \langle W_C, \sim^{\mathcal{M}}, V_C^{\mathcal{M}} \rangle$ *together with the local set-learning protocol* $P : W \to$ Prot(𝔼) *generates an ETL forest* For$(\mathcal{M}, P)$ *of the form:*

$$\mathcal{H} = \langle W^{\mathcal{H}}, \mathbb{E}, H, \sim, V \rangle, \text{ where:}$$

1. $W^{\mathcal{H}} := W_C$,

2. *H is defined inductively as follows:*

$H_0 := W^{\mathcal{H}}$;
$H_{n+1} := \{(we_1 \dots e_{n+1}) \mid (we_1 \dots e_n) \in H_n;\ \mathcal{M}_C^{P,\epsilon|n}, w \models \mathrm{pre}(e_{n+1})$ *and* $(e_1 \dots e_{n+1}) \in P(w)\}$;
$H := \bigcup_{0 \le k < \omega} H_k$;

3. *If* $w, v \in W^{\mathcal{H}}$, *then* $w \sim v$ *iff* $w \sim^{\mathcal{M}} v$;

4. $whe \sim_a vh'e'$ *iff* $whe, vh'e' \in H_k$, $wh \sim vh'$, *and* $e = e'$;

5. *Finally, the valuation* $V : \textsc{Prop} \cup \textsc{Prop}_{\textsc{nom}} \to \mathcal{P}(H)$ *is defined in the following way:*

   - *for every* $p \in \textsc{Prop}$, $wh \in V(p)$ *iff* $w \in V_C^{\mathcal{M}}(p)$;
   - *for every* $q_i \in \textsc{Prop}_{\textsc{nom}}$, $wh \in V(q_i)$ *iff* $w \in V_C^{\mathcal{M}}(i)$.

The above construction is in strict correspondence with the general case of generated epistemic temporal forest of Definition 5.2.6. Our concept allows a slight simplification in point 4 because of the very simple structure of our public announcement events.

At this point we have the temporal structures that correspond to the learning situation. The next step is to give a temporal characterization of forests that satisfy the identifiability condition.

5.2.5 Finite Identifiability in ETL

In this section we will give a general characterization of finite identification in the language of an epistemic temporal logic (see Emerson & Halpern, 1986; Fagin et al., 1995; Parikh & Ramanujam, 2003). The aim of this section is to give a formula of epistemic temporal logic that characterizes learnable classes of sets.

Epistemic Temporal Language

Syntax The syntax of our epistemic temporal language $\mathcal{L}_{\mathrm{ETL}^*}$ is defined in the following way.

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid K\varphi \mid F\varphi \mid A\varphi$$

where p ranges over a countable set of proposition letters PROP. $K\varphi$ reads: 'the agent knows that φ'. Symbol F stands for future, and we define G to mean $\neg F \neg$. $A\varphi$ means: 'in all infinite continuations conforming to the protocol, φ holds'.

Semantics $\mathcal{L}_{\mathrm{ETL}^*}$ is interpreted over epistemic temporal frames, $\mathcal{H}$, and pairs of the form (ε, h), the former being a maximal, infinite history in our trees, and the latter a finite prefix of ε (see Van der Meyden & Wong, 2003; Parikh & Ramanujam, 2003).

Definition 5.2.13. *We give the semantics of* $\mathcal{L}_{\mathrm{ETL}^*}$. *We skip the boolean clauses. We take* $h \sqsubseteq h'$ *to mean that* h *is an initial segment of* h', *and* $p \in \textsc{Prop}$.

$\mathcal{H}, \varepsilon, wh \models p$	iff	$wh \in V(p)$
$\mathcal{H}, \varepsilon, wh \models K\varphi$	iff	*for all* ε', vh' *if* $vh' \in K[wh]$ *then* $\mathcal{H}, \varepsilon', vh' \models \varphi$
$\mathcal{H}, \varepsilon, wh \models F\varphi$	iff	*there is* $\sigma \in \Sigma^*$ s.t. $wh' = wh\sigma$ and $\mathcal{H}, \varepsilon, wh' \models \varphi$
$\mathcal{H}, \varepsilon, wh \models A\varphi$	iff	*for all* $\varepsilon' \in P(w)$ such that $wh \sqsubset \varepsilon'$ we have $\mathcal{H}, \varepsilon', wh \models \varphi$

The modality 'A' refers to the particular infinite sequences that belong to the chosen protocol associated to w. It can be viewed as an operator that performs a global update on the overall temporal structure, 'accepting' only those infinite histories that conform to the protocol.

To give a temporal characterization of finite identifiability we need to express the following idea. In our epistemic temporal forest, for any starting, bottom node S_i it is the case that for all branches in the future there will be a point after which the agent will know that he started in S_i, which means that he will remain certain about the partition of the tree he is in. The designated propositional letters from $\textsc{Prop}_{\textsc{nom}}$ correspond to the partitions, which can also be viewed as underlying theories that allow predicting further events.[4] Formally, with respect to finite identifiability of sets, the following theorem holds.

[4]The characterization involving designated propositional letters can be replaced with one that uses nominals as markers of bottom nodes. For such an approach see Dégremont & Gierasimczuk, 2009.

Theorem 5.2.14. *The following are equivalent:*

1. *$\mathcal{C}$ is finitely identifiable.*

2. *For all $S_i \in W_{\mathcal{C}}$ and $\varepsilon \in P(S_i)$ the learner's knowledge about the initial state stabilizes to S_i on ε in the generated forest* $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P)$.

3. $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P) \models q_i \to AFGKq_i$.

Proof. ($1 \Leftrightarrow 2$) This equivalence restates the earlier result (Theorem 5.1.7) in terms of epistemic temporal forests.

($2 \Leftrightarrow 3$) Let us first observe that in our generated epistemic temporal forest $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P)$ the following holds:

$$S_i h \sim S_j h' \text{ iff } S_i \sim S_j \text{ and } h = h'. \tag{5.1}$$

Now let us analyze the structure of Clause 3. $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P) \models q_i$ stands for a choice of the partition of the forest, and hence, implicitly, for the initial node S_i; then, the temporal prefix AF stands for: 'on every infinite continuation of S_i consistent with the protocol there is a point'. Hence, it expresses that for all ε for S_i there is a special finite point, a point in which the epistemic temporal fragment of the formula GKq_i holds. Following this observation, to conclude the proof it suffices to show the following proposition:

Proposition 5.2.15. *Let $S_i \in W^{\mathcal{H}}$ and $S_i h \in H$. The following are equivalent:*

1. *For all $\sigma \in \Sigma^*$, such that $S_i h\sigma \in H$, $\mathcal{K}[S_i h\sigma] = \{S_i h\sigma\}$;*

2. $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P), S_i h \models GKq_i$.

($1 \Rightarrow 2$) Assume that $\mathcal{K}[S_i h] = \{S_i h\}$. By the definition of the valuation V, we get that $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P), S_i h \models q_i$. Then, by the assumption and the semantics of K, we get that $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P), S_i h \models Kq_i$. Finally, since $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P)$ satisfies Perfect Recall and by the definition of protocol P, we get that $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P), S_i h \models GKq_i$.

($2 \Rightarrow 1$) Now, assume that $\mathtt{For}(\mathcal{M}_{\mathcal{C}}, P), S_i h \models GKq_i$. Then, by the semantics of K and by (5.1) we get that for all $\sigma \in \Sigma^*$, such that $S_i h\sigma \in H$, $\mathcal{K}[S_i h\sigma] = \{S_i h\sigma\}$. □

In Chapter 4 we mentioned the adequacy of epistemic models and update with respect to the modeling of finite identification. However, we have been mostly concerned with identification in the limit. In the next section we will explore the language of a temporal logic that can express the condition of identifiability in the limit.

5.2.6 Identification in the Limit and DETL

In order to give a temporal characterization of identifiability in the limit we need to be able to express beliefs of the learner. Therefore, our temporal forests should include a plausibility ordering. In Chapter 4 we have shown that conditioning (update) is a universal learning method from truthful data. In other words, in the case of identifiability in the limit, eliminating the worlds of an epistemic plausibility model is enough to reach stable and true belief. This allows considering very specific temporal structures that result from updating a doxastic epistemic model with purely propositional information.[5]

Definition 5.2.16. An epistemic plausibility temporal forest $\mathcal{H}$ *is a tuple:*

$$\langle W, \Sigma, H, (\sim_a)_{a \in \mathcal{A}}, (\leq_a)_{a \in \mathcal{A}}, V \rangle.$$

where $W \neq \emptyset$ is a countable set of initial states; Σ is a countable set of events; $H \subseteq W\Sigma^*$ is a set of histories (sequences of events starting at states from W) closed under non-empty finite prefixes; for each $a \in A$, $\sim_a \subseteq H \times H$ is an equivalence relation, $\leq_a \subseteq H \times H$ is a plausibility preorder; and $V : \textsc{Prop} \to \mathcal{P}(H)$ is a valuation. We write wh to denote some finite history starting in the state w.

Doxastic Epistemic Temporal Language

Syntax Our doxastic epistemic temporal language of $\mathcal{L}_{\mathrm{DETL}^*}$ is defined by the following inductive syntax.

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid K\varphi \mid B\varphi \mid F\varphi \mid A\varphi$$

where p ranges over a countable set of proposition letters $\textsc{Prop}$. $K\varphi$ reads: 'the agent knows that φ', and $B\varphi$: 'the agent believes that φ'. Symbol F stands for future, G is defined as $\neg F \neg$. $A\varphi$ means: 'in all continuations φ'.

$\mathcal{L}_{\mathrm{DETL}^*}$ is interpreted over epistemic plausibility temporal forests. its semantics is for the most part the same as $\mathcal{L}_{\mathrm{ETL}^*}$. Below we give the semantics of the missing clause, the belief operator B.

Definition 5.2.17.

$$\mathcal{H}, wh \models B\varphi \quad \text{iff} \quad \textit{for all } vh', \textit{ if } vh' \in \min_{\leq} \mathcal{K}[wh], \textit{ then } \mathcal{H}, vh' \models \varphi$$

We again start with an initial learning epistemic model that corresponds to a class of sets and a local set-learning protocol. This time we want to add a plausibility ordering to generate an epistemic plausibility temporal forest. The construction is defined in the following way:

[5] For more complex actions performed on plausibility models in the context of the comparison between dynamic doxastic and doxastic temporal logic see Van Benthem & Dégremont, 2010.

Definition 5.2.18 (Learning Forest). *A learning model* $\mathcal{M}_C = \langle W_C, \sim^{\mathcal{M}}, V_C^{\mathcal{M}} \rangle$ *together with the local set-learning protocol* $P : W \to \text{Prot}(\mathbb{E})$ *and a plausibility preorder* $\leq^{\mathcal{M}} \subseteq W_C \times W_C$ *generates an DETL forest* $\texttt{For}(\mathcal{M}, P, \leq)$ *of the form:*

$$\mathcal{H} = \langle W^{\mathcal{H}}, \mathbb{E}, H, \sim, \leq, V \rangle, \text{ where:}$$

1. $W^{\mathcal{H}}$, $\mathbb{E}$, H, $\sim$ *and* V *are defined as in the generated epistemic temporal forest, Definition 5.2.12;*

2. *If* $w, v \in W^{\mathcal{H}}$ *and* $wh, vh' \in H$, *then* $wh \leq vh'$ *iff* $wh \sim vh'$ *and* $w \leq^{\mathcal{M}} v$.

As in the case of finite identifiability we will now provide a formula of doxastic epistemic temporal logic that characterizes identifiability in the limit.

Theorem 5.2.19. *The following are equivalent:*

1. $\mathcal{C}$ *is identifiable in the limit.*

2. *There exists a plausibility preorder* $\leq \subseteq W_C \times W_C$ *such that for all* $S_i \in W_C$ *and* $\varepsilon \in P(S_i)$ *the learner's belief about the initial state stabilizes to* S_i *on* ε *in the generated forest* $\texttt{For}(\mathcal{M}_C, P, \leq)$.

3. *There exists a plausibility preorder* $\leq \subseteq W_C \times W_C$ *such that* $\texttt{For}(\mathcal{M}_C, P, \leq) \models q_i \to AFGBq_i$.

Proof. ($1 \Leftrightarrow 2$) This equivalence follows from the existence of an appropriate preorder, defined in Section 4.5, and its adaptation to the notion of epistemic temporal forest.

($2 \Leftrightarrow 3$) The proof has a strategy similar to the proof of Theorem 5.2.14. This time the crucial observation is that in $\texttt{For}(\mathcal{M}_C, P, \leq)$, $S_i h \leq S_j h'$ iff $S_i \leq S_j$ and $h = h'$. □

Let us observe that the last clauses of Theorems 5.2.14 and 5.2.19 can be strengthened to exclude the condition of persistence of the doxastic-epistemic states. In our setting, once such a state is reached, it cannot disappear. In the above characterizations this can be reflected by dropping the temporal operator G.

The above theorems give simple syntactic temporal characterizations of finite and limiting learning in doxastic epistemic temporal logic. We do not provide any proof theory for these notions, any 'logic of learning'. However, we do perceive this as an interesting direction for future work. Moreover, we are especially interested in giving temporal characterizations of various learning-theoretic facts, e.g., the existence of tell-tale sets or the locking-sequence lemma (see Chapter 4). Further questions concern modifications of our syntactic temporal characterization and observing what notions of learning can be obtained this way.

5.2.7 Further Questions on Protocols

Uniform-No-Miracles states that any two histories that are not distinguishable from an agent's perspective cannot get distinguished by extending them with the same event (or two indistinguishable event states). In our learnability context a strengthening of this rule seems interesting.

Let us consider the problem of identification in a more general perspective. Objects to be learned do not have to be sets, in particular their protocols do not have to be order-independent. Except for sets, formal learning theory is also concerned, for example, with learnability of functions (see Section 2.1.3). Possible realities can even be more general, they can be classes of functions (scenarios of this kind are at the heart of many inductive inference games, as the card game *Eleusis*, see, e.g., Romesburg, 1978). Then the worlds can be identified with protocols that allow certain sequences of events that can be defined by some logical formula. In particular, events might be assumed to occur in a certain order. Let us consider the following example.

Example 5.2.20. *Let us take two possible worlds:* w_1 *and* w_2 *such that:*

1. *the protocol for* w_1 *allows all infinite sequences that contain all even numbers, and additionally require that whenever a number is* 8 *then the successor should be* 10;

2. *the protocol for* w_2 *allows all infinite sequences that contain all even numbers, and additionally require that whenever a number is* 8 *then the successor should be* 6.

As long as the learner receives even numbers different than 10 *he cannot distinguish between the two states, e.g., the two sequences,* h, h', *are in both protocols:*

- $h : 2, 4, 6, 8$
- $h' : 4, 2, 6, 8$

Therefore, we can say that whichever of the two is enumerated, $w_1 \sim w_2$. *However, complementing both of them with the same event,* 10, *leads to 'a miracle'—two hypotheses get to be distinguished.*

In principle, there is no reason why such 'miraculous' classes of hypotheses should be excluded from learnability considerations. Such cases show a strength of the protocol based temporal approach over the one-step simple DEL update. The latter is well-suited for set learning, because set-learning protocols are permutation closed and in this sense they are reducible to the precondition check. This is why we turned to a more liberal setting of epistemic temporal logic in which the 'miracle' of order-dependence is possible. What we observed is that with a protocol we can obtain not only factual, but also genuine 'procedural information' in the model. Therefore, sometimes we can distinguish between hypotheses not because a new fact comes in, but because of *the way in which* it comes in.

In general, thinking about learnability in terms of protocols leads to a setting in which the possible realities are identified with sets of scenarios of what should be expected to happen in the future. In this sense, the most general realities are sets—they allow any possible enumeration of their content. Functions allow only one particular sequence of events. In between there are a variety of possibilities for defining protocols that can be characterized in an arbitrary way. In general, our results in the previous sections are only the beginning of the logical study of the richness of possible learning protocols.

5.3 Conclusions and Perspectives

Our work provides a translation of scenarios from formal learning theory into the domain of dynamic epistemic logic and epistemic temporal logic. In particular, we characterized the process of identification in the syntax of dynamic doxastic epistemic logic. Moreover, in the more general context of learnability of protocols, we characterized learning in the syntax of a doxastic epistemic temporal language. Hence, we showed that the proposal of expressing learnability in languages of modal-temporal logics of knowledge and belief (see Van Benthem, 2010) can be made precise.

Our results again show that the two prominent approaches, learning theory and epistemic modal-temporal logics, can be joined together in order to describe the notions of belief and knowledge involved in inductive inference. We believe that bridging the two approaches benefits both sides. For formal learning theory, to create a logic for it is to provide additional syntactic insight into the process of inductive learning. For logics of epistemic and doxastic change, it enriches their present scope with different learning scenarios, i.e., not only those based on the incorporation of new data but also on generalization.

Moreover, as we indicated in the last section of this chapter the temporal logic based approach to inductive inference gives a straightforward framework for analyzing various domains of learning on a common ground. In terms of protocols, sets can be seen as classes of specific histories—their permutation-closed complete enumerations. Functions, on the other hand, can be seen as 'realities' that allow only one particular infinite sequence of events. We can think of many intermediate concepts that may be the object of learning. Interestingly, the identification of protocols, that seems to be a generalization of the set-learning paradigm provides what has been the original motivation for epistemic temporal logic from the start: identifying the current history that the agent is in, including its order of events, repetitions, and other constraints.

Further directions include extending our approach to other types of identification, e.g., identification of functions; finding a modal framework for learning

from both positive and negative information; studying systematically the effects of different restrictions on protocols. We are also interested in investigating various constraints one can enforce on learning functions (e.g., consistency, conservatism or set-drivenness) and comparing them to those of epistemic and doxastic agents in doxastic epistemic temporal logics. Modal logics of belief change are a natural framework to study a variety of notions that underly such concepts of learnability. Another important restriction on learning functions is computability. In the next chapter we will be concerned with computable learning functions in the case of finite identification—the convergence to irrevocable knowledge.