Chapter 12

Empirical Inqury Without Certainty

Problems in Σ^0_1 are computationally verifiable, problems in Π^0_1 are computationally refutable, and problems in Δ^0_1 are computationally decidable. What about problems in Σ^0_2, Π^0_2, and Δ^0_2? Should we just turn our backs on them because they lie beyond the realm of either positive or negative certainty?

Natural science faces a similar difficulty. We all believe that there are only finitely many kinds of fundamental particles because that is how many our current theories posit and we haven't seen any more than that. But there may be infinitely many kinds that appear at ever higher energies that haven't been observed at close hand yet. Is there any sense in which one could be said to find the truth about such empirically abstruse questions, which lie beyond the scope of absolute verification and refutation?

Human nature being what it is, you might expect scientists to be lulled into complacency that no new particles will be found during the long dry spells in which no new particles are observed. When new particles are discovered, there are sensational headlines about physics being shaken to the core until the recalcitrant particles are domesticated in the next round of textbook theories. That happened, for example, in 1974, when Samuel C. C. Ting discovered the unexpected "J" particle, which violated the extant quark model and precipitated the postulation of the conserved quantity "charm". Then after a quiescent period, the old psychological complacency begins to reassert itself until the next discovery of new particles, etc. So if there are just finitely many particles, eventually they will all be discovered and the ensuing complacency is never again disturbed, so we arrice at stable opinion that there are just finitely many particles. But if there are infinitely many, possibly appearing after ever longer dry spells, then opinion will be punctuated infinitely often by periods during which it is suspected that new particles will keep appearing forever. In other words, the natural human attitude *converges* to 1 $\iff$ it is true that there are but finitely many particles. One may then say that our native tendency toward credibility **verifies** the finiteness hypothesis **in the limit**.

Similarly, one may say that the hypothesis expressing an infinite variety of particle types is **refutable in the limit** in the sense that some method converges to 0 $\iff$ the hypothesis is false.

It would be nicer, of course, if you could converge to the truth value of the hypothesis whatever it happens to be. Say that a method **decides** a hypothesis **in the limit** $\iff$ it converges to the right answer whatever it happens to be. Then even though it would be possible to say *when* science has found the right answer, one would be assured that eventually it will produce and stick with the right answer, after an arbitrary number of mistakes and retractions of earlier answers.

Unfortunately, the hypothesis that there are at most finitely many particle types is not decidable in the limit, for let an arbitrary method be given. A skeptical demon can withhold new particles until the given method converges to "finitely many particle types" (if the method refuses to, then no further particles appear and it fails to converge to the truth). Then new particles are presented according to some schedule until the method takes the bait and converges to "infinitely many particle types" (again, it the method refuses to, then infinitely many particles appear and the method fails to converge to the truth). So in the limit of inquiry, the method changes its mind infinitely often, so it fails to converge to the truth.

Why not apply the same concepts of fallible convergence to the truth to formal or mathematical questions that defy absolute refutation or verification? To do so, one must entertain *fallible* formal methods that find the right answer without *halting*. That sounds odd, but only in light of the persistent, but mistaken assumption that formal questions (relations of ideas) must always issue in certainty. That assumption is clearly false of questions such as whether W_i is infinite, which is neither effectively verifiable nor effectively refutable. Come to think of it, this question is formally quite analogous to the question whether there are infinitely many kinds of subatomic particles. If you were to do computational experiments on ϕ_i to determine if it halts on infinitely many inputs, the situation would be just as ambiguous as is the physicist's position with respect to particles, for experiments that result in halting might come ever farther apart.

Think of a fallible formal method as a computable means for producing successive guesses. This may be viewed as a two-place recursive function f, where $f(x,t)$ denotes the tth successive guess by f about the right value to output on input x. Call f a **guessing function**. The idea is that the guessing function my have some setbacks, but eventually it stabilizes to the right output for input x. Accordingly (Gold 1965), say that ψ is **limiting partial recursive** iff

$$(\exists \text{ total recursive} f)(\forall x, y)\ \psi(x) \simeq y \iff (\exists t)(\forall t' \geq t)\ f(x,t) = y.$$

Thus, ψ is limiting recursive just in case f eventually stabilizes the value of ψ and refuses to stabilize to any value if $\psi(x) \uparrow$. Let *Limpart* denote the set of all limiting partial recursive functions. A function is **limiting recursive** just in case it is a total limiting partial recursive function.

Then in direct analogy to the definitions of computational verifiability, refutability, and decidability, define:

> S is **computationally verifiable in the limit** $\iff$ S has a limiting partial recursive verification function.
>
> S is **computationally refutable in the limit** $\iff$ S has a limiting partial recursive refutation function.
>
> S is **computationally decidable in the limit** $\iff$ S has a limiting recursive characteristic function.

So now it makes perfectly good sense to have a fallible method for converging to the *formal* truth, just as in the empirical case. Decidability is still two-sided verifiability:

Proposition 12.1 *S is computably decidable in the limit iff S is both computably refutable in the limit and computably verifiable in the limit.*

Exercise 12.1 *Prove the preceding proposition.*

Recall that the empirical hypothesis "infinitely many particle types" is refutable in the limit but not decidable in the limit. Similarly, one may now show that the question whether W_i is infinite has a similar standing in the formal domain.

Proposition 12.2 *The formal problem Inf is formally refutable in the limit but is not formally decidable in the limit.*

Proof. Judging from the discussion of particles, the obvious strategy for the guessing function is to be pessimistic about W_x being infinite when no "new" items are discovered in W_x and to be optimistic each time a new item is discovered in W_x. The idea can be implemented as follows.

$$\begin{aligned} f(x,0) &= 0; \\ f(x,t+1) &= (\exists y \leq n+1) \left(\begin{array}{l} \text{(a) } U(x,t+1,(y)_0,\langle (y)_1 \rangle) \wedge \\ \text{(b) } (\forall t' \leq t)\ \neg U(x,t',(y)_0,\langle (y)_1 \rangle) \end{array} \right). \end{aligned}$$

Suppose that W_i is finite. Then there exists t_0 such that

$$(\forall w \in W_x)(\exists t', y \leq n_0)\ U(x,t',(y)_0,\langle w \rangle).$$

Furthermore,

$$(\forall w \notin W_x)(\forall t)(\forall y \leq t+1)\ \neg U(x,t+1,(y)_0,\langle (y)_1 \rangle).$$

So for all $t' \geq t_0$, either (a) fails at t' or (b) fails at t'. So $lim_{t\to\infty} f(x,t) = 0$, as required.

Suppose that W_i is finite. Then there are infinitely many m at which both (a) and (b) hold. Hence, Inf is computably refutable in the limit.

To see that Inf is not computably decidable in the limit, let total recursive guessing function f be given. You can use the recursion theorem to implement the obvious demon strategy against f. The demonic index d should add a new element to W_d at resource bound t if $f(d,t) = 0$ and should refuse to add a new element to W_d at resource bound t otherwise. This is pretty much the strategy employed by the empirical demon in the particle example. To implement this strategy, first one must effectively count the number of times f returns 0.

$$\begin{aligned} count_n(i,0) &= (f(i,0) = n); \\ count_n(i,t+1) &= count_n(i,t) + (f(i,t+1) = n). \end{aligned}$$

Then define

$$\psi(i,x) \simeq (\mu z)\ count_0(i,z) = x.$$

Apply the s-m-n theorem in the usual way to obtain total recursive g such that

$$\phi_{g(i)}(x) \simeq \psi(i,x).$$

Apply the Kleene recursion theorem to produce the demonic fixed-point d such that

$$\phi_{g(d)} = \phi_d.$$

Thus:

$$\phi_d(x) \downarrow \iff (\exists t)\ count_0(i,t) \geq x.$$

So if there are infinitely many t at which $f(d,t) = 0$, then W_d is infinite and f fails to converge to 1. And if there are but finitely many t at which $f(d,t) = 0$, then W_d is finite and f fails to converge to 0. So f fails to decide Inf in the limit. $\dashv$

Exercise 12.2 *By arguments closely analogous to those just given, show that Tot is refutable in the limit but not decidable in the limit. In addition to the proof, explain the respective strategies of the guessing method and of the computational demon informally.*

So it is clear that fallibility and convergence are just as applicable to formal questions as to empirical questions.

The theory of computability is filled with suggestive and useful analogies. You already know that

$$\begin{aligned} S \text{ is computably verifiable} &\iff S \in \Sigma^0_1; \\ S \text{ is computably refutable} &\iff S \in \Pi^0_1; \\ S \text{ is computably decidable} &\iff S \in \Delta^0_1. \end{aligned}$$

You might already suspect the following, natural extension of these results.

Proposition 12.3

$$\begin{array}{rcl} S \textit{ is computably verifiable in the limit} & \iff & S \in \Sigma^0_2; \\ S \textit{ is computably refutable in the limit} & \iff & S \in \Pi^0_2; \\ S \textit{ is computably decidable in the limit} & \iff & S \in \Delta^0_2. \end{array}$$

Proof. I prove just the first statement, leaving the other two to you. Suppose that S is computably verifiable in the limit. Let guessing function f witness this fact. Then

$$x \in S \iff (\exists t)(\forall t' \geq t)\ f(x, t') = 1.$$

Since $f(x, t')$ is recursive, the quantifier prefix of the preceding statement witnesses that $S \in \Sigma^0_2$.

Conversely, suppose that $S \in \Sigma^0_2$. So for some recursive relation R, you have

$$x \in S \iff (\exists n)(\forall m)\ R(x, n, m).$$

So if $x \in S$, there is some n *in virtue of which* $x \in S$. Construct a guessing function f according to the following idea. Function f provisionally concludes that $x \in S$ in virtue of 0 until some m is found such that $\neg R(x, 0, m)$. At that point, f loses confidence and concludes that $x \notin S$. Thereafter, f regains confidence that $x \in S$ in virtue of 1 until some m is encountered such that $\neg R(x, 1, m)$, and so forth. So think of the current n in virtue of which f thinks that $x \in S$ as f's current **reason** for concluding that $x \in S$. The current reason is the first unrefuted reason, in the following sense:

$$\begin{array}{rcl} reason(x, 0) & = & 0; \\ reason(x, n+1) & = & reason(x, n) + (\forall m \leq n+1)\ R(x, reason(x, n), m). \end{array}$$

The function *reason* is total recursive since it is defined by primitive recursive operations over a recursive relation. Then define the total recursive function

$$\begin{array}{rcl} f(x, 0) & = & 1; \\ f(x, t+1) & = & (reason(x, t) = reason(x, t+1)). \end{array}$$

So f is total recursive since it is defined in terms of primitive recursive operators over a total recursive function. Then

$$\begin{array}{rcl} x \in S & \iff & (\exists n)(\forall m)\ R(x, n, m) \\ & \iff & (\exists n)(\forall m \geq n)\ reason(x, n) = reason(x, m) \\ & \iff & (\exists n)(\forall m \geq n)\ reason(x, m) = reason(x, m+1) \\ & \iff & (\exists n)(\forall m \geq n)\ f(x, m) = 1. \end{array}$$

So f verifies S in the limit. $\dashv$

Exercise 12.3 *Prove the remaining two statements in the preceding proposition. Hint: use the first statement to prove the second statement and exercise 12 to prove the third statement.*

Corollary 12.4 *The guessing function f in the preceding theorem can be chosen to be primitive recursive.*

Proof. The relation R can be chosen to be primitive recursive and then since only primitive recursive operators are applied in the definition of f, we have that f is also primitive recursive. $\dashv$