1

Propositional Logic

1.1 Propositions and Connectives

Traditionally, logic is said to be the art (or study) of reasoning; so in order to describe logic in this tradition, we have to know what 'reasoning' is. According to some traditional views reasoning consists of the building of chains of linguistic entities by means of a certain relation '... follows from ...', a view which is good enough for our present purpose. The linguistic entities occurring in this kind of reasoning are taken to be *sentences*, i.e. entities that express a complete thought, or state of affairs. We call those sentences *declarative*. This means that, from the point of view of natural language our class of acceptable linguistic objects is rather restricted.

Fortunately this class is wide enough when viewed from the mathematician's point of view. So far logic has been able to get along pretty well under this restriction. True, one cannot deal with questions, or imperative statements, but the role of these entities is negligible in pure mathematics. I must make an exception for performative statements, which play an important role in programming; think of instructions as 'goto, if ... then, else ...', etc. For reasons given below, we will, however, leave them out of consideration.

The sentences we have in mind are of the kind '27 is a square number', 'every positive integer is the sum of four squares', 'there is only one empty set'. A common feature of all those declarative sentences is the possibility of assigning them a truth value, *true* or *false*. We do not require the actual determination of the truth value in concrete cases, such as for instance Goldbach's conjecture or Riemann's hypothesis. It suffices that we can 'in principle' assign a truth value.

Our so-called *two-valued* logic is based on the assumption that every sentence is either true or false, it is the cornerstone of the practice of truth tables.

Some sentences are minimal in the sense that there is no proper part which is also a sentence. e.g. $5 \in \{0, 1, 2, 5, 7\}$, or $2+2 = 5$; others can be taken apart into smaller parts, e.g. 'c is rational or c is irrational' (where c is some constant). Conversely, we can build larger sentences from smaller ones by using *connectives.* We know many connectives in natural language; the following list is by no means meant to be exhaustive: *and, or, not, if ... then ..., but, since, as, for, although, neither ... nor ...* . In ordinary discourse, and also in informal mathematics, one uses these connectives incessantly; however, in formal mathematics we will economise somewhat on the connectives we admit. This is mainly for reason of exactness. Compare, for example, the following two sentences: "π is irrational, but it is not algebraic", "Max is a Marxist, but he is not humourless". In the second statement we may discover a suggestion of some contrast, as if we should be surprised that Max is not humourless. In the first case such a surprise cannot be so easily imagined (unless, e.g. one has just read that almost all irrationals are algebraic); without changing the meaning one can transform this statement into "π is irrational and π is not algebraic". So why use (in a formal text) a formulation that carries vague, emotional undertones? For these and other reasons (e.g. of economy) we stick in logic to a limited number of connectives, in particular those that have shown themselves to be useful in the daily routine of formulating and proving.

Note, however, that even here ambiguities loom. Each of the connectives has already one or more meanings in natural language. We will give some examples:

1. John drove on and hit a pedestrian.
2. John hit a pedestrian and drove on.
3. If I open the window then we'll have fresh air.
4. If I open the window then $1 + 3 = 4$.
5. If $1 + 2 = 4$, then we'll have fresh air.
6. John is working or he is at home.
7. Euclid was a Greek or a mathematician.

From 1 and 2 we conclude that 'and' may have an ordering function in time. Not so in mathematics; "π is irrational and 5 is positive" simply means that both parts are the case. Time just does not play a role in formal mathematics. We could not very well say "π was neither algebraic nor transcendent before 1882". What we would want to say is "before 1882 it was unknown whether π was algebraic or transcendent".

In the examples 3-5 we consider the implication. Example 3 will be generally accepted, it displays a feature that we have come to accept as inherent to implication: there is a relation between the premise and conclusion. This feature is lacking in the examples 4 and 5. Nonetheless we will allow cases such as 4 and 5 in mathematics. There are various reasons to do so. One is the consideration that meaning should be left out of syntactical considerations. Otherwise syntax would become unwieldy and we would run into an esoteric practice of exceptional cases. This general implication, in use in mathematics, is called *material implication.* Some other implications have been studied under the names of *strict implication, relevant implication*, etc.

Finally 6 and 7 demonstrate the use of 'or'. We tend to accept 6 and to reject 7. One mostly thinks of 'or' as something exclusive. In 6 we more or less expect John not to work at home, while 7 is unusual in the sense that we as a rule do not use 'or' when we could actually use 'and'. Also, we normally hesitate to use a disjunction if we already know which of the two parts is the case, e.g. "32 is a prime or 32 is not a prime" will be considered artificial (to say the least) by most of us, since we already know that 32 is not a prime. Yet mathematics freely uses such superfluous disjunctions, for example "$2 \geq 2$" (which stands for "$2 > 2$ or $2 = 2$").

In order to provide mathematics with a precise language we will create an artificial, formal language, which will lend itself to mathematical treatment. First we will define a language for propositional logic, i.e. the logic which deals only with *propositions* (sentences, statements). Later we will extend our treatment to a logic which also takes properties of individuals into account.

The process of *formalization* of propositional logic consists of two stages: (1) present a formal language, (2) specify a procedure for obtaining *valid* or *true* propositions.

We will first describe the language, using the technique of *inductive definitions.* The procedure is quite simple: *First* give the smallest propositions, which are not decomposable into smaller propositions; *next* describe how composite propositions are constructed out of already given propositions.

Definition 1.1.1 *The language of propositional logic has an alphabet consisting of*

(i) *proposition symbols :* $p_0, p_1, p_2, \ldots,$
(ii) *connectives :* $\wedge, \vee, \rightarrow, \neg, \leftrightarrow, \perp,$
(iii) *auxiliary symbols :* (,).

The connectives carry traditional names:

$\wedge$	- *and*	- *conjunction*
$\vee$	- *or*	- *disjunction*
$\rightarrow$	- *if ..., then ...*	- *implication*
$\neg$	- *not*	- *negation*
$\leftrightarrow$	- *iff*	- *equivalence, bi-implication*
$\perp$	- *falsity*	- *falsum, absurdum*

The proposition symbols and $\perp$ stand for the indecomposable propositions, which we call *atoms*, or *atomic propositions.*

Definition 1.1.2 *The set PROP of propositions is the smallest set X with the properties*
(i) $p_i \in X (i \in N), \perp \in X,$
(ii) $\varphi, \psi \in X \Rightarrow (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), (\varphi \leftrightarrow \psi) \in X,$
(iii) $\varphi \in X \Rightarrow (\neg \varphi) \in X.$

The clauses describe exactly the possible ways of building propositions. In order to simplify clause *(ii)* we write $\varphi, \psi \in X \Rightarrow (\varphi \Box \psi) \in X$, where $\Box$ is one of the connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

A warning to the reader is in order here. We have used Greek letters φ, ψ in the definition; are they propositions? Clearly we did not intend them to be so, as we want only those strings of symbols obtained by combining symbols of the alphabet in a correct way. Evidently no Greek letters come in at all! The explanation is that φ and ψ are used as variables for propositions. Since we want to study logic, we must use a language to discuss it in. As a rule this language is plain, everyday English. We call the language used to discuss logic our *meta-language* and φ and ψ are *meta-variables* for propositions. We could do without meta-variables by handling (ii) and (iii) verbally: if two propositions are given, then a new proposition is obtained by placing the connective $\wedge$ between them and by adding brackets in front and at the end, etc. This verbal version should suffice to convince the reader of the advantage of the mathematical machinery.

Note that we have added a rather unusual connective, $\bot$. Unusual, in the sense that it does not connect anything. *Logical constant* would be a better name. For uniformity we stick to our present usage. $\bot$ is added for convenience, one could very well do without it, but it has certain advantages. One may note that there is something lacking, namely a symbol for the true proposition; we will indeed add another symbol, $\top$, as an abbreviation for the "true" proposition.

Examples.

$$(p_7 \rightarrow p_0),\ ((\bot \vee p_{32}) \wedge (\neg p_2)) \in PROP.$$
$$p_1 \leftrightarrow p_7,\ \neg\neg\bot,\ ((\rightarrow \wedge \notin PROP$$

It is easy to show that something belongs to *PROP* (just carry out the construction according to 1.1.2); it is somewhat harder to show that something does not belong to *PROP*. We will do one example:

$$\neg\neg\bot \notin PROP.$$

Suppose $\neg\neg\bot \in X$ and X satisfies (i), (ii), (iii) of Definition 1.1.2. We claim that $Y = X - \{\neg\neg\bot\}$ also satisfies (i), (ii) and (iii). Since $\bot, p_i \in X$, also $\bot, p_i \in Y$. If $\varphi, \psi \in Y$, then $\varphi, \psi \in X$. Since X satisfies (ii) $(\varphi\Box\psi) \in X$. From the form of the expressions it is clear that $(\varphi\Box\psi) \neq \neg\neg\bot$ (look at the brackets), so $(\varphi\Box\psi) \in X - \{\neg\neg\bot\} = Y$. Likewise one shows that Y satisfies (iii). Hence X is not the smallest set satisfying (i), (ii) and (iii), so $\neg\neg\bot$ cannot belong to *PROP*.

Properties of propositions are established by an inductive procedure analogous to definition 1.1.2: first deal with the atoms, and then go from the parts to the composite propositions. This is made precise in

Theorem 1.1.3 (Induction Principle) *Let A be a property, then $A(\varphi)$ holds for all $\varphi \in PROP$ if*

(*i*) $A(p_i)$, *for all* i, *and* $A(\bot)$,
(*ii*) $A(\varphi),\ A(\psi) \Rightarrow A((\varphi\Box\psi))$,
(*iii*) $A(\varphi) \Rightarrow A((\neg\varphi))$.

Proof. Let $X = \{\varphi \in PROP \mid A(\varphi)\}$, then X satisfies (i), (ii) and (iii) of definition 1.1.2. So $PROP \subseteq X$, i.e. for all $\varphi \in PROP$ $A(\varphi)$ holds. □

We call an application of theorem 1.1.3 a *proof by induction on* φ. The reader will note an obvious similarity between the above theorem and the principle of complete induction in arithmetic.

The above procedure for obtaining all propositions, and for proving properties of propositions is elegant and perspicuous; there is another approach, however, which has its own advantages (in particular for coding): consider propositions as the result of a linear step-by-step construction. E.g. $((\neg p_0) \rightarrow \bot)$ is constructed by assembling it from its basic parts by using previously constructed parts: $p_0 \ldots \bot \ldots (\neg p_0) \ldots ((\neg p_0) \rightarrow \bot)$. This is formalized as follows:

Definition 1.1.4 *A sequence $\varphi_0, \ldots, \varphi_n$ is called a formation sequence of φ if $\varphi_n = \varphi$ and for all $i \leq n$ φ_i is atomic, or*

$\varphi_i = (\varphi_j \Box \varphi_k)$ *for certain* $j, k < i$, *or*
$\varphi_i = (\neg\varphi_j)$ *for certain* $j < i$.

Observe that in this definition we are considering strings φ of symbols from the given alphabet; this mildly abuses our notational convention.

Examples. (a) $\bot, p_2, p_3, (\bot \vee p_2), (\neg(\bot \vee p_2)), (\neg p_3)$ and $p_3, (\neg p_3)$ are both formation sequences of $(\neg p_3)$. Note that formation sequences may contain 'garbage'.

(b) p_2 is a subformula of $((p_7 \vee (\neg p_2)) \rightarrow p_1)$; $(p_1 \rightarrow \bot)$ is a subformula of $(((p_2 \vee (p_1 \wedge p_0)) \leftrightarrow (p_1 \rightarrow \bot))$.

We now give some trivial examples of proof by induction. In practice we actually only verify the clauses of the proof by induction and leave the conclusion to the reader.

1. *Each proposition has an even number of brackets.*

Proof. (i) Each atom has 0 brackets and 0 is even.

(ii) Suppose φ and ψ have $2n$, resp. $2m$ brackets, then $(\varphi\Box\psi)$ has $2(n + m + 1)$ brackets.

(iii) Suppose φ has $2n$ brackets, then $(\neg\varphi)$ has $2(n + 1)$ brackets. □

2. *Each proposition has a formation sequence.*

Proof. (i) If φ is an atom, then the sequence consisting of just φ is a formation sequence of φ.

(ii) Let $\varphi_0, \ldots, \varphi_n$ and $\psi_0, \ldots, \psi_m$ be formation sequences of φ and ψ, then one easily sees that $\varphi_0, \ldots, \varphi_n,\ \psi_0, \ldots, \psi_m,\ (\varphi_n \Box \psi_m)$ is a formation sequence of $(\varphi \Box \psi)$.

(iii) Left to the reader. □

We can improve on 2:

Theorem 1.1.5 *PROP is the set of all expressions having formation sequences.*

Proof. Let F be the set of all expressions (i.e. strings of symbols) having formation sequences. We have shown above that $PROP \subseteq F$.

Let φ have a formation sequence $\varphi_0, \ldots, \varphi_n$, we show $\varphi \in PROP$ by induction on n.

$n = 0 : \varphi = \varphi_0$ and by definition φ is atomic, so $\varphi \in PROP$.

Suppose that all expressions with formation sequences of length $m < n$ are in $PROP$. By definition $\varphi_n = (\varphi_i \Box \varphi_j)$ for $i, j < n$, or $\varphi_n = (\neg \varphi_i)$ for $i < n$, or φ_n is atomic. In the first case φ_i and φ_j have formation sequences of length $i, j < n$, so by induction hypothesis $\varphi_i, \varphi_j \in PROP$. As $PROP$ satisfies the clauses of definition 1.1.2, also $(\varphi_i \Box \varphi_j) \in PROP$. Treat negation likewise. The atomic case is trivial. Conclusion $F \subseteq PROP$. □

Theorem 1.1.5 is in a sense a justification of the definition of formation sequence. It also enables us to establish properties of propositions by ordinary induction on the length of formation sequences.

In arithmetic one often defines functions by recursion, e.g. exponentiation is defined by $x^0 = 1$ and $x^{y+1} = x^y \cdot x$, or the factorial function by $0! = 1$ and $(x+1)! = x! \cdot (x+1)$.
The jusitification is rather immediate: each value is obtained by using the preceding values (for positive arguments). There is an analogous principle in our syntax.

Example. The number $b(\varphi)$ of brackets of φ, can be defined as follows:

$$\begin{cases} b(\varphi) & = 0 \text{ for } \varphi \text{ atomic,} \\ b((\varphi \Box \psi)) & = b(\varphi) + b(\psi) + 2, \\ b((\neg \varphi)) & = b(\varphi) + 2. \end{cases}$$

The value of $b(\varphi)$ can be computed by successively computing $b(\psi)$ for its subformulae ψ. □

We can give this kind of definitions for all sets that are defined by induction. The principle of "definition by recursion" takes the form of "there is a unique function such that ...". The reader should keep in mind that the basic idea is that one can 'compute' the function value for a composition in a prescribed way from the function values of the composing parts.

The general principle behind this practice is laid down in the following theorem.

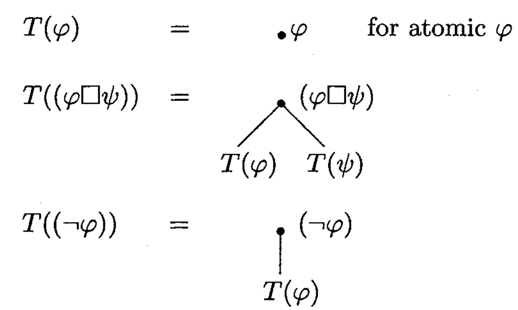
Theorem 1.1.6 (Definition by Recursion) *Let mappings $H_\Box : A^2 \to A$ and $H_\neg : A \to A$ be given and let H_{at} be a mapping from the set of atoms into A, then there exists exactly one mapping $F : PROP \to A$ such that*

$$\begin{cases} F(\varphi) & = H_{at}(\varphi) \text{ for } \varphi \text{ atomic,} \\ F((\varphi \Box \psi)) & = H_\Box(F(\varphi), F(\psi)), \\ F((\neg \varphi)) & = H_\neg(F(\varphi)). \end{cases}$$

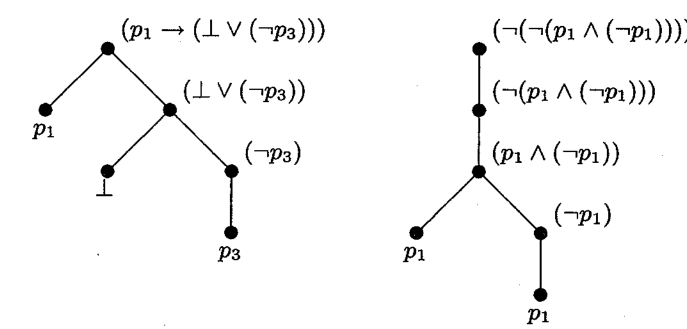
In concrete applications it is usually rather easily seen to be a correct principle. However, in general one has to prove the existence of a unique function satisfying the above equations. The proof is left as an exercise, cf. Exercise 11.

Here are some examples of definition by recursion:

1. The (parsing) *tree* of a proposition φ is defined by

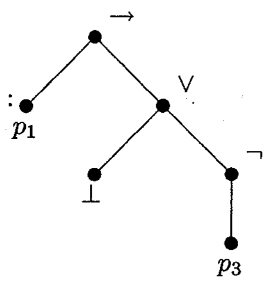


Examples. $T((p_1 \to (\bot \vee (\neg p_3))))$; $T(\neg(\neg(p_1 \wedge (\neg p_1))))$

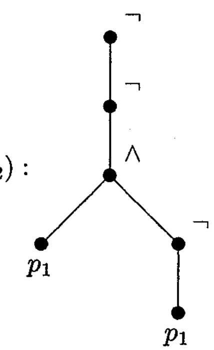


A simpler way to exhibit the trees consists of listing the atoms at the bottom, and indicating the connectives at the nodes.

$T(\varphi_1)$:



$T(\varphi_2)$:



2. The *rank* $r(\varphi)$ of a proposition φ is defined by
$$\begin{cases} r(\varphi) &= 0 \text{ for atomic } \varphi, \\ r((\varphi \Box \psi)) &= \max(r(\varphi), r(\psi)) + 1, \\ r((\neg\varphi)) &= r(\varphi) + 1. \end{cases}$$

We now use the technique of definition by recursion to define the notion of subformula.

Definition 1.1.7 *The set of subformulas $Sub(\varphi)$ is given by*
$$\begin{aligned} Sub(\varphi) &= \{\varphi\} \text{ for atomic } \varphi \\ Sub(\varphi_1 \Box \varphi_2) &= Sub(\varphi_1) \cup Sub(\varphi_2) \cup \{\varphi_1 \Box \varphi_2\} \\ Sub(\neg\varphi) &= Sub(\varphi) \cup \{\neg\varphi\} \end{aligned}$$

We say that ψ *is a subformula of* φ if $\psi \in Sub(\varphi)$.

Notational convention. In order to simplify our notation we will economise on brackets. We will always discard the outermost brackets and we will discard brackets in the case of negations. Furthermore we will use the convention that $\wedge$ and $\vee$ bind more strongly than $\rightarrow$ and $\leftrightarrow$ (cf. $\cdot$ and $+$ in arithmetic), and that $\neg$ binds more strongly than the other connectives.

Examples. $\neg\varphi \vee \varphi$ stands for $((\neg\varphi) \vee \varphi)$,
$\neg(\neg\neg\neg\varphi \wedge \bot)$ stands for $(\neg((\neg(\neg(\neg\varphi))) \wedge \bot))$,
$\varphi \vee \psi \rightarrow \varphi$ stands for $((\varphi \vee \psi) \rightarrow \varphi)$,
$\varphi \rightarrow \varphi \vee (\psi \rightarrow \chi)$ stands for $(\varphi \rightarrow (\varphi \vee (\psi \rightarrow \chi)))$.

Warning. Note that those abbreviations are, properly speaking, not propositions.

In the proposition $(p_1 \rightarrow p_1)$ only one atom is used to define it, it is however used twice and it occurs at two places. For some purpose it is convenient to distinguish between *formulas* and *formula occurrences.* Now the definition of subformula does not tell us what an occurrence of φ in ψ is, we have to add some information. One way to indicate an occurrence of φ is to give its place in the tree of ψ, e.g. an occurrence of a formula in a given formula ψ is a pair (φ, k), where k is a node in the tree of ψ. One might even code k as a sequence of 0's and 1's, where we associate to each node the following sequence: $\langle\,\rangle$ (the

empty sequence) to the top node, $\langle s_0, \ldots, s_{n-1}, 0\rangle$ to the left immediate descendant of the node with sequence $\langle s_0, \ldots, s_{n-1}\rangle$ and $\langle s_0, \ldots, s_{n-1}, 1\rangle$ to the second immediate descendant of it (if there is one). We will not be overly formal in handling occurrences of formulas (or symbols, for that matter), but it is important that it can be done.

The introduction of the rank function above is not a mere illustration of the 'definition by recursion', it also allows us to prove facts about propositions by means of plain *complete induction* (or *mathematical induction*). We have, so to speak, reduced the tree structure to that of the straight line of natural numbers. Note that other 'measures' will do just as well, e.g. the number of symbols. For completeness sake we will spell out the *Rank-Induction Principle*:

Theorem 1.1.8 (Induction on rank-Principle) If for all φ $[A(\psi)$ for all ψ with rank less than $r(\varphi)] \Rightarrow A(\varphi)$, then *$A(\varphi)$ holds for all $\varphi \in PROP$*

Let us show that induction on φ and induction on the rank of φ are equivalent.[1]
First we introduce a convenient notation for the rank-induction: write $\varphi \prec \psi$ $(\varphi \preceq \psi)$ for $r(\varphi) < r(\psi)$ $(r(\varphi) \leq r(\psi))$. So $\forall \psi \preceq \varphi A(\psi)$ stands for "$A(\psi)$ holds for all ψ with rank at most $r(\varphi)$"
The *Rank-Induction Principle* now reads

$$\forall\varphi(\forall\psi \prec \varphi A(\psi) \Rightarrow A(\varphi)) \Rightarrow \forall\varphi A(\varphi)$$

We will now show that the rank-induction principle follows from the induction principle. Let $\qquad \forall\varphi(\forall\psi \prec \varphi A(\psi) \Rightarrow A(\varphi)) \qquad (\dagger)$
be given.In order to show $\forall\varphi A(\varphi)$ we have to indulge in a bit of induction loading. Put $B(\varphi) := \forall\psi \preceq \varphi A(\psi)$. Now show $\forall\varphi B(\varphi)$ by induction on φ.

1. for atomic φ $\forall\psi \prec \varphi A(\psi)$ is vacuously true, hence by $(\dagger)$ $A(\varphi)$ holds. Therefore $A(\psi)$ holds for all ψ with rank ≤ 0. So $B(\varphi)$
2. $\varphi = \varphi_1 \Box \varphi_2$. Induction hypothesis: $B(\varphi_1), B(\varphi_2)$. Let ρ be any proposition with $r(\rho) = r(\varphi) = n + 1$ (for a suitable n). We have to show that ρ and all propositions with rank less than $n + 1$ have the property A. Since $r(\varphi) = max(r(\varphi_1), r(\varphi_2)) + 1$, one of φ_1 and φ_2 has rank n — say φ_1. Now pick an arbitrary ψ with $r(\psi) \leq n$, then $\psi \preceq \varphi_1$. Therefore, by $B(\varphi_1)$, $A(\psi)$. This shows that $\forall\psi \prec \rho A(\psi)$, so by $(\dagger)$ $A(\rho)$ holds. This shows $B(\varphi)$
3. $\varphi = \neg\varphi_1$. Similar argument.

An application of the induction principle yields $\forall\varphi B(\varphi)$, and as a consequence $\forall\varphi A(\varphi)$.

For the converse we assume the premisses of the induction principle. In order to apply the rank-induction principle we have to show $(\dagger)$. We distinguish the following cases:

[1] The reader may skip this proof at first reading. He will do well to apply induction on rank naively.

1. φ atomic. Then (†) holds trivially.
2. $\varphi = \varphi_1 \square \varphi_2$. Then $\varphi_1, \varphi_2 \preceq \varphi$ (see exercise 6). Our assumption is $\forall \psi \prec \varphi A(\psi)$, so $A(\varphi_1)$ and $A(\varphi_2)$. Therefore $A(\varphi)$.
3. $\varphi = \neg \varphi_1$. Similar argument.

This establishes (†). So by rank-induction we get $\forall \varphi A(\varphi)$.

Exercises

1. Give formation sequences of
   $(\neg p_2 \to (p_3 \vee (p_1 \leftrightarrow p_2))) \wedge \neg p_3$,
   $(p_7 \to \neg \bot) \leftrightarrow ((p_4 \wedge \neg p_2) \to p_1)$,
   $(((p_1 \to p_2) \to p_1) \to p_2) \to p_1$.

2. Show that $((\to \notin PROP$.

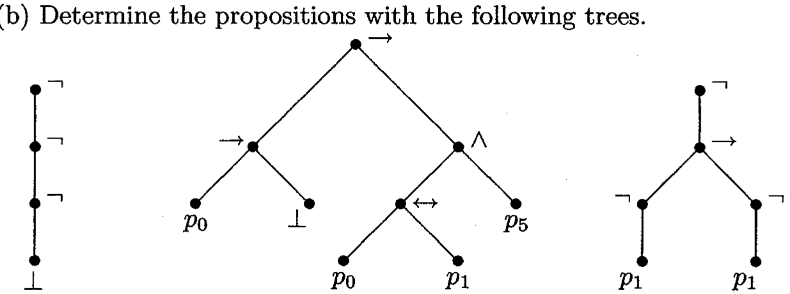
3. Show that the relation "is a subformula of" is transitive.

4. Let φ be a subformula of ψ. Show that φ occurs in each formation sequence of ψ.

5. If φ occurs in a shortest formation sequence of ψ then φ is a subformula of ψ.

6. Let r be the rank function.
   (a) Show that $r(\varphi) \leq$ number of occurrences of connectives of φ,
   (b) Give examples of φ such that $<$ or $=$ holds in (a),
   (c) Find the rank of the propositions in exercise 1.
   (d) Show that $r(\varphi) < r(\psi)$ if φ is a proper subformula of ψ.

7. (a) Determine the trees of the propositions in exercise 1,
   (b) Determine the propositions with the following trees.



8. Let $\#(T(\varphi))$ be the number of nodes of $T(\varphi)$. By the "number of connectives in φ" we mean the number of occurrences of connectives in φ. (In general $\#(A)$ stands for the number of elements of a (finite) set A) .
   (a) If φ does not contain $\bot$, show: number of connectives of $\varphi+$ number of atoms of $\varphi \leq \#(T(\varphi))$.
   (b) $\#(\text{sub}(\varphi)) \leq \#(T(\varphi))$.
   (c) A branch of a tree is a maximal linearly ordered set.
   The length of a branch is the number of its nodes minus one. Show that $r(\varphi)$ is the length of a longest branch in $T(\varphi)$.
   (d) Let φ not contain $\bot$. Show: the number of connectives in φ + the number of atoms of $\varphi \leq 2^{r(\varphi)+1} - 1$.

9. Show that a proposition with n connectives has at most $2n+1$ subformulas.
10. Show that for PROP we have a unique decomposition theorem: for each non-atomic proposition σ either there are two propostions φ and ψ such that $\sigma = \varphi \square \psi$, or there is a proposition φ such that $\sigma = \neg \varphi$.
11. (a) Give an inductive definition of the function F, defined by recursion on $PROP$ from the functions H_{at}, $H_{\square}$, $H_{\neg}$, as a set F^* of pairs.
    (b) Formulate and prove for F^* the induction principle.
    (c) Prove that F^* is indeed a function on $PROP$.
    (d) Prove that it is the unique function on $PROP$ satisfying the recursion equations.

1.2 Semantics

The task of interpreting propositional logic is simplified by the fact that the entities considered have a simple structure. The propositions are built up from rough blocks by adding connectives.
The simplest parts (atoms) are of the form "grass is green", "Mary likes Goethe", "$6-3=2$", which are simply *true* or *false*. We extend this assignment of *truth values* to composite propositions, by reflection on the meaning of the logical connectives.

Let us agree to use 1 and 0 instead of 'true' and 'false'. The problem we are faced with is how to interprete $\varphi \square \psi$, $\neg \varphi$, given the truth values of φ and ψ.

We will illustrate the solution by considering the in-out-table for Messrs. Smith and Jones.

Conjunction. A visitor who wants to see both Smith and Jones wants the table to be in the position shown here, i.e.

	in	out
Smith	×	
Jones	×	

"Smith is in"∧"Jones is in" is true iff "Smith is in" is true and "Jones is in" is true.

We write $v(\varphi) = 1$ (resp. 0) for "φ is true" (resp. false). Then the above consideration can be stated as $v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = v(\psi) = 1$, or $v(\varphi \wedge \psi) = \min(v(\varphi), v(\psi))$.

One can also write it in the form of a *truth table*:

∧	0	1
0	0	0
1	0	1

One reads the truth table as follows: the first argument is taken from the leftmost column and the second argument is taken from the top row.

Disjunction. If a visitor wants to see one of the partners, no matter which one, he wants the table to be in one of the positions

	in	out
Smith	×	
Jones		×

	in	out
Smith		×
Jones	×	

	in	out
Smith	×	
Jones	×	

In the last case he can make a choice, but that is no problem, he wants to see at least one of the gentlemen, no matter which one.

In our notation, the interpretation of ∨ is given by

$$v(\varphi \vee \psi) = 1 \quad \text{iff} \quad v(\varphi) = 1 \quad \text{or} \quad v(\psi) = 1.$$

Shorter: $v(\varphi \vee \psi) = \max(v(\varphi), v(\psi))$.

In truth table form:

∨	0	1
0	0	1
1	1	1

Negation. The visitor who is solely interested in our Smith will state that "Smith is not in" if the table is in the position:

	in	out
Smith		×

So "Smith is not in" is true if "Smith is in" is false. We write this as $v(\neg\varphi) = 1$ iff $v(\varphi) = 0$, or $v(\neg\varphi) = 1 - v(\varphi)$.

In truth table form:

¬	
0	1
1	0

Implication. Our legendary visitor has been informed that "Jones is in if Smith is in". Now he can at least predict the following positions of the table

	in	out
Smith	×	
Jones	×	

	in	out
Smith		×
Jones		×

If the table is in the position

	in	out
Smith	×	
Jones		×

then he knows that the information was false.

The remaining case,

	in	out
Smith		×
Jones	×	

, cannot be dealt with in such a simple way. There evidently is no reason to consider the information false, rather 'not very helpful', or 'irrelevant'. However, we have committed ourselves to the position that each statement is true or false, so we decide to call "If Smith is in, then Jones is in" true also in this particular case. The reader should realize that we have made a deliberate choice here; a choice that will prove a happy one in view of the elegance of the system that results. There is no compelling reason, however, to stick to the notion of implication that we just introduced. Various other notions have been studied in the literature, for mathematical purpose our notion (also called 'material implication') is however perfectly suitable.

Note that there is just one case in which an implication is false (see the truth table below), one should keep this observation in mind for future application – it helps to cut down calculations.

In our notation the interpretation of implication is given by $v(\varphi \rightarrow \psi) = 0$ iff $v(\varphi) = 1$ and $v(\psi) = 0$.

Its truth table is:

→	0	1
0	1	1
1	0	1

Equivalence. If our visitor knows that "Smith is in if and only if Jones is in", then he knows that they are either both in, or both out. Hence $v(\varphi \leftrightarrow \psi) = 1$ iff $v(\varphi) = v(\psi)$.

The truth table of ↔ is:

↔	0	1
0	1	0
1	0	1

Falsum. An absurdity, such as "$0 \neq 0$", "some odd numbers are even", "I am not myself", cannot be true. So we put $v(\bot) = 0$.

Strictly speaking we should add one more truth table, i.e. the table for $\top$, the opposite of *falsum*.

Verum. This symbol stands for manifestly true propostion such as $1 = 1$; we put $v(\top) = 1$ for all v.

We collect the foregoing in

Definition 1.2.1 *A mapping $v : PROP \to \{0,1\}$ is a valuation if*

$$\begin{array}{ll} v(\varphi \wedge \psi) & = \min(v(\varphi), v(\psi)), \\ v(\varphi \vee \psi) & = \max(v(\varphi), v(\psi)), \\ v(\varphi \to \psi) = 0 & \Leftrightarrow \quad v(\varphi) = 1 \text{ and } v(\psi) = 0, \\ v(\varphi \leftrightarrow \psi) = 1 & \Leftrightarrow \quad v(\varphi) = v(\psi), \\ v(\neg\varphi) & = 1 - v(\varphi) \\ v(\bot) & = 0. \end{array}$$

If a valuation is only given for atoms then it is, by virtue of the definition by recursion, possible to extend it to all propositions, hence we get:

Theorem 1.2.2 *If v is a mapping from the atoms into $\{0,1\}$, satisfying $v(\bot) = 0$, then there exists a unique valuation $[\![\cdot]\!]_v$, such that $[\![\varphi]\!]_v = v(\varphi)$ for atomic φ.*

It has become common practice to denote valuations as defined above by $[\![\varphi]\!]$, so will adopt this notation. Since $[\![\cdot]\!]$ is completely determined by its values on the atoms, $[\![\varphi]\!]$ is often denoted by $[\![\varphi]\!]_v$. Whenever there is no confusion we will delete the index v.

Theorem 1.2.2 tells us that each of the mappings v and $[\![\cdot]\!]_v$ determines the other one uniquely, therefore we call v also a valuation (or an *atomic valuation*, if necessary). From this theorem it appears that there are many valuations (cf. Exercise 4).

It is also obvious that the *value* $[\![\varphi]\!]_v$ *of* φ *under* v only depends on the values of v on its atomic subformulae:

Lemma 1.2.3 *If $v(p_i) = v'(p_i)$ for all p_i occurring in φ, then $[\![\varphi]\!]_v = [\![\varphi]\!]_{v'}$.*

Proof. An easy induction on φ. □

An important subset of *PROP* is that of all propositions φ which are *always true*, i.e. true under all valuations.

Definition 1.2.4 *(i) φ is a tautology if $[\![\varphi]\!]_v = 1$ for all valuations v,*
(ii) $\models \varphi$ stands for 'φ is a tautology',
(iii) Let Γ be a set of propositions, then $\Gamma \models \varphi$ iff for all v: $([\![\psi]\!]_v = 1$ for all $\psi \in \Gamma) \Rightarrow [\![\varphi]\!]_v = 1$.

In words: $\Gamma \models \varphi$ holds iff φ is true under all valuations that make all ψ in Γ true. We say that φ is semantical consequence of Γ. We write $\Gamma \not\models \varphi$ if $\Gamma \models \varphi$ is not the case.

Convention. $\varphi_1, \ldots, \varphi_n \models \psi$ stands for $\{\varphi_1, \ldots, \varphi_n\} \models \psi$.

Note that "$[\![\varphi]\!]_v = 1$ for all v" is another way of saying "$[\![\varphi]\!] = 1$ for all valuations".

Examples. (i) $\models \varphi \to \varphi$; $\models \neg\neg\varphi \to \varphi$; $\models \varphi \vee \psi \leftrightarrow \psi \vee \varphi$,
(ii) $\varphi, \psi \models \varphi \wedge \psi$; $\varphi, \varphi \to \psi \models \psi$; $\varphi \to \psi, \neg\psi \models \neg\varphi$.

One often has to substitute propositions for subformulae; it turns out to be sufficient to define substitution for atoms only.

We write $\varphi[\psi/p_i]$ for the proposition obtained by replacing all occurrences of p_i in φ by ψ. As a matter of fact, substitution of ψ for p_i defines a mapping of *PROP* into *PROP*, which can be given by recursion (on φ).

Definition 1.2.5

$$\begin{array}{ll} \varphi[\psi/p_i] & = \begin{cases} \varphi \text{ if } \varphi \text{ atomic and } \varphi \neq p_i \\ \psi \text{ if } \varphi = p_i \end{cases} \\ (\varphi_1 \square \varphi_2)[\psi/p_i] & = \varphi_1[\psi/p_i] \square \varphi_2[\psi/p_i] \\ (\neg\varphi)[\psi/p_i] & = \neg\varphi[\psi/p_i]. \end{array}$$

The following theorem spells out the basic property of the substitution of equivalent propositions.

Theorem 1.2.6 (Substitution Theorem) *If $\models \varphi_1 \leftrightarrow \varphi_2$, then $\models \psi[\varphi_1/p] \leftrightarrow \psi[\varphi_2/p]$, where p is an atom.*

The substitution theorem is actually a consequence of a slightly stronger

Lemma 1.2.7 $[\![\varphi_1 \leftrightarrow \varphi_2]\!]_v \leq [\![\psi[\varphi_1/p] \leftrightarrow \psi[\varphi_2/p]]\!]_v$ *and*
$\models (\varphi_1 \leftrightarrow \varphi_2) \to (\psi[\varphi_1/p] \leftrightarrow \psi[\varphi_2/p])$

Proof. Induction on ψ. We only have to consider $[\![\varphi_1 \leftrightarrow \varphi_2]\!]_v = 1$ (why?).

- ψ atomic. If $\psi = p$, then $\psi[\varphi_i/p] = \varphi_i$ and the result follows immediately. If $\psi \neq p$, then $\psi[\varphi_i/p] = \psi$, and $[\![\psi[\varphi_1/p] \leftrightarrow \psi[\varphi_2/p]]\!]_v = [\![\psi \leftrightarrow \psi]\!]_v = 1$.
- $\psi = \psi_1 \square \psi_2$. Induction hypothesis: $[\![\psi_i[\varphi_1/p]]\!]_v = [\![\psi_i[\varphi_2/p]]\!]_v$. Now the value of $[\![(\psi_1 \square \psi_2)[\varphi_i/p]]\!]_v = [\![\psi_1[\varphi_i/p] \square \psi_2[\varphi_i/p]]\!]_v$ is uniquely determined by its parts $[\![\psi_j[\varphi_i/p]]\!]_v$, hence $[\![(\psi_1 \square \psi_2)[\varphi_1/p]]\!]_v = [\![(\psi_1 \square \psi_2)[\varphi_2/p]]\!]_v$.
- $\psi = \neg\psi_1$. Left to the reader.

The proof of the second part essentially uses the fact that $\models \varphi \to \psi$ iff $[\![\varphi]\!]_v \leq [\![\psi]\!]_v$ for all v(cf. Exercise 6). $\square$

The proof of the substitution theorem now immediately follows . $\square$

The substitution theorem says in plain english that *parts may be replaced by equivalent parts.*

There are various techniques for testing tautologies. One such (rather slow) technique uses truth tables. We give one example:

$$(\varphi \to \psi) \leftrightarrow (\neg\psi \to \neg\varphi)$$

φ	ψ	$\neg\varphi$	$\neg\psi$	$\varphi \to \psi$	$\neg\psi \to \neg\varphi$	$(\varphi \to \psi) \leftrightarrow (\neg\psi \to \neg\varphi)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

The last column consists of 1's only. Since, by lemma 1.2.3 only the values of φ and ψ are relevant, we had to check 2^2 cases. If there are n (atomic) parts we need 2^n lines.

One can compress the above table a bit, by writing it in the following form:

$(\varphi$	$\to$	$\psi)$	$\leftrightarrow$	$(\neg\psi$	$\to$	$\neg\varphi)$
0	1	0	1	1	1	1
0	1	1	1	0	1	1
1	0	0	1	1	0	0
1	1	1	1	0	1	0

Let us make one more remark about the role of the two 0-ary connectives, $\bot$ and $\top$. Clearly, $\models \top \leftrightarrow (\bot \to \bot)$, so we can define $\top$ from $\bot$. On the other hand, we cannot define $\bot$ from $\top$ and $\to$; we note that from $\top$ we can never get anything but a proposition equivalent to $\top$ by using $\wedge, \vee, \to$, but from $\bot$ we can generate $\bot$ and $\top$ by means of applying $\wedge, \vee, \to$.

Exercises

1. Check by the truth table method which of the following propositions are tautologies
   (a) $(\neg\varphi \vee \psi) \leftrightarrow (\psi \to \varphi)$
   (b) $\varphi \to ((\psi \to \sigma) \to ((\varphi \to \psi) \to (\varphi \to \sigma)))$
   (c) $(\varphi \to \neg\varphi) \leftrightarrow \neg\varphi$
   (d) $\neg(\varphi \to \neg\varphi)$
   (e) $(\varphi \to (\psi \to \sigma)) \leftrightarrow ((\varphi \wedge \psi) \to \sigma)$
   (f) $\varphi \vee \neg\varphi$ (*principle of the excluded third*)
   (g) $\bot \leftrightarrow (\varphi \wedge \neg\varphi)$
   (h) $\bot \to \varphi$ (*ex falso sequitur quodlibet*)

2. Show (a) $\varphi \models \varphi$;
   (b) $\varphi \models \psi$ and $\psi \models \sigma \Rightarrow \varphi \models \sigma$;
   (c) $\models \varphi \to \psi \Leftrightarrow \varphi \models \psi$.

3. Determine $\varphi[\neg p_0 \to p_3/p_0]$ for $\varphi = p_1 \wedge p_0 \to (p_0 \to p_3)$; $\varphi = (p_3 \leftrightarrow p_0) \vee (p_2 \to \neg p_0)$.

4. Show that there are $2^{\aleph_0}$ valuations.

5. Show $[\![\varphi \wedge \psi]\!]_v = [\![\varphi]\!]_v \cdot [\![\psi]\!]_v$,
   $[\![\varphi \vee \psi]\!]_v = [\![\varphi]\!]_v + [\![\psi]\!]_v - [\![\varphi]\!]_v \cdot [\![\psi]\!]_v$,
   $[\![\varphi \to \psi]\!]_v = 1 - [\![\varphi]\!]_v + [\![\varphi]\!]_v \cdot [\![\psi]\!]_v$,
   $[\![\varphi \leftrightarrow \psi]\!]_v = 1 - |[\![\varphi]\!]_v - [\![\psi]\!]_v|$.

6. Show $[\![\varphi \to \psi]\!]_v = 1 \quad \Leftrightarrow \quad [\![\varphi]\!]_v \leq [\![\psi]\!]_v$.

1.3 Some Properties of Propositional logic

On the basis of the previous sections we can already prove a lot of theorems about propositional logic. One of the earliest discoveries in modern propositional logic was its similarity with algebras.
Following Boole, an extensive study of the algebraic properties was made by a number of logicians. The purely algebraic aspects have since then been studied in the so-called *Boolean Algebra.*

We will just mention a few of those algebraic laws.

Theorem 1.3.1 *The following propositions are tautologies:*

$$(\varphi \vee \psi) \vee \sigma \leftrightarrow \varphi \vee (\psi \vee \sigma) \qquad (\varphi \wedge \psi) \wedge \sigma \leftrightarrow \varphi \wedge (\psi \wedge \sigma)$$

associativity

$$\varphi \vee \psi \leftrightarrow \psi \vee \varphi \qquad \varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$$

commutativity

$$\varphi \vee (\psi \wedge \sigma) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \sigma) \qquad \varphi \wedge (\psi \vee \sigma) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \sigma)$$

distributivity

$$\neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi \qquad \neg(\varphi \wedge \psi) \leftrightarrow \neg\varphi \vee \neg\psi$$

De Morgan's laws

$$\varphi \vee \varphi \leftrightarrow \varphi \qquad \varphi \wedge \varphi \leftrightarrow \varphi$$

idempotency

$$\neg\neg\varphi \leftrightarrow \varphi$$

double negation law

Proof. Check the truth tables or do a little computation. E.g. De Morgan's law: $[\![\neg(\varphi \vee \psi)]\!] = 1 \Leftrightarrow [\![\varphi \vee \psi]\!] = 0 \Leftrightarrow [\![\varphi]\!] = [\![\psi]\!] = 0 \Leftrightarrow [\![\neg\varphi]\!] = [\![\neg\psi]\!] = 1 \Leftrightarrow [\![\neg\varphi \wedge \neg\psi]\!] = 1$.
So $[\![\neg(\varphi \vee \psi)]\!] = [\![\neg\varphi \wedge \neg\psi]\!]$ for all valuations, i.e $\models \neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi$.
The remaining tautologies are left to the reader. □

In order to apply the previous theorem in "logical calculations" we need a few more equivalences. This is demonstrated in the simple equivalence $\models \varphi \wedge (\varphi \vee \psi) \leftrightarrow \varphi$ (exercise for the reader). For, by the distributive law $\models \varphi \wedge (\varphi \vee \psi) \leftrightarrow (\varphi \wedge \varphi) \vee (\varphi \wedge \psi)$ and $\models (\varphi \wedge \varphi) \vee (\varphi \wedge \psi) \leftrightarrow \varphi \vee (\varphi \wedge \psi)$, by idempotency and the substitution theorem. So $\models \varphi \wedge (\varphi \vee \psi) \leftrightarrow \varphi \vee (\varphi \wedge \psi)$. Another application of the distributive law will bring us back to start, so just applying the above laws will not eliminate ψ!

We list therefore a few more convenient properties.

Lemma 1.3.2 *If* $\models \varphi \rightarrow \psi$, *then* $\models \varphi \wedge \psi \leftrightarrow \varphi$ *and*
$\models \varphi \vee \psi \leftrightarrow \psi$

Proof. By Exercise 6 of section 1.2 $\models \varphi \rightarrow \psi$ implies $[\![\varphi]\!]_v \leq [\![\psi]\!]_v$ for all v. So $[\![\varphi \wedge \psi]\!]_v = \min([\![\varphi]\!]_v, [\![\psi]\!]_v) = [\![\varphi]\!]_v$ and $[\![\varphi \vee \psi]\!]_v = \max([\![\varphi]\!]_v, [\![\psi]\!]_v) = [\![\psi]\!]_v$ for all v. □

Lemma 1.3.3 *(a)* $\models \varphi \quad \Rightarrow \quad \models \varphi \wedge \psi \leftrightarrow \psi$
(b) $\models \varphi \quad \Rightarrow \quad \models \neg\varphi \vee \psi \leftrightarrow \psi$
(c) $\models \bot \vee \psi \leftrightarrow \psi$
(d) $\models \top \wedge \psi \leftrightarrow \psi$

Proof. Left to the reader. □

The following theorem establishes some equivalences involving various connectives. It tells us that we can "define" up to logical equivalence all connectives in terms of $\{\vee, \neg\}$, or $\{\rightarrow, \neg\}$, or $\{\wedge, \neg\}$, or $\{\rightarrow, \bot\}$.
That is, we can find e.g. a proposition involving only $\vee$ and $\neg$, which is equivalent to $\varphi \leftrightarrow \psi$, etc.

Theorem 1.3.4 *(a)* $\models (\varphi \leftrightarrow \psi) \leftrightarrow (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$,
(b) $\models (\varphi \rightarrow \psi) \leftrightarrow (\neg\varphi \vee \psi)$,
(c) $\models \varphi \vee \psi \leftrightarrow (\neg\varphi \rightarrow \psi)$,
(d) $\models \varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$,
(e) $\models \varphi \wedge \psi \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$,
(f) $\models \neg\varphi \leftrightarrow (\varphi \rightarrow \bot)$,
(g) $\models \bot \leftrightarrow \varphi \wedge \neg\varphi$.

Proof. Compute the truth values of the left-hand and right-hand sides. □

We now have enough material to handle logic as if it were algebra. For convenience we write $\varphi \approx \psi$ for $\models \varphi \leftrightarrow \psi$.

Lemma 1.3.5 $\approx$ *is an equivalence relation on PROP, i.e.*
$\varphi \approx \varphi$ *(reflexitivity)*,
$\varphi \approx \psi \quad \Rightarrow \quad \psi \approx \varphi$ *(symmetry)*,
$\varphi \approx \psi$ *and* $\psi \approx \sigma \quad \Rightarrow \quad \varphi \approx \sigma$ *(transitivity)*.

Proof. Use $\models \varphi \leftrightarrow \psi$ iff $[\![\varphi]\!]_v = [\![\psi]\!]_v$ for all v. □

We give some examples of algebraic computations, which establish a chain of equivalences.

1. 1. $\models [\varphi \rightarrow (\psi \rightarrow \sigma)] \leftrightarrow [\varphi \wedge \psi \rightarrow \sigma]$,
   $\varphi \rightarrow (\psi \rightarrow \sigma) \approx \neg\varphi \vee (\psi \rightarrow \sigma)$, (1.3.4(b))
   $\neg\varphi \vee (\psi \rightarrow \sigma) \approx \neg\varphi \vee (\neg\psi \vee \sigma)$, (1.3.4(b) and subst. thm.)
   $\neg\varphi \vee (\neg\psi \vee \sigma) \approx (\neg\varphi \vee \neg\psi) \vee \sigma$, (ass.)
   $(\neg\varphi \vee \neg\psi) \vee \sigma \approx \neg(\varphi \wedge \psi) \vee \sigma$, (*De Morgan* and subst. thm.)
   $\neg(\varphi \wedge \psi) \vee \sigma \approx (\varphi \wedge \psi) \rightarrow \sigma$, (1.3.4(b))
   So $\varphi \rightarrow (\psi \rightarrow \sigma) \approx (\varphi \wedge \psi) \rightarrow \sigma$.
   We now leave out the references to the facts used, and make one long string. We just calculate till we reach a tautology.
2. 2. $\models (\varphi \rightarrow \psi) \leftrightarrow (\neg\psi \rightarrow \neg\varphi)$,
   $\neg\psi \rightarrow \neg\varphi \approx \neg\neg\psi \vee \neg\varphi \approx \psi \vee \neg\varphi \approx \neg\varphi \vee \psi \approx \varphi \rightarrow \psi$
3. 3. $\models \varphi \rightarrow (\psi \rightarrow \varphi)$,
   $\varphi \rightarrow (\psi \rightarrow \varphi) \approx \neg\varphi \vee (\neg\psi \vee \varphi) \approx (\neg\varphi \vee \varphi) \vee \neg\psi$.

We have seen that $\vee$ and $\wedge$ are associative, therefore we adopt the convention, also used in algebra, to delete brackets in iterated disjunctions and conjunctions; i.e. we write $\varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \varphi_4$, etc. This is alright, since no matter how we restore (syntactically correctly) the brackets, the resulting formula is determined uniquely up to equivalence.

Have we introduced *all* connectives so far? Obviously not. We can easily invent new ones. Here is a famous one, introduced by Sheffer: $\varphi|\psi$ stands for "not both φ and ψ". More precise: $\varphi|\psi$ is given by the following truth table

Sheffer stroke

|	0	1
0	1	1
1	1	0

Let us say that an n-ary logical connective \$ is *defined* by its truth table, or by its valuation function, if $[\![\$(p_1, \ldots, p_n)]\!] = f([\![p_1]\!], \ldots, [\![p_n]\!])$ for some function f.

Although we can apparently introduce many new connectives in this way, there are no surprises in stock for us, as all of those connectives are definable in terms of $\vee$ and $\neg$:

Theorem 1.3.6 *For each n-ary connective \$ defined by its valuation function, there is a proposition τ, containing only $p_1, \ldots, p_n$, $\vee$ and $\neg$, such that $\models \tau \leftrightarrow \$(p_1, \ldots, p_n)$.*

Proof. Induction on n. For $n = 1$ there are 4 possible connectives with truth tables

$\$_1$	
0	0
1	0

$\$_2$	
0	1
1	1

$\$_3$	
0	0
1	1

$\$_4$	
0	1
1	0

One easily checks that the propositions $\neg(p \vee \neg p)$, $p \vee \neg p$, p and $\neg p$ will meet the requirements.

Suppose that for all n-ary connectives propositions have been found.

Consider $\$(p_1, \ldots, p_n, p_{n+1})$ with truth table:

p_1	$p_2 \cdots$	p_n	p_{n+1}	$\$(p_1, \ldots, p_n, p_{n+1})$
0	0	0	0	i_1
.	.	0	1	i_2
.	0	1	.	.
.	1	1	.	.
0	.	.	.	.
.	1	.	.	.
⋯	⋯	⋯	⋯	⋯
1	0	.	.	.
.	.	.	.	.
.	.	.	.	.
.	0	.	.	.
.	1	0	.	.
.	.	0	.	.
1	.	1	0	.
.	.	1	1	$i_{2^{n+1}}$

where $i_k \leq 1$.

We consider two auxiliary connectives $\$_1$ and $\$_2$ defined by

$\$_1(p_2, \ldots, p_{n+1}) = \$(\bot, p_2, \ldots, p_{n+1})$ and
$\$_2(p_2, \ldots, p_{n+1}) = \$(\top, p_2, \ldots, p_{n+1})$, where $\top = \neg\bot$

(as given by the upper and lower half of the above table).

By the induction hypothesis there are propositions σ_1 and σ_2, containing only $p_2, \ldots, p_{n+1}$, $\vee$ and $\neg$ so that $\models \$_i(p_2, \ldots, p_{n+1}) \leftrightarrow \sigma_i$.
From those two propositions we can construct the proposition τ:
$\tau := (p_1 \to \sigma_2) \wedge (\neg p_1 \to \sigma_1)$.

Claim $\models \$(p_1, \ldots, p_{n+1}) \leftrightarrow \tau$.

If $[\![p_1]\!]_v = 0$, then $[\![p_1 \to \sigma_2]\!]_v = 1$, so $[\![\tau]\!]_v = [\![\neg p_1 \to \sigma_1]\!]_v = [\![\sigma_1]\!]_v = [\![\$_1(p_2, \ldots, p_{n+1})]\!]_v = [\![\$(p_1, p_2, \ldots, p_{n+1})]\!]_v$, using $[\![p_1]\!]_v = 0 = [\![\bot]\!]_v$.

The case $[\![p_1]\!]_v = 1$ is similar.

Now expressing $\to$ and $\wedge$ in terms of $\vee$ and $\neg$ (1.3.4), we have $[\![\tau']\!] = [\![\$(p_1, \ldots, p_{n+1})]\!]$ for all valuations (another use of lemma 1.3.5), where $\tau' \approx \tau$ and τ' contains only the connectives $\vee$ and $\neg$. □

For another solution see Exercise 7.

The above theorem and theorem 1.3.4 are pragmatic justifications for our choice of the truth table for $\to$: we get an extremely elegant and useful theory. Theorem 1.3.6 is usually expressed by saying that $\vee$ and $\neg$ form a *functionally complete* set of connectives. Likewise $\wedge$, $\neg$ and $\to$, $\neg$ and $\bot$, $\to$ form functionally complete sets.

In analogy to the $\sum$ and $\prod$ from algebra we introduce finite disjunctions and conjunctions:

Definition 1.3.7

$$\begin{cases} \bigwedge_{i \leq 0} \varphi_i = \varphi_0 \\ \bigwedge_{i \leq n+1} \varphi_i = \bigwedge_{i \leq n} \varphi_i \wedge \varphi_{n+1} \end{cases} \qquad \begin{cases} \bigvee_{i \leq 0} \varphi_i = \varphi_0 \\ \bigvee_{i \leq n+1} \varphi_i = \bigvee_{i \leq n} \varphi_i \vee \varphi_{n+1} \end{cases}$$

Definition 1.3.8 *If $\varphi = \bigwedge_{i \leq n} \bigvee_{j \leq m_i} \varphi_{ij}$, where φ_{ij} is atomic or the negation of an atom, then φ is a conjunctive normal form. If $\varphi = \bigvee_{i \leq n} \bigwedge_{j \leq m_i} \varphi_{ij}$, where φ_{ij} is atomic or the negation of an atom, then φ is a disjunctive normal form.*

The normal forms are analogous to the well-known normal forms in algebra: $ax^2 + byx$ is "normal", whereas $x(ax + by)$ is not. One can obtain normal forms by simply "multiplying", i.e. repeated application of distributive laws. In algebra there is only one "normal form"; in logic there is a certain duality between $\wedge$ and $\vee$, so that we have two normal form theorems.

Theorem 1.3.9 *For each φ there are conjunctive normal forms $\varphi^\wedge$ and disjunctive normal forms $\varphi^\vee$, such that $\models \varphi \leftrightarrow \varphi^\wedge$ and $\models \varphi \leftrightarrow \varphi^\vee$.*

Proof. First eliminate all connectives other than $\bot$, $\wedge$, $\vee$ and $\neg$. Then prove the theorem by induction on the resulting proposition in the restricted language of $\bot, \wedge$, $\vee$ and $\neg$. In fact, $\bot$ plays no role in this setting; it could just as well be ignored.

(a) φ is atomic. Then $\varphi^\wedge = \varphi^\vee = \varphi$.

(b) $\varphi = \psi \wedge \sigma$. Then $\varphi^\wedge = \psi^\wedge \wedge \sigma^\wedge$. In order to obtain a disjunctive normal form we consider $\psi^\vee = \bigvee \psi_i$, $\sigma^\vee = \bigvee \sigma_j$, where the ψ_i's and σ_j's are conjunctions of atoms and negations of atoms.

Now $\varphi = \psi \wedge \sigma \approx \psi^\vee \wedge \sigma^\vee \approx \bigvee_{i,j} (\psi_i \wedge \sigma_j)$.

The last proposition is in normal form, so we equate $\varphi^\vee$ to it.

(c) $\varphi = \psi \vee \sigma$. Similar to (b).

(d) $\varphi = \neg\psi$. By induction hypothesis ψ has normal forms $\psi^\vee$ and $\psi^\wedge$. $\neg\psi \approx \neg\psi^\wedge \approx \neg \bigvee \bigwedge \psi_{ij} \approx \bigwedge \bigvee \neg\psi_{ij} \approx \bigwedge \bigvee \psi'_{ij}$, where $\psi'_{ij} = \neg\psi_{ij}$ if ψ_{ij} is atomic, and $\psi_{ij} = \neg\psi'_{ij}$ if ψ_{ij} is the negation of an atom. (Observe $\neg\neg\psi_{ij} \approx \psi_{ij}$). Clearly $\bigwedge \bigvee \psi'_{ij}$ is a conjunctive normal form for φ. The disjunctive normal form is left to the reader.

For another proof of the normal form theorems see Exercise 7. □

When looking at the algebra of logic in theorem 1.3.1, we saw that $\vee$ and $\wedge$ behaved in a very similar way, to the extent that the same laws hold for both. We will make this 'duality' precise. For this purpose we consider a language with only the connectives $\vee$, $\wedge$ and $\neg$.

Definition 1.3.10 *Define an auxiliary mapping* $^*: PROP \to PROP$ *recursively by*

$$\begin{aligned} \varphi^* &= \neg\varphi \text{ if } \varphi \text{ is atomic,} \\ (\varphi \wedge \psi)^* &= \varphi^* \vee \psi^*, \\ (\varphi \vee \psi)^* &= \varphi^* \wedge \psi^*, \\ (\neg\varphi)^* &= \neg\varphi^*. \end{aligned}$$

Example. $((p_0 \wedge \neg p_1) \vee p_2)^* = (p_0 \wedge \neg p_1)^* \wedge p_2^* = (p_0^* \vee (\neg p_1)^*) \wedge \neg p_2 = (\neg p_0 \vee \neg p_1^*) \wedge \neg p_2 = (\neg p_0 \vee \neg\neg p_1) \wedge \neg p_2 \approx (\neg p_0 \vee p_1) \wedge \neg p_2$.

Note that the effect of the *-translation boils down to taking the negation and applying De Morgan's laws.

Lemma 1.3.11 $[\![\varphi^*]\!] = [\![\neg\varphi]\!]$

Proof. Induction on φ. For atomic φ $[\![\varphi^*]\!] = [\![\neg\varphi]\!]$.
$[\![(\varphi \wedge \psi)^*]\!] = [\![\varphi^* \vee \psi^*]\!] = [\![\neg\varphi \vee \neg\psi]\!]) = [\![\neg(\varphi \wedge \psi)]\!])$.
$[\![(\varphi \vee \psi)^*]\!]$ and $[\![(\neg\varphi)^*]\!]$ are left to the reader. □

Corollary 1.3.12 $\models \varphi^* \leftrightarrow \neg\varphi$.

Proof. Immediate from Lemma 1.3.11. □

So far this is not the proper duality we have been looking for. We really just want to interchange $\wedge$ and $\vee$. So we introduce a new translation.

Definition 1.3.13 *The duality mapping* $^d: PROP \to PROP$ *is recursively defined by*

$$\begin{aligned} \varphi^d &= \varphi \text{ for } \varphi \text{ atomic,} \\ (\varphi \wedge \psi)^d &= \varphi^d \vee \psi^d, \\ (\varphi \vee \psi)^d &= \varphi^d \wedge \psi^d, \\ (\neg\varphi)^d &= \neg\varphi^d. \end{aligned}$$

Theorem 1.3.14 (Duality Theorem) $\models \varphi \leftrightarrow \psi \Leftrightarrow \models \varphi^d \leftrightarrow \psi^d$.

Proof. We use the $*$-translation as an intermediate step. Let us introduce the notion of simultaneous substitution to simplify the proof:
$\sigma[\tau_0, \dots, \tau_n / p_0, \dots, p_n]$ is obtained by substituting τ_i for p_i for all $i \le n$ simultaneously (see Exercise 15). Observe that $\varphi^* = \varphi^d[\neg p_0, \dots, \neg p_n / p_0, \dots, p_n]$, so $\varphi^*[\neg p_0, \dots, \neg p_n / p_0, \dots, p_n] = \varphi^d[\neg\neg p_0, \dots, \neg\neg p_n / p_0, \dots, p_n]$, where the atoms of φ occur among the $p_0, \dots, p_n$.

By the Substitution Theorem $\models \varphi^d \leftrightarrow \varphi^*[\neg p_0, \dots, \neg p_n / p_0, \dots, p_n]$. The same equivalence holds for ψ.

By Corollary 1.3.12 $\models \varphi^* \leftrightarrow \neg\varphi$, $\models \psi^* \leftrightarrow \neg\psi$. Since $\models \varphi \leftrightarrow \psi$, also $\models \neg\varphi \leftrightarrow \neg\psi$. Hence $\models \varphi^* \leftrightarrow \psi^*$, and therefore $\models \varphi^*[\neg p_0, \dots, \neg p_n / p_0, \dots, p_n] \leftrightarrow \psi^*[\neg p_0, \dots, \neg p_n / p_0, \dots, p_n]$.

Using the above relation between φ^d and φ^* we now obtain $\models \varphi^d \leftrightarrow \psi^d$. The converse follows immediately, as $\varphi^{dd} = \varphi$. □

The duality Theorem gives us one identity for free for each identity we establish.

Exercises

1. Show by 'algebraic' means

$\models (\varphi \to \psi) \leftrightarrow (\neg\psi \to \neg\varphi)$, *Contraposition,*
$\models (\varphi \to \psi) \wedge (\psi \to \sigma) \to (\varphi \to \sigma)$, *transitivity of* $\to$,
$\models (\varphi \to (\psi \wedge \neg\psi)) \to \neg\varphi$,
$\models (\varphi \to \neg\varphi) \to \neg\varphi$,
$\models \neg(\varphi \wedge \neg\varphi)$,
$\models \varphi \to (\psi \to \varphi \wedge \psi)$,
$\models ((\varphi \to \psi) \to \varphi) \to \varphi$. *Peirce's Law* .

2. Simplify the following propositions (i.e. find a simpler equivalent proposition).
(a) $(\varphi \to \psi) \wedge \varphi$, (b) $(\varphi \to \psi) \vee \neg\varphi$, (c) $(\varphi \to \psi) \to \psi$,
(d) $\varphi \to (\varphi \wedge \psi)$, (e) $(\varphi \wedge \psi) \vee \varphi$, (f) $(\varphi \to \psi) \to \varphi$.

3. Show that $\{\neg\}$ is not a functionally complete set of connectives. Idem for $\{\to, \vee\}$ (hint: show that each formula φ with only $\to$ and $\vee$ there is a

valuation v such that $[\![\varphi]\!]_v = 1$).

4. Show that the Sheffer stroke, $|$, forms a functionally complete set (hint: $\models \neg\varphi \leftrightarrow \varphi \mid \varphi$).

5. Show that the connective $\downarrow$ (φ *nor* ψ), with valuation function $[\![\varphi\downarrow\psi]\!] = 1$ iff $[\![\varphi]\!] = [\![\psi]\!] = 0$, forms a functionally complete set.

6. Show that $|$ and $\downarrow$ are the only binary connectives $\$$ such that $\{\$\}$ is functionally complete.

7. The functional completeness of $\{\vee, \neg\}$ can be shown in an alternative way. Let $\$$ be an n-ary connective with valuation function $[\![\$(p_1, \ldots, p_n)]\!] = f([\![p_1]\!], \ldots, [\![p_n]\!])$. We want a proposition τ (in $\vee, \neg$) such that $[\![\tau]\!] = f([\![p_1]\!], \ldots, [\![p_n]\!])$.
   Suppose $f([\![p_1]\!], \ldots, [\![p_n]\!]) = 1$ at least once. Consider all tuples $([\![p_1]\!], \ldots, [\![p_n]\!])$ with $f([\![p_1]\!], \ldots, [\![p_n]\!]) = 1$ and form corresponding conjunctions $\bar{p}_1 \wedge \bar{p}_2 \wedge \ldots \wedge \bar{p}_n$ such that $\bar{p}_i = p_i$ if $[\![p_i]\!] = 1$, $\bar{p}_i = \neg p_i$ if $[\![p_i]\!] = 0$. Then show $\models (\bar{p}_1^1 \wedge \bar{p}_2^1 \wedge \ldots \wedge \bar{p}_n^1) \vee \ldots \vee (\bar{p}_1^k \wedge \bar{p}_2^k \wedge \ldots \wedge \bar{p}_n^k) \leftrightarrow \$(p_1, \ldots, p_n)$, where the disjunction is taken over all n-tuples such that $f([\![p_1]\!], \ldots, [\![p_n]\!]) = 1$.
   Alternatively, we can consider the tuples for which $f([\![p_1]\!], \ldots, [\![p_n]\!]) = 0$. Carry out the details. Note that this proof of the functional completeness at the same time proves the Normal Form Theorems.

8. Let the ternary connective $\$$ be defined by $[\![\$(\varphi_1, \varphi_2, \varphi_3)]\!] = 1 \Leftrightarrow [\![\varphi_1]\!] + [\![\varphi_2]\!] + [\![\varphi_3]\!] \geq 2$ (the majority connective). Express $\$$ in terms of $\vee$ and $\neg$.

9. Let the binary connective $\#$ be defined by

#	0	1
0	0	1
1	1	0

   Express $\#$ in terms of $\vee$ and $\neg$.

10. Determine conjunctive and disjunctive normal forms for $\neg(\varphi \leftrightarrow \psi)$, $((\varphi \to \psi) \to \psi) \to \psi$, $(\varphi \to (\varphi \wedge \neg\psi)) \wedge (\psi \to (\psi \wedge \neg\varphi))$.

11. Give a criterion for a conjunctive normal form to be a tautology.

12. Prove $\bigwedge\limits_{i \leq n} \varphi_i \vee \bigwedge\limits_{j \leq m} \psi_j \approx \bigwedge\limits_{\substack{i \leq n \\ j \leq m}} (\varphi_i \vee \psi_j)$ and

$$\bigvee_{i \leq n} \varphi_i \wedge \bigvee_{j \leq m} \psi_j \approx \bigvee_{\substack{i \leq n \\ j \leq m}} (\varphi_i \wedge \psi_j)$$

13. The set of all valuations, thought of as the set of all $0 - 1$–sequences, forms a topological space, the so-called Cantor space $\mathcal{C}$. The basic open sets are finite unions of sets of the form $\{v \mid [\![p_{i_1}]\!]_v = \ldots = [\![p_{i_n}]\!]_v = 1$ and $[\![p_{j_1}]\!]_v = \ldots = [\![p_{j_m}]\!]_v = 0\}$, $i_k \neq j_p$ for $k \leq n$; $p \leq m$.
    Define a function $[\![\ ]\!] : PROP \to \mathcal{P}(\mathcal{C})$ (subsets of Cantor space) by: $[\![\varphi]\!] = \{v \mid [\![\varphi]\!]_v = 1\}$.
    (a) Show that $[\![\varphi]\!]$ is a basic open set (which is also closed),
    (b) $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$; $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$; $[\![\neg\varphi]\!] = [\![\varphi]\!]^{\wedge}$,
    (c) $\models \varphi \Leftrightarrow [\![\varphi]\!] = C$; $[\![\bot]\!] = \emptyset$; $\models \varphi \to \psi \Leftrightarrow [\![\varphi]\!] \subseteq [\![\psi]\!]$.
    Extend the mapping to sets of propositions Γ by $[\![\Gamma]\!] = \{v \mid [\![\varphi]\!]_v = 1$ for all $\varphi \in \Gamma\}$. Note that $[\![\Gamma]\!]$ is closed.
    (d) $\Gamma \models \varphi \Leftrightarrow [\![\Gamma]\!] \subseteq [\![\varphi]\!]$.

14. We can view the relation $\models \varphi \to \psi$ as a kind of ordering. Put $\varphi \sqsubset \psi := \models \varphi \to \psi$ and $\not\models \psi \to \varphi$.
    (i) for each φ, ψ such that $\varphi \sqsubset \psi$, find σ with $\varphi \sqsubset \sigma \sqsubset \psi$,
    (ii) find $\varphi_1, \varphi_2, \varphi_3, \ldots$ such that $\varphi_1 \sqsubset \varphi_2 \sqsubset \varphi_3 \sqsubset \varphi_4 \sqsubset \ldots$,
    (iii) show that for each φ, ψ with φ and ψ incomparable, there is a least σ with $\varphi, \psi \sqsubset \sigma$.

15. Give a recursive definition of the simultaneous substitution $\varphi[\psi, \ldots, \psi_n/p_1, \ldots, p_n]$ and formulate and prove the appropriate analogue of the Substitution Theorem (theorem 1.2.6).