80-310/610 Logic and Computation
Exercise Set 3
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Exercise 1 *Van Dalen 1.4.5.*

$$\begin{array}{rcl} \Gamma \vdash \phi & \Rightarrow & \Gamma \cup \Delta \vdash \phi; \\ \Gamma \vdash \phi \wedge \Delta \wedge \phi \vdash \psi & \Rightarrow & \Gamma \cup \Delta \vdash \psi. \end{array}$$

Answer: Suppose $\Gamma \vdash \phi$. Then there exists $\mathcal{D} \in$ der such that hyps$(\mathcal{D}) \subseteq \Gamma$ and conc$(\mathcal{D}) = \phi$. So hyps$(\mathcal{D}) \subseteq \Gamma \cup \Delta$. So in virtue of $\mathcal{D}$, $\Gamma \cup \Delta \vdash \phi$.

Suppose $\Gamma \vdash \phi$ and $\Delta, \phi \vdash \psi$. Then there exist $\mathcal{D}, \mathcal{D}' \in$ der such that

$$\begin{array}{rcl} \text{hyps}(\mathcal{D}) & \subseteq & \Gamma; \\ \text{hyps}(\mathcal{D}') & \subseteq & \Delta \cup \{\phi\}; \\ \text{conc}(\mathcal{D}) & = & \phi; \\ \text{conc}(\mathcal{D}') & = & \psi. \end{array}$$

Then clamp $\mathcal{D}'$ onto ϕ in $\mathcal{D}'$ to obtain $\mathcal{D}''$. Then hyps$(\mathcal{D}'') = \Gamma \cup \Delta$. So $\Gamma \cup \Delta \vdash \psi$.

Exercise 2 *Van Dalen 1.4.6. Give a recursive definition of the set of hypotheses of a derivation.*

Answer I'm writing out the derivations horizontally rather than vertically, which doesn't make any difference formally. That makes it easier to typeset constructions involving proofs. Thus,

$$\mathcal{D}(\phi \wedge \psi)|\phi$$

is the $\wedge E$ rule.

$$\begin{array}{rcl} h(\phi) & = & \{\phi\}; \\ h(\mathcal{D}\phi, \mathcal{D}'\psi|(\phi \wedge \psi)) & = & h(\mathcal{D}) \cup h(\mathcal{D}'); \\ h(\mathcal{D}(\phi \wedge \psi)|\phi) & = & h(\mathcal{D}); \\ h([\phi]\mathcal{D}\psi|(\phi \rightarrow \psi)) & = & h(\mathcal{D}) \setminus \{\phi\}; \\ h(\mathcal{D}\phi \rightarrow \psi, \mathcal{D}\phi|\psi) & = & h(\mathcal{D}) \cup h(\mathcal{D}'); \\ h(\mathcal{D}\bot|\phi) & = & h(\mathcal{D}); \\ h([\neg\phi]\mathcal{D}\bot|\phi) & = & h(\mathcal{D}) \setminus \{\phi\}. \end{array}$$

Exercise 3 *Van Dalen 1.4.7. Give a recursive definition of substitution of a formula in a derivation. Show that a substituted derivation is a derivation. Show that if $\Gamma \vdash \phi$ then $\Gamma[\delta/p] \vdash \phi[\delta/p]$.*

Answer Recall that we already have substitution of a prop for an atom: $\phi[\delta/p]$. For derivations use angle brackets instead $\mathcal{D}\langle\delta/p\rangle$.

$$\begin{array}{rcl} \phi\langle\delta/p\rangle & = & \phi[\delta/p]; \\ (\mathcal{D}\phi, \mathcal{D}'\psi|(\phi \wedge \psi))\langle\delta/p\rangle & = & \mathcal{D}\langle\delta/p\rangle\phi[\delta/p], \mathcal{D}'\langle\delta/p\rangle\psi[\delta/p]|\phi[\delta/p]; \\ & \vdots & \\ ([\neg\phi]\mathcal{D}\bot|\phi)\langle\delta/p\rangle & = & ([\neg\phi[\delta/p]]\mathcal{D}\langle\delta/p\rangle\bot|\phi[\delta/p])[\delta/p]. \end{array}$$

Next, show that if $\mathcal{D}$ is a derivation then $\mathcal{D}[\delta/p]$ is as well. By induction on derivations of course. In the base case if ϕ is a derivation, so is $\phi\langle\delta/p\rangle = \phi[\delta/p]$, since $\phi[\delta/p]$ is a proposition. For an example of the induction, consider the RAA case. Suppose that $[\neg\phi]\mathcal{D}\bot|\phi)$ is a derivation. Then

$$([\neg\phi]\mathcal{D}\bot|\phi)\langle\delta/p\rangle = ([\neg\phi[\delta/p]]\mathcal{D}\langle\delta/p\rangle\bot|\phi[\delta/p])[\delta/p].$$

By the IH, $\mathcal{D}\langle\delta/p\rangle$ is a derivation. Since the same substitutions are performed on the formulas that must match, it follows that $([\neg\phi[\delta/p]]\mathcal{D}\langle\delta/p\rangle\bot|\phi[\delta/p])[\delta/p]$ is a derivation.

Finally, suppose that $\Gamma \vdash \phi$. Then there exists derivation $\mathcal{D}$ such that $h(\mathcal{D}) \subseteq \Gamma$ and the conclusion of $\mathcal{D}$ is ϕ. But then $\mathcal{D}\langle\delta/p\rangle$ is a derivation whose hypotheses are in $\Gamma[\delta/p]$ and whose conclusion is $\phi[\delta/p]$. So $\Gamma[\delta/p] \vdash \phi[\delta/p]$.