

80-310/610 Logic and Computation

Exercise Set 12

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Exercise 1 (3.2.1.iii, vi) For (iii), observe that: $\mathcal{A} \in \text{Mod}(\Gamma \cup \Delta)$ iff for each $\phi \in \Gamma$, $\mathcal{A} \models \phi$ and for each $\psi \in \Delta$, $\mathcal{A} \models \psi$ iff $\mathcal{A} \in \text{Mod}(\Gamma)$ and $\mathcal{A} \in \text{Mod}(\Delta)$ iff $\mathcal{A} \in \text{Mod}(\Gamma) \cap \text{Mod}(\Delta)$.

N.B. Van Dalen's notation $\text{Mod}(\Gamma) \cap \text{Mod}(\Delta)$ is naughty because $\text{Mod}(\Gamma)$ is not a set. But as long as we understand by $\mathcal{A} \in \text{Mod}(\Gamma) \cap \text{Mod}(\Delta)$ nothing more than $\mathcal{A} \in \text{Mod}(\Gamma)$ and $\mathcal{A} \in \text{Mod}(\Delta)$, there is no harm.

For (vi), suppose that $\mathcal{A} \in \text{Mod}(\Gamma) \cup \text{Mod}(\Delta)$. Then $\mathcal{A} \in \text{Mod}(\Gamma)$ or $\mathcal{A} \in \text{Mod}(\Delta)$. So either (i) for each $\phi \in \Gamma$, $\mathcal{A} \models \phi$, or (ii) for each $\psi \in \Delta$, $\mathcal{A} \models \psi$. Observe that $\Gamma \cap \Delta \subseteq \Gamma$ and $\Gamma \cap \Delta \subseteq \Delta$. So whether (i) is true or (ii) is true, for each $\phi \in \Gamma \cap \Delta$, $\mathcal{A} \models \phi$, so $\mathcal{A} \in \text{Mod}(\Gamma \cap \Delta)$.

The converse fails. Here is a really easy counterexample. Let p, q be distinct propositional variables. Let $\Gamma = \{p\}$ and let $\Delta = \{q\}$. Then $\Gamma \cap \Delta = \emptyset$, so every structure is a model of $\Gamma \cap \Delta$. Let \mathcal{A} make p false.

Exercise 2 (3.2.6) Following Van Dalen's advice, suppose that Γ axiomatizes the class of well-orderings (with respect to language $<$). Add constants $\{c_i : i \in \omega\}$ to the language and let

$$\Gamma^* = \Gamma \cup \{c_{i+1} < c_i : i < \omega\}.$$

Now apply compactness (theorem 3.2.1) as Van Dalen applied it in the proof of the upward Löwenheim-Skolem theorem (Theorem 3.2.4). That is, let $\Delta \subset \Gamma$ be finite. Then there exists some maximum k such that c_i occurs in Δ . Let

$$\Gamma_k = \Gamma \cup \{c_{i+1} < c_i : i \leq k\}.$$

Then Γ_n has a model, namely

$$\mathcal{N}_k = (N, <, c_0, \dots, c_k) = (N, <, 0, \dots, k-1).$$

Since $\Delta \subseteq \Gamma_n$, $\mathcal{N}_k \models \Delta$. So by compactness (Theorem 3.2.1), there exists \mathcal{B} such that $\mathcal{B} \models \Gamma^*$. But \mathcal{B} has an infinite descending chain because Γ^* says so. Contradiction. So the class of all well-orderings is not characterized by any first-order theory.

Exercise 3 (3.2.8) Let Γ consist of the axioms:

1. $\forall x P(x, x)$;
2. $\forall x, y, z P(x, y) \wedge P(y, z) \rightarrow P(x, z)$;
3. $\forall x \exists y P(x, y)$.

Suppose for reductio that \mathcal{A} has finite domain and $\mathcal{A} \models \Gamma$. Let R be the denotation of P in \mathcal{A} . Say that sequence $\beta = (a_0, a_1, \dots, a_n)$ is a chain of length n in R if and only if for each i such that $0 \leq i < n$, $R(a_i, a_{i+1})$. Claim: For each chain in R of length $n > 0$, $R(a_0, a_n)$. The base case is trivial since $n > 0$. Consider chain $\beta = (a_0, a_1, \dots, a_n, a_{n+1})$. By the IH, $R(a_0, a_n)$. Since β is a chain, $R(a_n, a_{n+1})$. Since \mathcal{A} makes axiom 2 true, $R(a_0, a_{n+1})$. Say that a chain of length n is a cycle if, furthermore, $R(a_n, a_0)$ (if $n = 0$, then understand that $a_n = a_0$). Suppose that R has cycle β of length n . Then by the claim, $R(a_0, a_n)$ and $R(a_n, a_0)$, so since \mathcal{A} satisfies axiom 2, $R(a_0, a_0)$ and, hence, fails to satisfy axiom 1. Contradiction. So R has no cycle. Suppose that \mathcal{A} satisfies axiom 3. Start at arbitrary $a_0 \in |\mathcal{A}|$ and keep extending the chain from a_0 according to axiom 3. Since $|\mathcal{A}|$ is finite, eventually the same domain element occurs twice. That is a cycle in R . Contradiction.

Exercise 4 (3.2.12) As suggested, prove the contrapositive. Let $T_0 \subseteq T_1 \subseteq \dots \subseteq T_n \subseteq \dots$ and let $T^* = \bigcup_i T_i$. Note that the inclusions are proper, so the nested theories always get larger. Suppose for reductio that T^* is finitely axiomatizable. Then by Lemma 3.2.9, T^* is axiomatized by a finite subset $\Delta \subset T^*$ since, trivially, T^* axiomatizes itself. There exists n such that $\Delta \subseteq T_n$. Then $cn(\Delta) = T_n$. But since Δ axiomatizes T^* , it is also the case that $cn(\Delta) = T^*$. Hence, $T_n = T^*$. Hence, for each $i \geq n$, $T_i = T_n$.

Exercise 5 (bonus, 3.2.18) This is a bit of real mathematics that is almost an immediate consequence of the completeness theorem, which makes it clear that the theorem has some real mathematical content. The argument again proceeds like the argument of the upward Löwenheim-Skolem theorem, using compactness (which is an immediate consequence of the completeness theorem).

Name each node of the graph G by a constant. Add non-identity statements for all the constants. Specify the edges in the graph with binary predicate R and make sure to add in $R(b, a)$ whenever you put in $R(a, b)$ and never put in $R(a, a)$. Add the 3-coloring postulates specified in the exercise and call the resulting theory Γ_G . Now let G be a graph. Suppose that each finite sub-graph of G is 3-colorable. A sub-graph $G' \subseteq G$ is the restriction $G|S$ of G to some subset S of nodes (constants), so it corresponds to the theory $\Gamma_{G|S}$ in which each statement $R(a, b)$ involving a constant missing from S is deleted from Γ_G . So each theory $\Gamma_{G|S}$ is satisfiable if S is finite. Let Δ be a finite subset of Γ_G . There exists finite S such that $\Delta \subseteq \Gamma_{G|S}$, so Δ has a model. So by compactness (Theorem 3.2.1), Γ_G has a model. So G is 3-colorable.