**80-310/610 Logic and Computation**

Exercise Set 11

Kevin T. Kelly

**Exercise 1 (2.5.2.i)** *Let $x \notin FV(\psi)$. Let $T_1 \subseteq T_2 \subseteq \ldots \subseteq T_n \subseteq \ldots$. Let $T^* = \bigcup_i T_i$. First, one must show that $T^*$ is a* theory, *which means that $T^*$ is closed under derivability. Suppose that $T^* \vdash \phi$. Then there exists derivation $(\Gamma\mathcal{D}\phi)$, where $\Gamma$ is a finite subset of $T^*$. So there exists $k$ such that $\Gamma \subseteq T_k$. But $T_k$ is a theory, so $T_k$ is closed under derivability, so $\phi \in T_i \subseteq T^*$. Hence, $T^*$ is closed under derivability.*

*Next, one must show that if each $T_i$ is consistent, then $T^*$ is consistent. For the contrapositive, suppose that $T^*$ is inconsistent. Then $T^* \vdash \perp$. So there exists a derivation $(\Gamma\mathcal{D}\perp)$ where $\Gamma$ is a finite subset of $T^*$. So there exists $k$ such that $\Gamma \subseteq T_k$. So $T_k \vdash \perp$, so $T_k$ is not consistent. So it is not the case that each $T_i$ is consistent.*

**Exercise 2 (Bonus)** *Let $T = cn\{\exists x\exists y(P(x, y)\}$, which is consistent (e.g., one element domain in which $P$ is interpreted by reflexive relation $R(a, a)$). For some new constant $c$ the Henkin sentence $(\exists x\exists yP(x, y)) \to \phi(c, y)$ is in $T^*$. Then $(T^*)^*$ contains*

$$((\exists x\exists y(P(x, c) \vee P(x, y))) \to (\exists yP(c', c) \vee P(c', y)),$$

*where $c'$ is not in the vocabulary of $T^*$. Since $T$ is consistent, so is $T^*$ by Lemma 3.1.7, so by the model existence lemma 3.1.11, there exists $\mathcal{A}$ such that $\mathcal{A} \models (\exists x\exists yP(x, y)) \in T \subseteq T^*$. Now form structure $\mathcal{B}$ by adding new object $b$ to the domain of $\mathcal{A}$ such that that $R(b, a)$ fails for each $a$ in $|\mathcal{A}| \cup \{b\}$ and let $b$ be the interpretation of $c'$. Now $T$ remains true in $\mathcal{B}$ and each false existential statement remains false, so each Henkin conditional in $T^*$ with a false antecedent remains true in $\mathcal{B}$. Furthermore, each Henkin conditional in $T^*$ with a true consequent remains true in $\mathcal{B}$, so $\mathcal{B} \models T^*$. But $\mathcal{B} \not\models ((\exists x\exists y(P(x, c) \vee P(x, y))) \to (\exists yP(c', c) \vee P(c', y))$ because the antecedent is true and the consequent is false.*