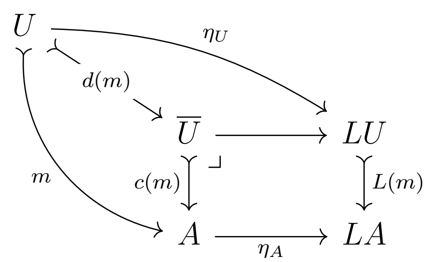
Category Theory — 80-514/814
Homework #4
Due Wednesday May 3, 9:00am

Let $\mathcal{E}$ be a topos, and let $\mathcal{F} \subseteq \mathcal{E}$ be a replete reflective subcategory whose reflector $L : \mathcal{E} \to \mathcal{E}$ (see previous homework sheet) preserves finite limits.

For subobjects $m : U \rightarrowtail A$ in $\mathcal{E}$ we define a factorization $m = c(m) \circ d(m)$ where $c(m) = \eta_A^*(L(m))$ and $d(m)$ is the mediating arrow as in the following diagram.



1. Show the following.

   (a) If $m, n \in \mathrm{Sub}(A)$ and $m \leq n$, then $c(m) \leq c(n)$.

   (b) $m \leq c(m)$ and $c(c(m)) \cong c(m)$ for all $m \in \mathrm{Sub}(A)$.

   (c) $f^*(c(m)) \cong c(f^*(m))$ for all $m \in \mathrm{Sub}(A)$ and $f : B \to A$.

These properties say that the mappings

$$\mathrm{Sub}(A) \to \mathrm{Sub}(A), \qquad m \mapsto c(m)$$

constitute a *universal closure operation*, i.e. an increasing and idempotent (pseudo-)natural transformation from the indexed preorder

$$\mathrm{Sub} : \mathcal{E}^{\mathrm{op}} \to \mathbf{Preord}$$

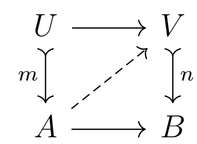
of subobjects to itself.

We say that $m \in \mathrm{Sub}(A)$ is *(L-)closed* if $c(m) \cong m$, and *(L-)dense* if $c(m) \cong \mathrm{id}_A$ in $\mathrm{Sub}(A)$. Pullback stability of c implies that the classes of closed and dense subobjects are closed under pullback.

2. (a) Show that $m : U \rightarrowtail A$ is dense iff $L(m)$ is an isomorphism.

   (b) Show that $c(m)$ is closed and $d(m)$ is dense for all $m : U \rightarrowtail A$.

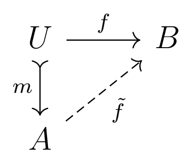
(c) Show that if m is dense and n is closed in a commutative square



then there exists a unique diagonal arrow making the two triangles commute.

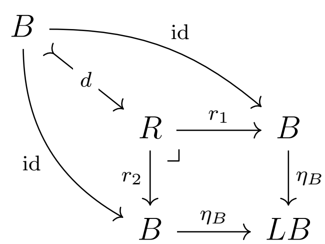
(d) Show that the factorization $m = c(m) \circ d(m)$ of $m : U \rightarrowtail A$ is determined up to isomorphism by the fact that $d(m)$ is dense and $c(m)$ is closed.

An object $B \in \mathcal{E}$ is called *L-separated*, if for every dense $m : U \rightarrowtail A$ and every $f : U \to B$ there exists at most one $\tilde{f} : A \to B$ with $\tilde{f} \circ m = f$,



and B is called a *L-sheaf* if for all m and f there exists exactly one such $\tilde{f}$.

3. (a) Show that $\eta_B : B \to LB$ is monic whenever B is separated. Hint: Show first that the inclusion $d : B \rightarrowtail R$ of B into the kernel of η_B is L-dense.



   (b) Show that B is a L-sheaf iff $B \in \mathcal{F}$.

   (c) Let $m : U \rightarrowtail A$ where $A \in \mathcal{F}$. Show that $U \in \mathcal{F}$ iff m is closed.

4. Let $m : B \rightarrowtail A$ be dense. Since closed subobjects are stable under pullback, the pullback map $m^* : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$ restricts to a map $\mathrm{Sub}_c(A) \to \mathrm{Sub}_c(B)$ between the sub-preorders of closed subobjects. Show that this restricted map is an equivalence of preorders. Hint: You will need the 'orthogonality' property from 2(c) at some point.

5. Let $j = \chi_{c(\mathtt{t})} : \Omega \to \Omega$ be the characteristic map of $c(\mathtt{t})$, where $\mathtt{t} : 1 \rightarrowtail \Omega$ is the generic subobject.

$$\begin{array}{ccccc} L1 & \longleftarrow & J & \longrightarrow & 1 \\ {\scriptstyle L\mathtt{t}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle c(\mathtt{t})} & \llcorner & \big\downarrow{\scriptstyle \mathtt{t}} \\ L\Omega & \xleftarrow[\eta_\Omega]{} & \Omega & \xrightarrow[j]{} & \Omega \end{array}$$

(a) Show that $\chi_{c(m)} = j \circ \chi_m$ for all subobjects $m : U \rightarrowtail A$.

(b) Show that the characteristic map $\chi_m : A \to \Omega$ of a subobject $m : U \rightarrowtail A$ factors through the equalizer $\Omega_j \rightarrowtail \Omega \underset{\mathrm{id}}{\overset{j}{\rightrightarrows}} \Omega$ of id and j iff m is closed.

(c) Show that $\Omega_j \in \mathcal{F}$.

(d) Show that Ω_j is a subobject classifier for $\mathcal{F}$.