Category Theory — 80-514/814
Homework #3
Due Monday April 17, 9:00am

1. A category $\mathbb{C}$ is called *balanced*, if every arrow $f : A \to B$ in $\mathbb{C}$ which is both a monomorphism and an isomorphism is already an isomorphism.

   (a) Give an example of a category which is balanced and a category which is not balanced.

   (b) Show that if every monomorphism is regular in a category $\mathbb{C}$, then $\mathbb{C}$ is balanced. (By duality, a category in which every epimorphism is regular is balanced.)

   (c) Show that in a balanced regular category, every epimorphism is regular.

   (d) Show that a category with finite limits and a subobject classifier is balanced.

   Thus every topos is balanced, which yields a simple proof that **Cat**, **Poset**, and **Top** are not toposes.

2. Which of the following toposes satisfy (SS) and/or (AC) (defined in the notes)?

   (a) **Set**

   (b) **Set** $\times$ **Set**

   (c) Presheaves on $(\bullet \to \bullet)$

   (d) Presheaves on $(\bullet \leftarrow \bullet \to \bullet)$

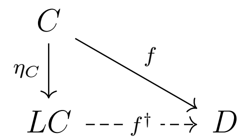
   (e) Presheaves on $(\bullet \to \bullet \leftarrow \bullet)$

   (f) Presheaves on $\mathbf{Z}_2$ (the group with two elements)

Recall that a *reflective subcategory* of a category $\mathbb{C}$ is a full subcategory $\mathbb{D} \subseteq \mathbb{C}$ such that the inclusion functor $J : \mathbb{D} \hookrightarrow \mathbb{C}$ has a left adjoint.

3. Show that a full subcategory $\mathbb{D} \subseteq \mathbb{C}$ is reflective iff for every $C \in \mathbb{C}$ there is an object $LC \in \mathbb{D}$ and a morphism $\eta_C : C \to LC$ such that for

every $D \in \mathbb{D}$ and $f : C \to D$ there is a unique $f^\dagger : LC \to D$ such that $f^\dagger \circ \eta_C = f$.



Hint: use the fact that a functor $U : \mathbb{B} \to \mathbb{A}$ has a left adjoint iff the 'comma category' $A \downarrow U$ has an initial object for all $A \in \mathbb{A}$.

In the situation of the exercise we can extend the assignment $C \mapsto LC$ to a functor $L : \mathbb{C} \to \mathbb{C}$ by setting $Lf = (\eta_{C'} \circ f)^\dagger$ for $f : C \to C'$, and it is easy to see that with this definition the arrows η_C constitute a natural transformation $\eta : \mathrm{id}_\mathbb{C} \to L$[1]. We call L the *reflector* of the reflective subcategory, and η its *unit*.

In the following we will use the characterization of reflective subcategories in terms of L and η to show that reflective subcategories of toposes are toposes, provided that L preserves finite limits.

4. A full subcategory $\mathbb{D} \subseteq \mathbb{C}$ is called *replete* if it is closed under isomorphism, i.e. $D \in \mathbb{D}$ and $C \cong D$ implies $C \in \mathbb{D}$.

   Let $\mathbb{D} \subseteq \mathbb{C}$ be a replete reflective subcategory with reflector L unit η. Given $C \in \mathbb{C}$, show that $C \in \mathbb{D}$ iff η_C is an isomorphism.

Since every full subcategory $\mathbb{D} \subseteq \mathbb{C}$ is equivalent to the replete full subcategory obtained by closing the class of objects under isomorphism in $\mathbb{C}$, the assumption that $\mathbb{D}$ is replete in $\mathbb{C}$ is not a genuine restriction but more of a convenience, at least when reasoning about properties that are stable under equivalence of categories.

5. Let $\mathbb{D} \subseteq \mathbb{C}$ be replete and reflective with reflector L and unit η.

   (a) Let $C \in \mathbb{C}$, $D \in \mathbb{D}$, and $f, g : LC \to D$. Show that if $f \circ \eta_C = g \circ \eta_C$ then $f = g$.

[1] We could also say that L is a functor of type $\mathbb{C} \to \mathbb{D}$, but then we would have to give the type of η as $\mathrm{id}_\mathbb{C} \to UL$, where $U : \mathbb{D} \hookrightarrow \mathbb{C}$ is the inclusion functor, and it is more convenient to leave the inclusion functor implicit to avoid cluttering up the notation.

(b) Let $\mathbb{J}$ be a category, let $D : \mathbb{J} \to \mathbb{D}$, and let

$$(A, (\alpha_i : A \to D_i)_{i \in \mathbb{J}})$$

be a limiting cone of D in $\mathbb{C}$[2]. Show that $\eta_A : A \to LA$ is an isomorphism, and thus $\mathbb{D}$ is closed under limits in $\mathbb{C}$.

In particular, if $\mathbb{C}$ has finite limits then so has $\mathbb{D}$.

6. Let $\mathbb{C}$ be a cartesian closed category and let $\mathbb{D} \subseteq \mathbb{C}$ be a replete and reflective subcategory whose reflector L preserves finite products, i.e. the canonical arrow

$$\mu_{A,B} = \langle Lp_1, Lp_2 \rangle \ : \ L(A \times B) \to LA \times LB$$

is an isomorphism for all $A, B \in \mathbb{C}$. Let $C \in \mathbb{C}$ and $D \in \mathbb{D}$, and consider the arrow

$$f = \left(L(D^C) \times C \xrightarrow{\mathrm{id} \times \eta} L(D^C) \times LC \xrightarrow{\mu^{-1}} L(D^C \times C) \xrightarrow{\varepsilon^\dagger} D \right)$$

Show that $\Lambda f : L(D^C) \to D^C$ is inverse to η_{D^C}, and thus $D^C \in \mathbb{D}$. Hint: Show first that $\Lambda f \circ \eta_{D^C} = \mathrm{id}$ by using the equation $\Lambda f \circ g = \Lambda(f \circ (g \times \mathrm{id}))$ and the equivalence $\Lambda h = k$ iff $h = \varepsilon \circ (k \times \mathrm{id})$.

We have shown that reflective subcategories of CCCs with finite-product preserving reflector are *exponential ideals*, meaning that $D^C \in \mathbb{D}$ whenever $D \in \mathbb{D}$ and $D \in \mathbb{C}$. In particular, such reflective subcategories are cartesian closed themselves.

To show that reflective subcategories of toposes with finite-limit-preserving reflectors are toposes, it remains to show that they have subobject classifiers. This will be established on the next homework sheet.

[2]We view D as a diagram with 'index-category' $\mathbb{J}$ and therefore use subscript notation for the values of D.