Category Theory — 80-514/814
Homework #2
Due Wednesday February 15, 9:00am

1. Let (Σ, E) be the algebraic theory with *no* operations and *no* equations. Show that the Lawvere theory $\mathbf{L}(\Sigma, E)$ is equivalent to the opposite of the category **FinSet** of finite sets and functions.

   Hint: construct a functor $F : \mathbf{L}(\Sigma, E)^{\mathrm{op}} \to \mathbf{FinSet}$ which maps objects $[n]$ of $\mathbf{L}(\Sigma, E)$ to sets with n elements, and show that it is an equivalence.

2. An epimorphism $e : B \to A$ in a category $\mathbb{C}$ is called *extremal*, if in any factorization $e = mf$ where m is a monomorphism, m is already an isomorphism.

   In other words, an epimorphism is called extremal if it does not factor through any non-trivial subobjects of its codomain.

   (a) Show that every regular epimorphism is a cover (as defined in the lecture), and that every cover is an extremal epimorphism.

   (b) Show that if $\mathbb{C}$ has pullbacks, then every extremal epimorphism is a cover.

   (c) Show that if $\mathbb{C}$ has equalizers, then every morphism which is left orthogonal to all monomorphisms is an epimorphism.

   We shall see that *in a regular category*, every cover is a regular epimorphism, thus the three concepts coincide.

   Recall that $\mathbb{C}$ is said to have *image factorizations*, if every morphism f in $\mathbb{C}$ factors as $f = me$ where e is a cover and m is a mono.

   (d) Show that if $\mathbb{C}$ has image factorizations, then for every object $A \in \mathbb{C}$, the inclusion functor

   $$I : \mathrm{Sub}(A) \to \mathbb{C}/A$$

   from the preorder of subobjects of A to the slice category of $\mathbb{C}$ over A has a left adjoint.

(e) Show that if $\mathbb{C}$ has finite limits and I has a left adjoint, then $\mathbb{C}$ has image factorizations.

3. Consider the partially ordered sets $A = \{0 \leq 2\}$, $B = \{0 \leq 1 \leq 2\}$, and $C = \{a \leq b, c \leq d\}$ (so $C \cong A + A$). Let $f : A \to B$ be the obvious inclusion, and let $g : C \to B$ be the monotone map given by $g(a) = 0$, $g(b) = g(c) = 1$, $g(d) = 2$. Show that g is a cover, but the pullback f^*g of g along f isn't. Thus the category **Poset** of posets and monotone maps is not regular (although it has finite limits and image factorizations).

4. Show that the following natural deduction rules for equality and existential quantification can be derived from the rules given in class (and in the notes).

$$\frac{}{\vec{x} \mid \Gamma \vdash t = t} \qquad \frac{\vec{x} \mid \Gamma \vdash \varphi[s/y] \qquad \vec{x} \mid \Gamma \vdash s = t}{\vec{x} \mid \Gamma \vdash \varphi[t/y]}$$

$$\frac{\vec{x} \mid \Gamma, \vdash \varphi[t/y]}{\vec{x} \mid \Gamma \vdash \exists y \,.\, \varphi} \qquad \frac{\vec{x} \mid \Gamma \vdash \exists y \,.\, \varphi \qquad \vec{x}, y \mid \Gamma, \varphi \vdash \psi}{\vec{x} \mid \Gamma \vdash \psi}$$

Here, the formulas of Γ as well as ψ and the terms s, t are assumed to be in context $\vec{x}$, and φ is in context $\vec{x}, y$.

5. Let $f, g : A \to B$ be two parallel arrows in a regular category $\mathbb{R}$. Show that $f = g$ iff the judgment $(x : A \mid \vdash fx = gx)$ holds in the internal language of $\mathbb{R}$.

6. Let $e : A \twoheadrightarrow B$ be a cover in a regular category $\mathbb{R}$, let $r_1, r_2 : R \to A$ be its kernel pair[1], and let $g : A \to C$ such that $g \circ r_1 = g \circ r_2$.

(a) Show that the judgment $(a, a' \mid ea = ea' \vdash ga = ga')$ holds in the internal logic.

(b) Show that the subobject $R = [\![b, c \mid \exists a \,.\, ea = b \wedge ga = c]\!] \in \mathrm{Sub}(B \times C)$ is the graph of a morphism $h : B \to C$ satisfying $he = g$.

Since e is epic, h is unique with the property $he = g$, thus we have shown that e is the coequalizer of its kernel pair and thus a regular epi.

[1] Recall that the kernel pair of e is given by the pullback $\begin{array}{ccc} R & \longrightarrow & B \\ \downarrow{\scriptstyle \langle r_1, r_2 \rangle} & & \downarrow{\scriptstyle \delta} \\ A \times A & \xrightarrow[e \times e]{} & B \times B \end{array}$.