Category Theory — 80-514/814
Homework #1
Due Wednesday February 1, 9:00am

1. (a) Give the definitions of equalizers and coequalizers in a category $\mathbb{C}$.

   (b) Determine the equalizer and coequalizer of the functions

$$f, g : \{a, b, c, d\} \to \{0, 1, 2, 3, 4, 5, 6\}$$

   given by

$$\begin{aligned} f(a) &= 0 & g(a) &= 1 \\ f(b) &= 1 & g(b) &= 2 \\ f(c) &= 3 & g(c) &= 4 \\ f(d) &= 5 & g(d) &= 5 \end{aligned}$$

   in the category **Set** of sets and functions. No proof required, but it's helpful to draw a picture.

2. (a) Give the definition of initial and terminal objects in a category $\mathbb{C}$.

   (b) Show that if S and T are both terminal in $\mathbb{C}$, then they are isomorphic.

3. Which of the following categories have initial and/or terminal objects, and what are they? Answer as many as you can, no proofs necessary.

   (a) The category **Set** of sets and functions.

   (b) The category **Rel** of sets and relations.

   (c) The category **Grp** of groups and group homomorphisms.

   (d) The category **Rng** of rings and ring homomorphisms.

   (e) The category **Field** of fields and field homomorphisms.

   (f) The category **Inj** of sets and injective functions.

   (g) The category **Surj** of sets and surjective functions.

   (h) The *slice category* **Set**$/I$ for a fixed set I.

4. *The following exercise is lengthy and difficult, and moreover the definition of* $\mathbf{Grp}(\mathbb{C})$ *requires the notion of internal morphism that we haven't defined yet. So don't despair if you can't solve it immediately.*

   Assume that $\mathbb{C}$ and $\mathbb{D}$ are categories with finite products.

   (a) Let $F : \mathbb{C} \to \mathbb{D}$ be a functor preserving finite products. Show that F gives rise to a functor $\tilde{F} : \mathbf{Grp}(\mathbb{C}) \to \mathbf{Grp}(\mathbb{D})$, i.e. that 'finite product preserving functors map internal groups to internal groups'.

   (b) Let $F, G : \mathbb{C} \to \mathbb{D}$ be finite-product-preserving functors, and let $\eta : F \to G$ be a natural transformation. Show that η induces a natural transformation

   $$\tilde{\eta} : \tilde{F} \to \tilde{G} : \mathbf{Grp}(\mathbb{C}) \to \mathbf{Grp}(\mathbb{D}).$$

   (c) Let $\mathbb{X}$ be an arbitrary category, and let $\mathbb{C}^{\mathbb{X}}$ be the category of functors $\mathbb{X} \to \mathbb{C}$ and natural transformations between them. The category $\mathbb{C}^{\mathbb{X}}$ has finite products, since $\mathbb{C}$ has them. Describe how they are constructed (no proof necessary).

   (d) Given an object $X \in \mathbb{X}$, show that the 'evaluation functor'

   $$\begin{aligned} E_X : \mathbb{C}^{\mathbb{X}} &\to \mathbb{C} \\ E_X(F) &= F(X) \\ E_X(\eta) &= \eta_X \end{aligned}$$

   preserves finite products.

   (e) Show that any morphism $f : X \to Y$ in $\mathbb{X}$ gives rise to a natural transformation

   $$E_f : E_X \to E_Y$$

   (f) Combine the previous results to construct a functor

   $$H : \mathbf{Grp}(\mathbb{C}^{\mathbb{X}}) \to \mathbf{Grp}(\mathbb{C})^{\mathbb{X}}$$

   One can show that the functor H is an equivalence of categories

   $$\mathbf{Grp}(\mathbb{C}^{\mathbb{X}}) \simeq \mathbf{Grp}(\mathbb{C})^{\mathbb{X}},$$

   i.e. that 'groups in functor categories are functors into groups'. The same works for models of arbitrary algebraic theories (Σ, E).