Categorical logic – lecture notes

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Last time: encoding of first-order logic in system with λ-abstractions and equality.

Recall that we encoded

$$\forall x.\varphi \equiv ((\lambda x.\varphi) = (\lambda x.\top))$$

This needs to satisfy

$$\frac{\Delta x \mid \Gamma \vdash \varphi}{\Delta \mid \Gamma \vdash \forall x.\varphi}$$

$$\frac{\Delta \mid \Gamma \vdash \forall x.\varphi}{\Delta \mid \Gamma \vdash \varphi[t/x]}$$

To see this, note that we have

$$\cfrac{\cfrac{\cfrac{}{\Delta x \mid \Gamma, \varphi \vdash \top} \qquad \cfrac{\Delta x \mid \Gamma \vdash \varphi}{\Delta x \mid \Gamma, \top \vdash \varphi}}{\Delta x \mid \Gamma \vdash \varphi = \top}}{\Delta \mid \Gamma \vdash (\lambda x.\varphi) = (\lambda x.\top)}$$

and

$$\cfrac{\cfrac{\cfrac{\Delta \mid \Gamma \vdash (\lambda x.\varphi) = (\lambda x.\top)}{\Delta \mid \Gamma \vdash (\lambda x.\varphi)t = (\lambda x.\top)t}}{\Delta \mid \Gamma \vdash \varphi[t/x] = \top}}{\Delta \mid \Gamma \vdash \varphi[t/x]}$$

We also need encodings of $\wedge$, $\top$, $\Rightarrow$, $\forall$, $\vee$, $\bot$, $\exists$. For example we can take

$$\varphi \vee \psi \equiv \forall p : \Omega.(\varphi \Rightarrow p) \Rightarrow (\psi \Rightarrow p) \Rightarrow p$$

Internal logic of a topos

Let $\mathcal{E}$ be a topos. Its internal language is given by:

- A higher-order signature $\Sigma(\varphi) = (\Sigma(\mathcal{E})_0, \Sigma(\mathcal{E})^c)$, which is definied in a similar way as for cartesian closed categoires. Specifically, we have
  - $\Sigma(\mathcal{E})_0 = \mathrm{ob}(\mathcal{E})$
  - For $A \in \tau(\Sigma(\mathcal{E})_0)$, define $\Sigma(\mathcal{E})^c_t = \mathrm{hom}(1, [\![A]\!])$.
- There is a canonical $\Sigma(\mathcal{E})$-structure $\mathbb{M}(\mathcal{E})$ in $\mathcal{E}$ defined by
  - $X_{\mathbb{M}(\mathcal{E})} = X$ for $X \in \Sigma(\mathcal{E})$; and
  - $f_{\mathbb{M}(\mathcal{E})} = f$ for $f \in \Sigma(\mathcal{E})^c_t$.

We can prove a lot of things using the internal language of a topos.

Recall that for $f : A \to B$ in $\mathcal{E}$ we define

$$\ulcorner f \urcorner = \Lambda(1 \times A \xrightarrow{p} A \xrightarrow{f} B) : 1 \to B^A$$

Observation. For $f : A \to B$ in $\mathcal{E}$, we have

$$[\![x : A \mid \ulcorner f \urcorner(x) : B]\!] = f$$

Theorem 1. For $A \in \mathcal{E}$, the poset $(\mathrm{Hom}(A, \Omega), \leq)$ (which is equivalent to $\mathrm{Sub}(A)$) is a Heyting algebra, with operations $\wedge$, $\top$, $\vee$, $\bot$, $\Rightarrow$ given by

$$[\![x : A \mid \varphi : \Omega]\!] \wedge [\![x : A \mid \psi : \Omega]\!] = [\![x : A \mid \varphi \wedge \psi : \Omega]\!]$$

$$[\![x : A \mid \varphi : \Omega]\!] \vee [\![x : A \mid \psi : \Omega]\!] = [\![x : A \mid \varphi \vee \psi : \Omega]\!]$$

$$[\![x : A \mid \varphi : \Omega]\!] \Rightarrow [\![x : A \mid \psi : \Omega]\!] = [\![x : A \mid \varphi \Rightarrow \psi : \Omega]\!]$$

$$\top = [\![x : A \mid \top]\!] \qquad \bot = [\![x : A \mid \bot]\!]$$

Proof. We check that $\wedge$ is indeed a binary meet, that is for $p, q, r : A \to \Omega$ we have

$$p \wedge q \leq p, \quad p \wedge q \leq q, \quad \text{and if } r \leq p \text{ and } r \leq q, \text{ then } r \leq p \wedge q$$

Assume $p = [\![\varphi]\!]$, $q = [\![\psi]\!]$ and $r = [\![\theta]\!]$. We have

- $p \wedge q \leq p$ if and only if $(x : A \mid \varphi \wedge \psi \vdash \varphi)$ is valid—this is a tautology;

- $p \wedge q \leq q$ if and only if $(x : A \mid \varphi \wedge \psi \vdash \psi)$ is valid—this is a tautology.

The remaining condition is equivalent to the validity of the following rule of inference:

$$\frac{x \mid \theta \vdash \varphi \qquad x \mid \theta \vdash \psi}{x \mid \theta \vdash \varphi \wedge \psi}$$

This is valid, since it is the introduction rule for the conjunction operator, so by soundness we're done.

Likewise for the other connectives. □

Theorem 2. For $f : B \to A$, the function

$$- \circ f : \mathrm{Hom}(A, \Omega) \to \mathrm{Hom}(B, \Omega)$$

preserves Heyting algebra structure.

Proof. This follows from

$$[\![a : A \mid \varphi]\!] \circ f = [\![a : A \mid \varphi]\!] \circ [\![b : B \mid \ulcorner f \urcorner(b)]\!] = [\![b : B \mid \varphi[\ulcorner f \urcorner(b)/a]]\!]$$

From this, we have, for instance

$$\begin{aligned} ([\![a : A \mid \varphi]\!] \wedge [\![a : A \mid \psi]\!]) \circ f &= [\![a : A \mid \varphi \wedge \psi]\!] \circ f \\ &= [\![b : B \mid (\varphi \wedge \psi)[\ulcorner f \urcorner(b)/a]]\!] \\ &= [\![b : B \mid \varphi[\ulcorner f \urcorner(b)/a] \wedge \psi[\ulcorner f \urcorner(b)/a]]\!] \\ &= [\![b : B \mid \varphi[\ulcorner f \urcorner(b)/a]]\!] \wedge [\![b : B \mid \psi[\ulcorner f \urcorner(b)/a]]\!] \\ &= ([\![a : A \mid \varphi]\!] \circ f) \wedge ([\![a : A \mid \psi]\!] \circ f) \end{aligned}$$

Theorem 3. Let $A, B \in \mathcal{E}$ and let $p_1 : A \times B \to A$ be the projection morphism. Then the functor

$$- \circ p_1 : \mathrm{Hom}(A, \Omega) \to \mathrm{Hom}(A \times B, \Omega)$$

has left and right adjoints $\exists_B, \forall_B$.

Proof. Take

$$\exists_B [\![\psi]\!] = [\![a \mid \exists b.\psi]\!] \quad \text{and} \quad \forall_B [\![\psi]\!] = [\![a \mid \forall b.\psi]\!]$$

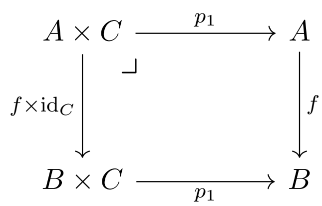
Note that the following rules of inference are valid:

$$\frac{\Delta x \mid \varphi \vdash \psi}{\overline{\Delta \mid \varphi \vdash \forall x.\psi}}$$

$$\frac{\Delta x \mid \psi \vdash \varphi}{\Delta \mid \exists x.\psi \vdash \varphi}$$

So in the internal logic we have $f \leq \forall_B g$ if and only if $f \circ p_1 \leq g$ and $\exists_B g \leq f$ if and only if $g \leq f \circ p_1$.

Theorem 4 (Beck–Chevalley)**.** Consider a pullback diagram:



Then given $[\![b, c \mid \varphi]\!] : B \times C \to \Omega$, we have

$$\exists_C([\![b, c \mid \varphi]\!] \circ (f \times \mathrm{id})) = (\exists_C [\![b, c \mid \varphi]\!]) \circ f$$

Proof.

$$\begin{aligned}
\exists_C([\![b, c \mid \varphi]\!] \circ (f \times \mathrm{id})) &= \exists_C([\![b, c \mid \varphi]\!] \circ [\![a, c \mid (\ulcorner f \urcorner(a), c)]\!] \\
&= \exists_C([\![a, c \mid \varphi[\ulcorner f \urcorner(a)/b]]\!] \\
&= [\![a \mid \exists c.\varphi[\ulcorner f \urcorner(a)/b]]\!]
\end{aligned}$$

A similar computation for the expression on the right-hand side yields the same result. Likewise for $\forall$. □

Theorem 5. The topos $\mathcal{E}$ is a regular category.

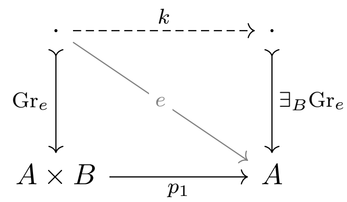
Proof. Let $e : B \to A$. Consider the graph

$$\mathrm{Gr}_e = [\![a, b \mid e(b) = a]\!]$$

Note that $\exists_B \mathrm{Gr}_e = [\![a \mid \exists b.e(b) = a]\!]$. Now

$$p_1^*(\exists_B \mathrm{Gr}_e) \geq \mathrm{Gr}_e$$

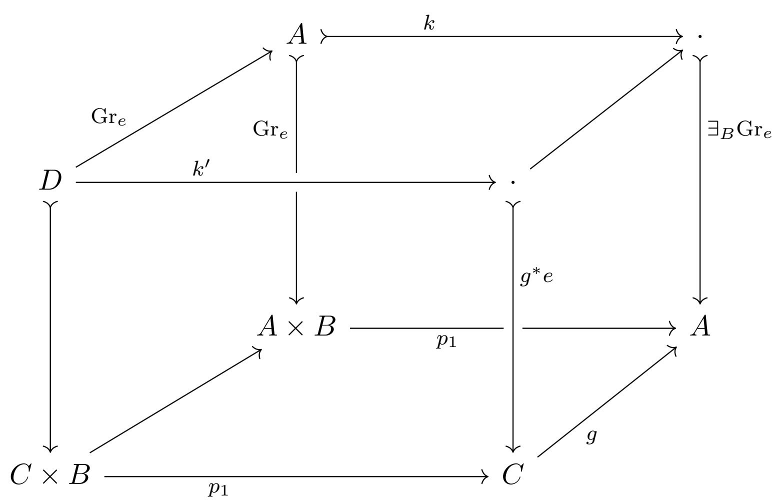
. Thus we get a map from Gr_e into the pullback of $\exists_B \mathrm{Gr}_e$ along p_1, and hence (equivalently) a map k as in: as in:



We claim that this provides the required image factorisation of e.

By the universal property of $\exists_{p_1} \dashv p_1^*$, we have $\exists_B \mathrm{Gr}_e$ is the least subobject of A that e factors through. This implies, in particular, that k is an extremal epimorphism ($\equiv$ cover, since $\mathcal{E}$ has finite limits). This gives the required image factorisation.

It remains to show that this factorisation is stable under pullback.



The map $A \to A \times B$ is $[\![a, b \mid \ulcorner e \urcorner(b) = a]\!]$. The map $D \to C \times B$ is $[\![c, b \mid \ulcorner e \urcorner(b) = \ulcorner g \urcorner(c)]\!]$. The map $\cdot \to C$ is $[\![c \mid \exists b. \ulcorner e \urcorner(b) = g(c)]\!]$.