Categorical logic – lecture notes

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Triposes

'Tripos' is a pun for a couple of reasons, one of which being that it stands for 'topos representing indexed poset'.

An $\mathbb{C}$-indexed poset is a functor $\mathbb{C}^{\text{op}} \to \mathbf{Pos}$—we'll typically focus on set-indexed posets $\mathbf{Set}^{\text{op}} \to \mathbf{Pos}$. More specifically, we will be interested in indexed *preorders*, of which the canonical example is the subobject functor $\text{sub} : \mathcal{E}^{\text{op}} \to \mathbf{PreOrd}$, assigning to each object A its subset preorder $(\text{sub}(A), \leq)$ and to each morphism $f : B \to A$ the pullback functor $f^* : \text{sub}(A) \to \text{sub}(B)$.

Let $\mathcal{E}$ be a topos.

(I) All subobject preorders are Heyting algebras.

(II) For all $f : B \to A$, the pullback functor $f^* : \text{sub}(A) \to \text{sub}(B)$ preserves Heyting algebra structure, and has left and right adjoints $\exists_f \dashv f^* \dashv \forall_f$ which satisfy the Beck–Chevalley condition.

(III) For every object A of $\mathcal{E}$ and every subobject $m : U \to A$, there is a unique $f : A \to \Omega$ such that $f^*(t : 1 \to \Omega) \cong m$ in $\text{sub}(A)$.

Definition. A **tripos** (on **Set**) is a functor $\mathcal{P} : \mathbf{Set}^{\text{op}} \to \mathbf{PreOrd}$ such that

(I) All $\mathcal{P}(A)$ are Heyting algebras, the elements of which we call **predicates**.

(II) For all functions $f : B \to A$, the functor $\mathcal{P}(f) = f^* : \text{sub}(A) \to \text{sub}(B)$ preserves Heyting algebra structure, and has left and right adjoints $\exists_f \dashv f^* \dashv \forall_f$ which satisfy the Beck–Chevalley condition: for all pullbacks

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

we have

$$\exists_q p^* \cong g^* \exists_f \quad \text{and} \quad \forall_q p^* \cong g^* \forall_f$$

(III) There is a set Prop and a predicate $\mathsf{tr} \in \mathcal{P}(\mathsf{Prop})$, called the **generic predicate**, such that for all A and all $\varphi \in \mathcal{P}(A)$, there is $f : A \to \mathsf{Prop}$ (not necessarily unique) such that $f^*(\mathsf{tr}) \cong \varphi$.

We will prove that every tripos gives rise to a topos and that there are interesting triposes arising from realisability.

Examples.

(I) The covariant power set functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Poset} \hookrightarrow \mathbf{PreOrd}$ is a tripos.

(II) If A is a complete Heyting algebra then

$$[-, A] : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Poset} \hookrightarrow \mathbf{PreOrd}$$

defined by

$$I \mapsto (A^I, \leq) \qquad \text{and} \qquad f \mapsto - \circ f$$

is a tripos, where $\leq$ is the pointwise ordering. Adjoints arise from completeness, and the generic predicate is given by $\mathrm{id}_A \in [A, A]$.

(III) The **effective tripos**. See below.

The effective tripos

The idea is that the effective tripos will give a model of logic based on computability; it is motivated by *Kleene realisability*.

Definition. Write PR for the set of all partial recursive functions. The elements of PR are particular partial functions $f : \mathbb{N} \rightharpoonup \mathbb{N}$.

Definition. The **effective tripos** $\mathrm{eff} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{PreOrd}$ is defined by

$$\mathrm{eff}(I) = (\mathcal{P}(\mathbb{N})^I, \leq)$$

where we view the power set $\mathcal{P}(\mathbb{N})$ as being a set of *truth values* or *realisers* of formulae—that is, for $\varphi : I \to \mathcal{P}(\mathbb{N})$, the subset $\varphi(i) \subseteq \mathbb{N}$ is the set of realisers of φ. The ordering $\leq$ is defined for $\varphi, \psi : I \to \mathcal{P}(\mathbb{N})$ by letting $\varphi \leq \psi$ if and only if there exists $\alpha \in \mathsf{PR}$ such that, for all $i \in I$ and all $n \in \varphi(i)$, we have $\alpha(n){\downarrow}$ and $\alpha(n) \in \psi(i)$. For $f : J \to I$, we define the reindexing $f^* = - \circ f : \mathrm{eff}(I) \to \mathrm{eff}(J)$. The generic predicate is defined by $\mathrm{id}_{\mathcal{P}(\mathbb{N})} \in \mathrm{eff}(\mathcal{P}(\mathbb{N}))$.

Lemma. The reindexing maps f^* are monotone.

Proof. This follows from the pointwise nature of the definition of $\leq$ on $\mathcal{P}(\mathbb{N})^I$. ☐

In what follows, fix a primitive recursive pairing function $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with projections $p, q : \mathbb{N} \to \mathbb{N}$, and let $(\phi_n)_{n \in \mathbb{N}}$ be an effective enumeration of partial recursive functions.

Moreover, for $U, V \subseteq \mathbb{N}$, define

$$U \wedge V = \{\langle n, m \rangle \mid n \in U,\ m \in V\} \subseteq \mathbb{N}$$

$$U \vee V = \{\langle n, 0 \rangle \mid n \in U\} \cup \{\langle m, 1 \rangle \mid m \in V\} \subseteq \mathbb{N}$$

$$U \Rightarrow V = \{e \in \mathbb{N} \mid \forall n \in U,\ \varphi_e(n)\!\downarrow \text{ and } \varphi_e(n) \in V\}$$

Lemma. For all sets I, the preorder $\mathrm{eff}(I)$ is a Heyting algebra.

Proof. For all $i \in I$, define

$$\bot(i) = \varnothing$$

$$\top(i) = \mathbb{N}$$

$$(\varphi \wedge \psi)(i) = \varphi(i) \wedge \psi(i)$$

$$(\varphi \vee \psi)(i) = \varphi(i) \vee \psi(i)$$

$$(\varphi \Rightarrow \psi)(i) = \varphi(i) \Rightarrow \psi(i)$$

where $\wedge, \vee, \Rightarrow$ are defined on subsets of $\mathbb{N}$ as above the statement of the theorem.

We verify that implication condition, namely that

$$\varphi \wedge \psi \leq \theta \quad \Leftrightarrow \quad \varphi \leq \psi \Rightarrow \theta$$

Spelling this out, we have to check the following biimplication

$$\frac{\exists a\, \forall i\, \forall n \in \varphi(i)\, \forall m \in \psi(i)\, \alpha(\langle n, m \rangle)\!\downarrow \wedge\, \alpha(\langle n, m \rangle) \in \theta(i)}{\exists \beta\, \forall i\, \forall n \in \varphi(i)\, \beta(n)\!\downarrow \wedge\, \beta(n) \in \psi(i) \Rightarrow \theta(i)}$$

Spelling this out even further, note that

$$\psi(i) \Rightarrow \theta(i) \quad \text{if and only if} \quad \forall m \in \psi(i)\, \phi_{\beta(n)}(m)\!\downarrow \wedge\, \phi_{\beta(n)}(m) \in \theta(i)$$

This follows from a theorem of recursion theory, namely the s_{mn} theorem. □

Lemma. The predicate $\mathrm{id} \in \mathrm{eff}(\mathcal{P}(\mathbb{N}))$ satisfies the condition required for the generic predicate of a tripos. □

Lemma. The reindexing maps f^* for all $f : J \to I$ have left and right adjoints.

Proof. Given $\varphi : J \to \mathcal{P}(\mathbb{N})$, define

$$(\exists_f \varphi)(i) = \bigcup_{f(j)=i} \varphi(j) \quad \text{and} \quad (\forall_f \varphi)(i) = \bigcap_{f(j)=i} \mathbb{N} \Rightarrow \varphi(j)$$

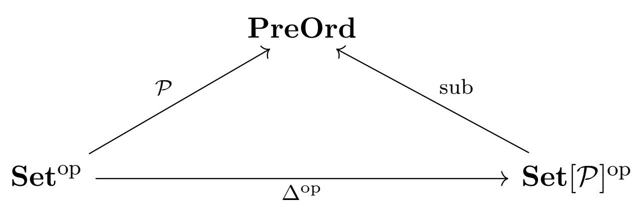
A technicality in the definition of $\forall_f$ requires us to throw in a dummy λ-abstraction, which is why we see $\mathbb{N} \Rightarrow \varphi(j)$ instead of simply $\varphi(j)$ in its definition. The details are omitted. □

The following theorem is now immediate.

Theorem. The effective tripos $\mathrm{eff} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{PreOrd}$ is indeed a tripos. □

The tripos-to-topos construction

The idea behind the tripos-to-topos construction is that a tripos $\mathcal{P}$ represents a non-standard logic on the category of sets. We seek to enlarge the base category to a new base category $\mathbf{Set}[\mathcal{P}]$, such that $\mathcal{P}$ in some sense becomes the subobject fibration on $\mathbf{Set}[\mathcal{P}]$. Thus, we'll have a functor $\Delta : \mathbf{Set} \to \mathbf{Set}[\mathcal{P}]$ such that the following diagram commutes up to natural isomorphism:



Let $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{PreOrd}$ be a tripos, and define a category $\mathbf{Set}[\mathcal{P}]$ as follows.

- The objects of $\mathbf{Set}[\mathcal{P}]$ are pairs (A, ρ), where A is a set and $\rho \in \mathcal{P}(A \times A)$, such that ρ is a partial equivalence relation on A with respect to $\mathcal{P}$, in the sense that

$$(a, b, c : A \mid \rho\, a\, b,\ \rho\, b\, c \vdash \rho\, a\, c) \quad \text{and} \quad (a, b : A \mid \rho\, a\, b \vdash \rho\, b\, a)$$

  hold in the internal language of $\mathcal{P}$, which is defined as expected.

- The morphisms $(A, \rho) \to (B, \sigma)$ are equivalence classes of predicates $\varphi \in \mathcal{P}(A \times B)$ under

$$(a, b \mid \varphi\, a\, b \vdash \rho\, a\, a \wedge \sigma\, b\, b)$$
$$(a, a', b, b' \mid \rho\, a\, a',\ \sigma\, b\, b',\ \varphi\, a\, b \vdash \varphi\, a'\, b')$$
$$(a\, b\, b' \mid \varphi\, a\, b,\ \varphi\, a\, b' \vdash b = b')$$
$$(a \mid \rho\, a\, a \vdash \exists b. \varphi\, a\, b)$$

  where $\varphi, \psi \in \mathcal{P}(A \times B)$ are identified as morphisms of $\mathbf{Set}[\mathcal{P}]$ if and only if $\varphi \cong \psi$ in $\mathcal{P}(A \times B)$.

- Composition is defined as follows. For

$$(A, \rho) \xrightarrow{[\varphi]} (B, \sigma) \xrightarrow{[\gamma]} (C, \tau)$$

  let

$$[\gamma] \circ [\varphi] = [\![a, c \mid \exists b. \varphi\, a\, b \wedge \psi\, b\, c]\!]$$

- The identity is defined by $\mathrm{id}_{(A,\rho)} = [\rho] : (A, \rho) \to (A, \rho)$.

Theorem. Let $\mathcal{P}$ be a tripos. Then $\mathbf{Set}[\mathcal{P}]$ is a topos.

Proof. Write $\mathcal{E} = \mathbf{Set}[\mathcal{P}]$.

The terminal object of $\mathcal{E}$ is $(1, \top)$.

Binary products in $\mathcal{E}$ are defined by

$$(A, \rho) \times (B, \sigma) = (A \times B, \rho \bowtie \sigma)$$

where $(\rho \bowtie \sigma) = [\![a, b, a', b' \mid \rho\, a\, a' \wedge \sigma\, b\, b']\!]$.

Equalisers are given by

$$(A, \rho|_\theta) \rightarrowtail (A, \rho) \underset{[\gamma]}{\overset{[\varphi]}{\rightrightarrows}} (B, \sigma)$$

where $\theta \in \mathcal{P}(A)$ is given by $\theta = [\![a \mid \exists b. \varphi\, a\, b \wedge \gamma\, a\, b]\!]$ and

$$(\rho|_\theta) = [\![a, a' \mid \theta\, a \wedge \rho\, a\, a']\!]$$

The subobject classifier is given by $\Omega = (\mathsf{Prop}, \tau)$, where

$$\tau = (p, q : \mathsf{Prop} \mid (\mathsf{tr}(p) \Rightarrow \mathsf{tr}(q)) \wedge (\mathsf{tr}(q) \Rightarrow \mathsf{tr}(p)))$$

Exponentials are given by

$$(B, \sigma)^{(A,\rho)} = (\mathsf{Prop}^{A \times B}, \tau)$$

where, informally speaking, $\tau(r, s)$ says that r, s are equivalent functional relations with respect to $\mathcal{P}$ which are comptaible with ρ and σ. ☐

This construction also gives rise to a functor $\Delta : \mathbf{Set} \to \mathbf{Set}[\mathcal{P}]$. We let $\Delta(I) = (I, \mathrm{eq}_I)$, where $\mathrm{eq}_I = \exists_{\delta_I}(\top)$.