Categorical logic – lecture notes

Clive Newstead, Wednesday 29th March 2017

Last time: toposes are regular.

Furthermore, toposes are coherent. This is immediate: we've already shown that the subobject lattices are Heyting algebras, and that f^* preserves the Heyting algebra structure.

Lemma. Toposes are Heyting categories.

Proof. We show that for every $f : A \to B$, the functor

$$f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$$

has a right adjoint $\forall_f$. To see this, define

$$\forall_f [\![a : A \mid \varphi(a)]\!] = [\![b : B \mid \forall a : A.\ (f(a) = b) \Rightarrow \varphi(a)]\!]$$

We have for $[\![b : B \mid \psi(b)]\!] \in \mathrm{Sub}(B)$:

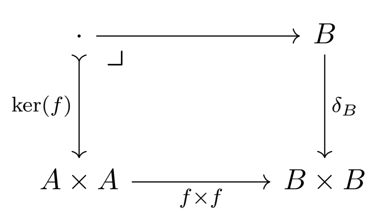
$$\begin{array}{c} a : A \mid \psi[f(a)] \vdash \varphi[a] \\ \hline\hline a : A, b : B \mid \psi[b], f(a) = b \vdash \varphi[a] \\ \hline\hline a : A, b : B \mid \psi[b] \vdash f(a) = b \Rightarrow \varphi[a] \\ \hline\hline b : B \mid \psi[b] \vdash \forall a. f(a) = b \Rightarrow \varphi[a] \end{array}$$

Hence $f^* \dashv \forall_f$, as required. □

Definition. A subobject $R \in \mathrm{Sub}_{\mathcal{E}}(A \times A)$ is an **equivalence relation** if it is so in the internal logic of $\mathcal{E}$. Specifically, if and only if the following hold in the internal logic:

$$a : A \mid\ \vdash R\,a\,a, \qquad a, b, c : A \mid R\,a\,b,\ R\,b\,c \vdash R\,a\,c, \qquad a, b : A \mid R\,a\,b \vdash R\,b\,a$$

Every kernel pair is an equivalence relation:



where $\ker(f) = [\![a, a' : A \mid f(a) = f(a')]\!]$.

Definition. A regular category $\mathbb{R}$ is **exact** if every equivalence relation is a kernel (and hence the components have a coequaliser).

Theorem. Every topos $\mathcal{E}$ is exact.

Proof. Let $R \in \mathrm{Sub}(A \times A)$ be an equivalence relation. Consider the morphism f defined by

$$f = [\![a : A \mid \lambda b : A.\ R\,a\,b]\!] : A \to (A \Rightarrow \Omega)$$

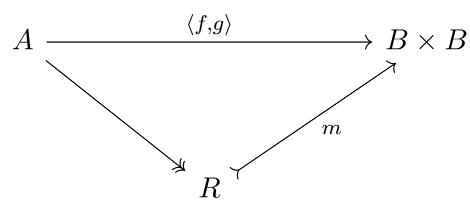
Intuitively, this maps every $a \in A$ to its equivalence class under R. We prove that $R = \ker(f)$.

It suffices to check that $R = [\![a, b : A \mid f(a) = f(b)]\!]$, and so we can work in the internal language of $\mathcal{E}$. Specifically, we must prove

$$(a, b : A \mid R\,a\,b \dashv\vdash \{c \mid R\,a\,c\} = \{c \mid R\,b\,c\})$$

which is straightforward using the fact that R is an equivalence relation.

To get more general coequalisers of $f, g : A \to B$, take the image factorisation



Let

$$\bar{R} = [\![b, c : B \mid \forall S : \Omega.\text{“}S \text{ is an eq. rel.”} \wedge ((R \subseteq S) \Rightarrow S\,b\,c)]\!] \in \mathrm{Sub}(B)$$

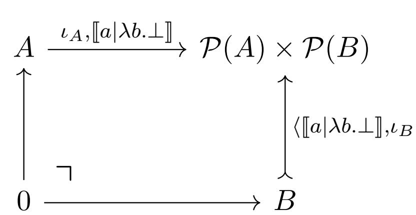
Then $f_S : B \to \mathcal{P}(B)$ is a coequaliser of f and g. $\square$

Lemma. Every topos $\mathcal{E}$ has coproducts.

Proof. Observe that for $A \in \mathcal{E}$, we have a map $\iota_A : A \to \mathcal{P}(A)$ given by

$$\iota_A = [\![a : A \mid \lambda b.\ a = b]\!] : A \to \mathcal{P}(A) = (A \Rightarrow \Omega)$$

and moreover ι_A is monic. Now given objects A and B, we have



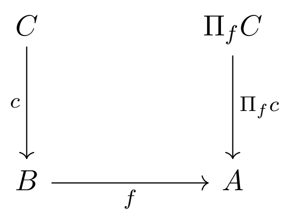
Taking the join of $A \to \mathcal{P}(A) \times \mathcal{P}(B)$ and $B \to \mathcal{P}(A) \times \mathcal{P}(B)$ gives $A + B$ as required. $\square$

Lemma. Toposes have dependent products, so are locally cartesian closed.

Proof. Essentially repeat the proof for **Set**, but now in the internal logic of $\mathcal{E}$. The idea is as follows: find

$$f^* \dashv \Pi_f : \mathcal{E}/B \to \mathcal{E}/A$$

Given $f : B \to A$ and $c : C \to B$, we seek $\Pi_f c : \Pi_f C \to A$:



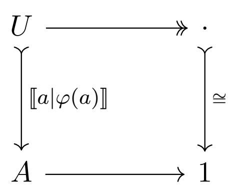
Intuitively, we take

$$\Pi_f c = \{(a, h : f^{-1}(a) \to C) \mid c \circ h = \mathrm{id}_{f^{-1}(a)}\}$$

In the internal language, we have to worry about defining relations to talk about the subset $f^{-1}(a) \subseteq B$ and so on—we skip over the details. $\square$

Existence in toposes

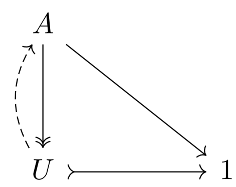
Consider $U \overset{[\![a|\varphi(a)]\!]}{\rightarrowtail} A$. If U has a point $c : 1 \to U$, then $(\mid \ \vdash \exists a.\varphi(a))$ holds. Conversely, if $(\mid \ \vdash \exists a.\varphi(a))$ holds, then we have



Since $\cdot \to 1$ is an isomorphism, the map $U \to 1$ is a cover—however, it does not necessarily split, so we don't necessarily obtain a point of U. (This failure of splitting in fact has geometric meaning in the context of sheaf toposes, e.g. the double-cover of $\mathbb{S}^1$ has no splitting.)

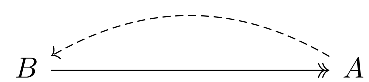
Definition. A topos $\mathcal{E}$ satisfies

- (**SS**) ("supports split") if for every $A \in \mathcal{E}$, there is a splitting as in the following diagram:



where U is the **support** of A, i.e. the subterminal object given by the image factorisation of $A \to 1$.

- (**AC**) ("axiom of choice") if every regular epimorphism in $\mathcal{E}$ splits

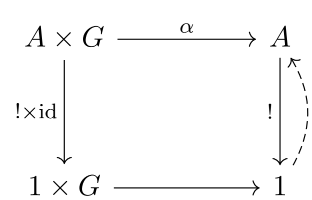


- (**IAC**) ("internal axiom of choice") if for all $[\![a, b, c \mid \varphi(a, b, c)]\!] \in \mathrm{Sub}(A \times B \times C)$, the following holds in the internal logic of $\mathcal{E}$.

$$(a \mid \forall b.\exists c.\varphi(a, b, c) \vdash \exists f : B \to C.\forall b : B.\ \varphi(a, b, f(b)))$$

Example.

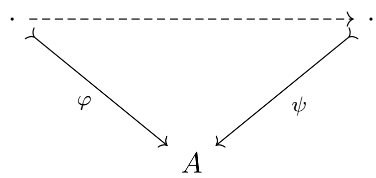
- For B a complete Boolean algebra, $\mathrm{Sh}(B)$ satisfies AA, AC and IAC.
- $\mathrm{Sh}(\mathbb{S}^1)$ satisfies neither AA, nor AC, nor IAC.
- Let G be a group, considered as a one-object category, and let $\widehat{G}$ be the category of presheaves on G ($\equiv$ the category of right group actions). Consider a set A and an action of G on A:



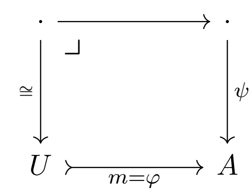
The map $A \to 1$ splits if and only if the action of G on A has a fixed point. Hence (SS) does not hold. However, (IAC) does hold.

Theorem. (AC) is equivalent to (IAC)+(SS).

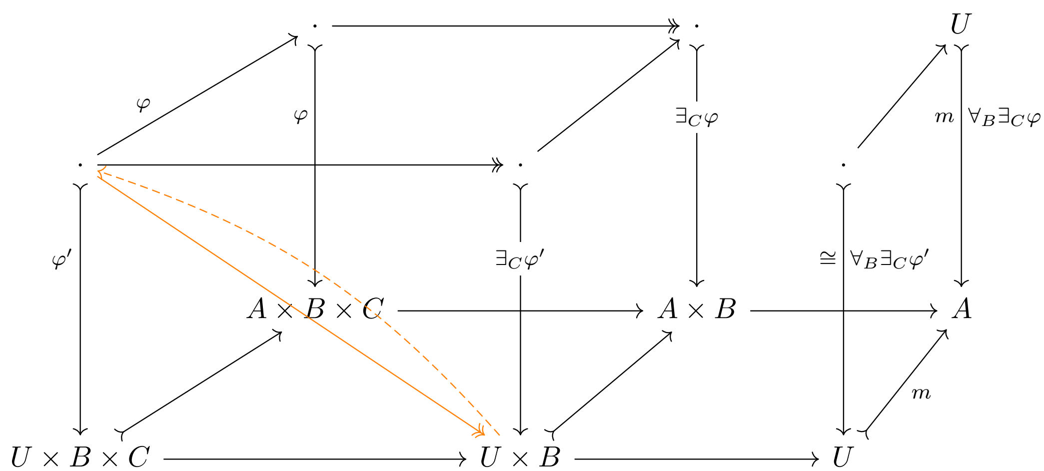
Proof. The proof that (AC) implies (SS) is easy. To see that (AC) implies (IAC), first observe that $(a : A \mid \varphi[a] \vdash \psi[a])$ holds if and only if there is a factorisation



which holds if and only if there is an isomorphism as in



which holds if and only if $(u : U \mid \psi[m(u)])$ holds in the internal logic of $\mathcal{E}$.



Use the splitting illustrated in orange to instantiate

$$(u : U \mid \ \vdash \exists f : B \to C. \forall b. \varphi[mu, b, f(b)])$$

To see that (IAC)+(SS) imply (AC), let $e : A \to B$ be epic. Then

$$(\mid \ \vdash \forall b. \exists a. f(a) = b)$$

holds, which by (IAC) implies that

$$(\mid \ \vdash \exists s : B \to A.\ \forall b.\ f(s(b)) = b)$$

holds, which by (SS) implies that there exists $s : B \to A$ such that

$$(\mid \; \vdash \forall b.\ f(s(b)) = b)$$

holds. □

Definition. A topos $\mathcal{E}$ is called **boolean** if one of the following equivalent statements holds:

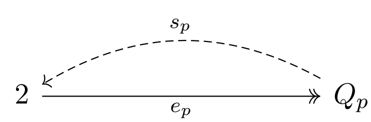
- The internal logic of $\mathcal{E}$ validates $(\Delta \mid \; \vdash \varphi \vee \neg\varphi)$ for arbitrary $(\Delta \mid \varphi)$
- The internal logic of $\mathcal{E}$ validates $(p : \Omega \mid \; \vdash p \vee \neg p)$
- The map $2 = 1 + 1 \xrightarrow{\langle t,f \rangle} \Omega$ is an isomorphism.

Theorem (Diaconescu?)**.** If a topos $\mathcal{E}$ satisfies (AC), then $\mathcal{E}$ is Boolean.

Idea of proof. For $p : \Omega$, define an equivalence relation

$$R_p = [\![x, y : 2 \mid x = y \vee p]\!] \in \mathrm{Sub}(2 \times 2)$$

Take the quotient map $e_p : 2 \to Q_p$. By (IAC), there exists $s_p : Q_p \to 2$ with $e_p \circ s_p = \mathrm{id}$:



Set $f_p = s_p \circ e_p : 2 \to 2$. Then $f_p(0) = f_p(1) \Leftrightarrow p$. The former is decidable, so that $p \vee \neg p$ is valid. □