Categorical Logic

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1 Algebraic theories and Lawvere theories

1.1 Signatures and structures

Definition 1.1 A *signature*[1] is a family $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ of sets. For $n \in \mathbb{N}$, the elements of Σ_n are called *n-ary operations*. $\diamondsuit$

Definition 1.2 Let $\mathbb{C}$ be a category with finite products and Σ a signature.

1. A *Σ-structure* A in $\mathbb{C}$ consists of

   - an object $A \in \mathbb{C}$ (we use the same letter for the structure and the underlying object), and
   - for every $n \in \mathbb{N}$ and $f \in \Sigma_n$ a morphism $f_A : A^n \to A$.

2. A *morphism of Σ-structures* A, B is an arrow $g : A \to B$ between the underlying objects such that the square

$$\begin{array}{ccc} A^n & \xrightarrow{g^n} & B^n \\ {\scriptstyle f_A}\downarrow & & \downarrow{\scriptstyle f_B} \\ A & \xrightarrow{g} & B \end{array}$$

   commutes for all $n \in \mathbb{N}$ and $f \in \Sigma_n$. $\diamondsuit$

It is easy to see that morphisms of Σ-structures commute, and we denote the category of Σ-structures in $\mathbb{C}$ and their morphisms by $\Sigma\text{-}\mathbf{Str}(\mathbb{C})$[2].

1.2 Interpretation of terms

Definition 1.3 The set $T(\Sigma)$ of *terms over* Σ is inductively defined as follows.

[1] These could more precisely be called 'algebraic signature', to distinguish them from the 'first-order signatures' in the next section, but I simply write signature since it's shorter.

[2] In the course I wrote $\mathbb{C}\text{-}\mathbf{Str}(\Sigma)$, but now I want the notation to be consistent with [1, Def. 1.2.1]

- *Variables*[3] $x, y, z, \dots$ are terms over Σ.
- If $f \in \Sigma_n$ and $t_1 \dots t_n$ are terms over Σ, then $f(t_1 \dots t_n)$ is a term over Σ.

For $n \in \mathbb{N}$ we write $T_n(\Sigma) \subseteq T(\Sigma)$ for the set of terms containing only the variables $x_1 \dots x_n$. ♢

Definition 1.4 Given a signature Σ, a Σ-structure A in a finite-product category $\mathbb{C}$, and a term $t \in T_n(\Sigma)$, the *interpretation*

$$[\![t]\!]_A : A^n \to A$$

of t w.r.t. to A is defined as follows by induction on the structure of t.

- $[\![x_i]\!]_A = (A^n \xrightarrow{p_i} A)$ (i-th projection for $1 \leq i \leq n$)
- $[\![f(t_1 \dots t_n)]\!]_A = (A^n \xrightarrow{\langle [\![t_1]\!]_A, \dots, [\![t_n]\!]_A \rangle} A^k \xrightarrow{f_A} A)$ (for $f \in \Sigma_k$) ♢

Strictly speaking, the notation $[\![t]\!]_A$ is ambiguous without specifying the n, since we can always view a term in n variables as a term in m variables for $m \geq n$. In the following the n will always be clear from the context; a more rigorous notation will be introduced in the next section, where we use 'terms in context', which are terms with explicit variable declarations.

Lemma 1.5 (Substitution Lemma) *Let Σ be a signature, and A a Σ-structure. Then for $t \in T_n(\Sigma)$ and $u_1 \dots u_n \in T_k(\Sigma)$ we have*

$$[\![t[u_1/x_1, \dots, u_n/x_n]]\!]_A = [\![t]\!]_A \circ \langle [\![u_1]\!]_A, \dots, [\![u_n]\!]_A \rangle.$$

Proof. By structural induction on t. ∎

1.3 Algebraic theories and models

Definition 1.6 1. An *algebraic theory* is a pair (Σ, E) where Σ is a signature, and $E = (E_n \subseteq T_n(\Sigma) \times T_n(\Sigma))_{n \in \mathbb{N}}$ is a set of families of *n-ary equations* (a pair $(t, u) \in E_n$ represents an equation $t = u$).

2. A *model* of an algebraic theory (Σ, E) in a category $\mathbb{C}$ with finite products is a Σ-structure A such that

$$[\![t]\!]_A = [\![u]\!]_A : A^n \to A$$

for all $n \in \mathbb{N}$ and $(t, u) \in E_n$ (in this situation we say that A *satisfies* the equation $t = u$).

3. The category Σ-**Mod**$(\mathbb{C})$ of models of (Σ, E) in $\mathbb{C}$ is the full subcategory of Σ-**Str**$(\mathbb{C})$ whose objects are the models of (Σ, E). ♢

[3] Formally we assume there is a given countable set $\{x_1, x_2, x_2, \dots\}$ of variables, but in practice we often write $x, y, z, \dots$ for variables instead of using subscripts.

1.4 Lawvere theories

Lawvere[4] theories are presentations of algebraic theories as 'syntactic categories'. They are more canonical than representations of theories via signatures and equations, since e.g. the theory of groups can be represented using different signatures, but the Lawvere theory is unique up to isomorphism of categories.

The central fact about Lawvere theories is that in the Lawvere-theoretic presentation, models of a theory (Σ, E) in a finite-product category $\mathbb{C}$ correspond to finite-product preserving functors from the associated Lawvere theory $\mathbf{L}(\Sigma, E)$ to $\mathbb{C}$, and morphisms of models correspond to natural transformations between such functors. This is known as *functorial semantics.*

Definition 1.7 Given an algebraic theory (Σ, E), the binary relation $=_E$ is the least binary relation on $T(\Sigma)$ closed under the following rules.

(ax) If $(t, u) \in E_n$ then $t =_E u$.

(refl) $t =_E t$

(trans) If $s =_E t$ and $t =_E u$ then $s =_E u$

(sym) If $s =_E t$ then $t =_E s$

(cong) If $t_1 =_E u_1, \ldots t_n =_E u_n$ then $s[t_1/x_1, \ldots, t_n/x_n] =_E s[u_1/x_1, \ldots, u_n/x_n]$

(subs) If $s =_E t$ then $s[u_1/x_1, \ldots, u_n/x_n] =_E t[u_1/x_1, \ldots, u_n/x_n]$

We write

$$T(\Sigma, E) \;=\; T(\Sigma)/{=_E} \qquad \text{and} \qquad T_n(\Sigma, E) \;=\; T_n(\Sigma)/{=_E}$$

for the set of terms (and the terms in n variables) modulo $=_E$. ◇

Intuitively, $=_E$ is the least *congruence relation* on on $T(\Sigma)$ containing E, or more precisely it is the least equivalence relation on $T(\Sigma)$ which contains E and is closed under the operations and substitution.

It is easy to see that the equations satisfied by a Σ-structure are closed under (refl), (trans), (sym), (cong), and (subs), whence we have the following lemma.

Lemma 1.8 *Let (Σ, E) be an algebraic theory, and let A be a model of (Σ, E) in a finite-product category $\mathbb{C}$. Then we have*

$$t =_E u \quad \Rightarrow \quad [\![t]\!]_A = [\![u]\!]_A$$

for all $n \in \mathbb{N}$ and $t, u \in T_n(\Sigma)$. ■

Definition 1.9 The *Lawvere theory* $\mathbf{L}(\Sigma, E)$ of an algebraic theory (Σ, E) is the category defined as follows:

[4] William Lawvere, born 1937, is commonly viewed as the founder of categorical logic.

- for each $n \in \mathbb{N}$ there is an object $[n] \in \mathbf{L}(\Sigma, E)$
- $\hom([n],[m]) = T_n(\Sigma, E)^m$ for $n, m \in \mathbb{N}$
- $(u_1 \dots u_n) \circ (t_1 \dots t_m) = (u_1[\vec{t}/\vec{x}], \dots, u_n[\vec{t}/\vec{x}])$ where

$$[k] \xrightarrow{(t_1 \dots t_m)} [m] \xrightarrow{(u_1 \dots u_n)} [n]$$

  is a composable pair of morphisms and $[\vec{t}/\vec{x}]$ is short for $[t_1/x_1, \dots, t_m/x_m]$
- $\mathrm{id}_{[n]} = (x_1 \dots x_n)$ ♢

Thus morphisms in $\mathbf{L}(\Sigma, E)$ are tuples of $=_E$-equivalence classes of terms, composition is simultaneous substitution, and identities are tuples of equivalence classes of variables. Since we are dealing with equivalence classes we have to check if the composition is well-defined, i.e. if substituting equivalent terms into equivalent terms yields equivalent terms. But this follows directly from (cong) and (subs) in Def. 1.7. Associativity and identity laws are easily verified, thus $\mathbf{L}(\Sigma, E)$ is a well-defined category. Moreover we have the following:

Lemma 1.10 *For every algebraic theory* (Σ, E)*, the Lawvere theory* $\mathbf{L}(\Sigma, E)$ *has strict finite products.*

Proof. $[0]$ is the terminal object, binary product spans are given by

$$[m] \xleftarrow{(x_1 \dots x_m)} [m+n] \xrightarrow{(x_{m+1}, \dots, x_{m+n})} [n].$$

for $m, n \in \mathbb{N}$, and the pairing operation is given by

$$\langle (\vec{t}), (\vec{u}) \rangle = (\vec{t}, \vec{u}) : [k] \to [m+n]$$

for $(\vec{t}) : [k] \to [m]$ and $(\vec{u}) : [k] \to [n]$. ■

Here's the central theorem about Lawvere theories.

Theorem 1.11 *For every algebraic theory* (Σ, E) *and finite-product category* $\mathbb{C}$*, the category* Σ-$\mathbf{Mod}(\mathbb{C})$ *of* (Σ, E)*-models in* $\mathbb{C}$ *is equivalent to the category* $\mathbf{FP}(\mathbf{L}(\Sigma, E), \mathbb{C})$ *of finite-product preserving functors from* $\mathbf{L}(\Sigma, E)$ *to* $\mathbb{C}$ *and arbitrary natural transformations between them*[5]*.*

Proof. The fine print: for simplicity we assume that $\mathbb{C}$ has strictly associative and unital finite products, and that the functors in $\mathbf{FP}(\mathbf{L}(\Sigma, E), \mathbb{C})$ strictly preserve finite products. By filling in suitable 'coherence isomorphisms' of the form $\gamma_{A,B} : F(A \times B) \xrightarrow{\cong} FA \times FB$ and $\gamma_1 : F1 \xrightarrow{\cong} 1$, we can obtain a proof that works without these 'strictness' assumptions, but this would only obscure the central ideas.

We construct a functor

$$I : \mathbf{FP}(\mathbf{L}(\Sigma, E), \mathbb{C}) \to \Sigma\text{-}\mathbf{Mod}(\mathbb{C})$$

[5] Thus $\mathbf{FP}(\mathbf{L}(\Sigma, E), \mathbb{C})$ is a full subcategory of the functor category $\mathbb{C}^{\mathbf{L}(\Sigma, E)}$.

as follows. Every (strictly) product preserving $F : \mathbf{L}(\Sigma, E) \to \mathbb{C}$ gets mapped to the Σ-structure IF whose underlying object is $F[1]$, and where operations $f \in \Sigma_n$ are interpreted as

$$f_{IF} \;=\; (F[n] \xrightarrow{F(f(\vec{x}):[n]\to[1])} F[1]).$$

With this definition we show easily that

$$[\![t]\!]_{IF} = F(t) : F[n] \to F[1]$$

for all $t \in T_n(\Sigma)$ (by induction on t), from which it follows that IF satisfies the equations in E, since for any $n \in \mathbb{N}$ and $(s, t) \in E_n$ we can argue

$$[\![s]\!]_{IF} = F(s) = F(t) = [\![t]\!]_{IF}$$

where the middle equation holds since we have s and t represent the same morphism in $\mathbf{L}(\Sigma, E)$.

We define the morphism part of I by $I(\eta) = \eta_{[1]}$ for $\eta : F \to G$, and to check that this is a morphism of Σ-structures we have to show that

$$\begin{array}{ccc} F[n] & \xrightarrow{\eta^n_{[1]}} & G[n] \\ {\scriptstyle f_{IF}}\downarrow & & \downarrow{\scriptstyle f_{IG}} \\ F & \xrightarrow{\eta_{[1]}} & G \end{array}$$

commutes for all $f \in \Sigma_n$. This follows from naturality of η since $f_{IF} = F(f(\vec{x}))$, $f_{IG} = G(f(\vec{x}))$, and $\eta^n_{[1]} = \eta_{[n]}$[6].

Clearly I preserves composition and identities, thus it is a well-defined functor. Furthermore we have

$$\eta_{[1]} = \theta_{[1]} \quad \Rightarrow \quad \eta_{[n]} = \eta^n_{[1]} = \theta^n_{[1]} = \theta_{[n]}$$

for $\eta, \theta : F \to G$ which shows that I is faithful.

To see that I is full, let $F, G \in \mathbf{FP}(\mathbf{L}(\Sigma, E), \mathbb{C})$ and let $g : F[1] \to G[1]$ be a morphism of Σ-structures. Any $\eta : F \to G$ with $\eta[1] = g$ must necessarily satisfy $\eta_{[n]} = g^n$, so we only have to show that this definition is natural, i.e. that

$$\begin{array}{ccc} F[n] & \xrightarrow{g^n} & G[n] \\ {\scriptstyle F(\vec{t})}\downarrow & & \downarrow{\scriptstyle G(\vec{t})} \\ F[m] & \xrightarrow{g^m} & G[m] \end{array}$$

commutes for all $t_1 \dots t_m \in T_n(\Sigma, E)$. Since any such square can be decomposed into n squares with exponent 1 in the lower row it is sufficient to consider the case $m = 1$, which is straightforward to prove by induction on the structure of t (using the fact that g is a morphism of Σ-structures).

[6] For the last equation note that we generally have $\eta_F \times \eta_b = \eta_{F \times G}$ for natural transformations between strictly finite-product preserving functors.

To show that I is essentially surjective, let A be a model of (Σ, E) in $\mathbb{C}$. We define a functor $F : \mathbf{L}(\Sigma, E) \to \mathbb{C}$ by $F[n] = A^n$ and $F(t_1 \dots t_n) = \langle [\![t_1]\!]_A, \dots [\![t_n]\!]_A \rangle$. We leave it to the reader to verify that F is well-defined and preserves finite products, and that $IF = A$. ■

2 First-order logic

Definition 2.1 1. A *first-order signature* is a pair

$$\Sigma = (\Sigma^{\text{fun}}, \Sigma^{\text{rel}}) = ((\Sigma_n^{\text{fun}})_{n \in \mathbb{N}}, (\Sigma_n^{\text{rel}})_{n \in \mathbb{N}})$$

of families of sets, where the elements of Σ_n^{fun} are called *n-ary function symbols*[7], and the elements of Σ_n^{rel} are called *n-ary relation symbols*.

2. The set of *first-order formulas* over a first-order signature Σ is inductively defined by the following grammar.

$$\varphi, \psi \quad ::= \quad t = u \mid R(t_1 \dots t_n) \mid \top \mid \varphi \wedge \psi \mid \exists x \,.\, \varphi \mid \bot \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \forall x \,.\, \varphi$$

where $t, u, t_1 \dots t_n \in T(\Sigma^{\text{fun}})$ and $R \in \Sigma_n^{\text{rel}}$.

The set of *Horn formulas* is generated by the initial segment of inductive clauses up to $\varphi \wedge \psi$, *regular formulas* also allow $\exists$ and *coherent formulas* additionally permit $\bot$ and $\varphi \vee \psi$.

3. The set $\text{FV}(\varphi)$ of *free variables* of a formula φ is defined in the usual way, where $\exists$ and $\forall$ bind variables.

4. A *formula in context* is an expression of the form

$$x_1 \dots x_n \mid \varphi$$

where $x_1 \dots x_n$ is a list of variables, and $\text{FV}(\varphi) \subseteq \{x_1 \dots x_n\}$.

5. A *sequent in context* is an expression of the form

$$x_1 \dots x_n \mid \varphi_1, \dots, \varphi_k \vdash \psi$$

where $\text{FV}(\varphi_1), \dots, \text{FV}(\varphi_k), \text{FV}(\psi) \subseteq \{x_1 \dots x_n\}$.

6. A *first-order theory* is a pair $\mathbb{T} = (\Sigma, A)$ where Σ is a first-order signature, and A is a set of sequents consisting of formulas generated from Σ.

The theory $\mathbb{T}$ is called a *Horn/regular/coherent theory*, if all formulas occurring in A are Horn/regular/coherent. ♢

[7] For algebraic theories we called them *n-ary operations*, but now I want to get the terminology consistent with [1, Def. D1.1.1].

2.1 Subobjects

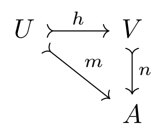
Given a first-order signature Σ and interpretations of function and relation symbols as functions and relations on a carrier set M, we can associate to each formula $(x_1 \dots x_n \mid \varphi)$ in n variables its *interpretation* $[\![\varphi]\!] \subseteq M^n$ which is the set of all valuations of the variables that make φ true relative to the given interpretations of function and relation symbols. The set $[\![\varphi]\!]$ can formally be defined by induction on the structure of φ, and this is the starting point of mathematical *model theory*.

In *categorical logic* we want to generalize this approach from the category of sets to other categories, and to do this we need a suitable categorical analogue of the notion of subset. The most obvious choice here (but not the only) is commonly called *subobject*:

Definition 2.2 Let $\mathbb{C}$ be a category.

1. A *subobject* of an object A in $\mathbb{C}$ is simply a monomorphism $m : U \rightarrowtail C$.

2. Given subobjects $U \xrightarrow{m} A$ and $V \xrightarrow{n} A$ of an object A, we say that 'm is included in n' (and write $m \leq n$), if there exists a map $h : U \to V$ such that $nh = m$.

   Since n is monic, such an h is necessarily unique and monic.

3. Since the inclusion relation on subobjects is reflexive and transitive, the subobjects of A form a *preorder* which we denote by $\mathrm{Sub}(A)$.

4. $\mathrm{Sub}(A)$ forms a full subcategory of the slice category $\mathbb{C}/A$, and we denote the inclusion functor by $I : \mathrm{Sub}(A) \hookrightarrow \mathbb{C}/A$.

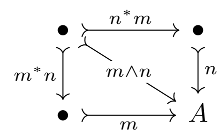
5. Since pullbacks of monos along arbitrary maps are monos, any morphism $f : A \to B$ in $\mathbb{C}$ induces a monotone map

   $$f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$$

   which maps any subobject of B to its pullback along f. We call f^* the *reindexing map* along f. $\diamondsuit$

Lemma 2.3 *Let $\mathbb{C}$ be a category with pullbacks. Then all preorders* $\mathrm{Sub}(A)$ *have finite* meets *(a.k.a. greatest lower bounds), and all reindexing maps preserve finite meets.*

Proof. Binary meets $m \wedge n$ of subobjects $m, n \in \mathrm{Sub}(A)$ are given by pullback as indicated in the following diagram,

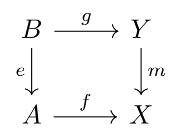


and largest elements ("nullary meets", denoted $\top$) are given by $\mathrm{id} : A \to A$. Proofs of pullback preservation are left as an exercise. ∎

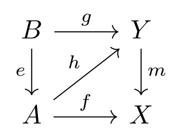
2.2 Regular categories

Definition 2.4 Let $\mathbb{C}$ be a category.

1. Given morphisms $e : B \to A$ and $m : Y \to X$ in $\mathbb{C}$, we say that 'e is *left-orthogonal* to m' (or equivalently that 'm is *right-orthogonal* to e'), if for every commutative square

   there exists a unique $h : A \to Y$ making the two triangles in

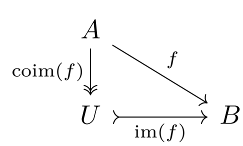
   

   commute.

2. A *cover* (or *strong epimorphism*) is an epimorphism $e : B \to A$ that is left orthogonal to all monomorphisms. ♢

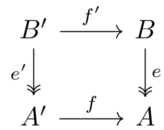
Definition 2.5 A *regular category* is a category $\mathbb{R}$ with finite limits in which

1. every morphism $f : A \to B$ factors

   into a cover $\mathrm{coim}(f)$ followed by a mono $\mathrm{im}(f)$, and

2. covers are stable under pullback, i.e. if e is a cover in a pullback square



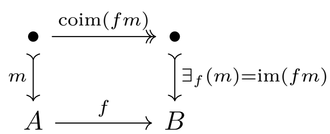
then e' is a cover as well. ♢

Regular categories come with a notion of structure-preserving functor that we introduce for later use:

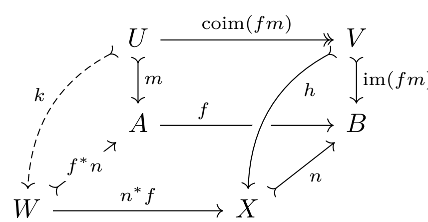
Definition 2.6 A *regular functor* between regular categories $\mathbb{Q}$, $\mathbb{R}$ is a functor $F : \mathbb{Q} \to \mathbb{R}$ which preserves finite limits and covers. ♢

When working with regular categories, we use the arrow symbol $\twoheadrightarrow$ for covers, and $\rightarrowtail$ for monomorphisms.

Lemma 2.7 *Let $f : A \to B$ in a regular category $\mathbb{R}$. The reindexing map f^* has a left adjoint $\exists_f : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$, given by $\exists_f(m : U \rightarrowtail A) = \mathrm{im}(f \circ m)$.*

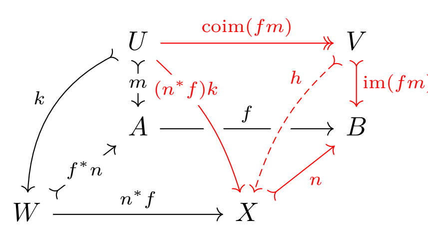


Proof. We have to show that $\mathrm{im}(fm) \leq n$ iff $m \leq f^*n$ for $m \in \mathrm{Sub}(A)$ and $n \in \mathrm{Sub}(B)$. Assume first that $\mathrm{im}(fm) \leq n$, i.e. there exists an $h : V \rightarrowtail X$ with $nh = \mathrm{im}(fn)$ as in the following diagram.



Then since the lower square is a pullback, there exists a unique $k : U \to W$ with $(f^*n)k = m$ and $(n^*f)k = h\,\mathrm{coim}(fm)$, which means that $m \leq f^*n$ in $\mathrm{Sub}(A)$.

Next assume that $m \leq f^*n$ in $\mathrm{Sub}(A)$, i.e. there exists $k : U \to W$ with $(f^*n)k = m$, and consider the following diagram.



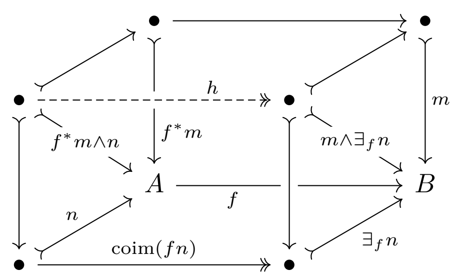
By orthogonality of covers and monos applied to the red square there exists a morphism $h : V \to X$ making the triangles commute, which implies $\mathrm{im}(fm) \leq n$ in $\mathrm{Sub}(B)$. ∎

Lemma 2.8 (Frobenius law) *Let $f : A \to B$ in a regular category $\mathbb{R}$. We have*

$$m \wedge \exists_f n \cong \exists_f(f^*m \wedge n)$$

for all $n \in \mathrm{Sub}(A)$ and $m \in \mathrm{Sub}(B)$.

Proof. Consider the following diagram, where the dashed arrow h is induced by the universal property of the pullback square on the right.



The left, right and back sides of the cube are pullbacks, whence the front is a pullback by double application of the pullback lemma. This means that h is a cover, which exhibits $m \wedge \exists_f n$ as image of $f \circ (f^*m \wedge n)$. Since $\exists_f(f^*m \wedge n)$ is also the image of $f \circ (f^*m \wedge n)$, we conclude that the two are isomorphic as subobjects of B. ∎

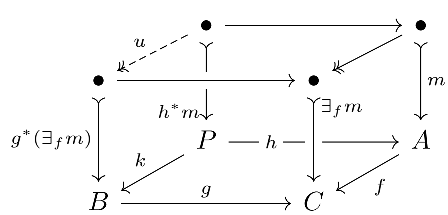
Lemma 2.9 (Beck-Chevalley condition for ∃) *If*

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

is a pullback square in a regular category $\mathbb{R}$ and $m \in \mathrm{Sub}(A)$, then

$$g^*(\exists_f m) \cong \exists_k(h^*m).$$

Proof. In the cube



the dashed arrow is induced by the universal property of the front pullback square. Analogously to the previous proof, the top square is a pullback since bottom, front, and back are, and hence u is a cover. This exhibits $g^*(\exists_f m)$ as image of $k \circ (h^* m)$, which implies the claimed isomorphism by uniqueness of image factorizations. ∎

2.3 Interpretation of regular logic in regular categories

Interpretation of first-order formulas over a first-order signature Σ is done by structural recursion, and depending on the logical connectives appearing in the formula we have to require certain structure on the underlying category.

To start we have to fix a *structure* M assigning interpretations to the symbols of the signature, and this can be done in any finite-limit category.

Definition 2.10 Let Σ be a first-order signature.

1. A Σ-*structure* M in a finite-limit category $\mathbb{C}$ consists of the folloing data.
   - an object $M \in \mathbb{C}$
   - a morphism $f_M : M^n \to M$ for every function symbol $f \in \Sigma_n^{\mathrm{fun}}$
   - a subobject $R_M \in \mathrm{Sub}(M^n)$ for every relation symbol $R \in \Sigma_n^{\mathrm{rel}}$
2. Given Σ-structures M and N in $\mathbb{C}$, a *morphism of* Σ-*structures* from M to N is a map $g : M \to N$ in $\mathbb{C}$ between the underlying objects such that

$$\begin{array}{ccc} A^n & \xrightarrow{g^n} & B^n \\ {\scriptstyle f_M}\downarrow & & \downarrow{\scriptstyle f_M} \\ A & \xrightarrow{g} & B \end{array}$$

   commutes for all function symbols $f \in \Sigma_n^{\mathrm{fun}}$, and

$$R_M \leq (g^n)^*(R_N) \quad \text{in } \mathrm{Sub}(M^n)$$

   for all relation symbols $R \in \Sigma_n^{\mathrm{rel}}$.

Σ-structures and morphisms of Σ-structures form a category Σ-$\mathbf{Str}(\mathbb{C})$. ◇

Definition 2.11 (Interpretation of regular formulas) Let Σ be a first-order signature and let M be a Σ-structure in a regular category $\mathbb{R}$. The interpretation

$$[\![x_1 \dots x_n \mid \varphi]\!]_M \in \mathrm{Sub}(M^n)$$

of regular formulas in context $(x_1 \dots x_n \mid \varphi)$ over Σ is defined by structural recursion as follows.

- $[\![\vec{x} \mid R(t_1 \dots t_k)]\!]_M = \langle [\![\vec{x} \mid t_1]\!]_M, \dots, [\![\vec{x} \mid t_k]\!]_M \rangle^*(R_M)$ where $R \in \Sigma_k^{\mathrm{rel}}$
- $[\![\vec{x} \mid t = u]\!]_M = \langle [\![\vec{x} \mid t]\!]_M, [\![\vec{x} \mid u]\!]_M \rangle^*(\delta_M)$

$$\frac{}{\vec{x} \mid \Gamma, \varphi \vdash \varphi} \qquad \frac{\vec{x} \mid \Gamma \vdash \varphi \qquad \vec{x} \mid \Gamma, \varphi \vdash \psi}{\vec{x} \mid \Gamma \vdash \psi}$$

$$\frac{\vec{x} \mid \Gamma, \varphi, \varphi \vdash \psi}{\vec{x} \mid \Gamma, \varphi \vdash \psi} \qquad \frac{\vec{x} \mid \Gamma \vdash \psi}{\vec{x} \mid \Gamma, \varphi \vdash \psi} \qquad \frac{\vec{x} \mid \Gamma, \varphi, \psi\Delta \vdash \theta}{\vec{x} \mid \Gamma, \psi, \varphi, \Delta \vdash \theta}$$

$$\frac{}{\vec{x} \mid \Gamma \vdash \top} \qquad \frac{\vec{x} \mid \Gamma \vdash \varphi \qquad \vec{x} \mid \Gamma \vdash \psi}{\vec{x} \mid \Gamma \vdash \varphi \wedge \psi}$$

$$\frac{\vec{x} \mid \Gamma \vdash \varphi}{\vec{y} \mid \Gamma[\vec{t}/\vec{x}] \vdash \varphi[\vec{t}/\vec{x}]} \qquad \frac{\vec{x} \mid \Gamma \vdash \varphi \wedge \psi}{\vec{x} \mid \Gamma \vdash \varphi} \qquad \frac{\vec{x} \mid \Gamma \vdash \varphi \wedge \psi}{\vec{x} \mid \Gamma \vdash \psi}$$

$$\frac{\vec{x}, y \mid \Gamma, \varphi \vdash \psi}{\vec{x} \mid \Gamma, \exists y \,.\, \varphi \vdash \psi} \qquad \frac{\vec{x}, y \mid \Gamma[y/z] \vdash \psi[y/z]}{\vec{x}, y, z \mid \Gamma, y = z \vdash \psi}$$

Table 1: Rules of regular logic

- $[\![\vec{x} \mid \top]\!]_M = \top$
- $[\![\vec{x} \mid \varphi \wedge \psi]\!]_M = [\![\vec{x} \mid \varphi]\!]_M \wedge [\![\vec{x} \mid \psi]\!]_M$
- $[\![\vec{x} \mid \exists y \,.\, \varphi]\!]_M = \exists_p [\![\vec{x}, y \mid \varphi]\!]_M$

In the second clause, δ_M is the *diagonal* $\langle \mathrm{id}, \mathrm{id} \rangle : M \to M \times M$, and in the last clause, $p : M^{n+1} \times M \to M^n$ is the projection which forgets the last component. ♢

Lemma 2.12 (Substitution Lemma) *Let Σ be a first-order signature and M a Σ-structure in a regular category $\mathbb{R}$. Then we have*

$$[\![\vec{y} \mid \varphi[\vec{t}/\vec{x}]]\!]_M = \langle [\![\vec{y} \mid u_1]\!]_M, \ldots, [\![\vec{y} \mid u_n]\!]_M \rangle^* [\![\vec{x} \mid \varphi]\!]_M$$

for formulas $(x_1 \ldots x_n \mid \varphi)$ and terms $(\vec{y} \mid u_1), \ldots, (\vec{y} \mid u_n)$.

Proof. By induction on φ. ■

Definition 2.13 Let Σ be a first-order signature and M a Σ-structure in a regular category $\mathbb{R}$. We say that M *satisfies* a judgment $(x_1 \ldots x_n \mid \varphi_1 \ldots \varphi_k \vdash \psi)$ (or equivalently that the judgment is *valid* in M), if

$$[\![\vec{x} \mid \varphi_1]\!]_M \wedge \cdots \wedge [\![\vec{x} \mid \varphi_k]\!]_M \leq [\![\vec{x} \mid \psi]\!]_M \quad \text{in } \mathrm{Sub}(M^n).$$

We say that M is a *model* of a regular theory $\mathbb{T} = (\Sigma, A)$, if M satisfies all judgments in A. We denote by $\mathbb{T}$-$\mathbf{Mod}(\mathbb{R})$ the full subcategory of Σ-$\mathbf{Str}(\mathbb{R})$ on models of $\mathbb{T}$. ♢

Theorem 2.14 (Soundness) *Let $\mathbb{T} = (\Sigma, A)$ be a regular theory, and let M be a model of $\mathbb{T}$ in a regular category $\mathbb{R}$. If a judgment $(\vec{x} \mid \Gamma \vdash \varphi)$ can be derived from the judgments in A using the rules of regular logic (Table 1), then it is satisfied in M.*

Proof. By induction on the derivation. ∎

2.4 The syntactic category of a regular theory

Syntactic categories of regular theories classify models of regular theories, in the same way that Lawvere theories classify models of algebraic theories. For the definition it is convenient to adapt the notion of α-equivalence familiar from languages with variable binding. Recall that α-equivalence means that we allow to rename bound variables, e.g.

$$(\forall x \,.\, \varphi) \;=_\alpha\; (\forall y \,.\, \varphi[y/x])$$

provided that y does not appear free in φ. For the definition of syntactic category we extend α-equivalence to the renaming of variables in *contexts*, i.e.

$$(\vec{x} \mid \varphi) \;=_\alpha\; (\vec{y} \mid \varphi[\vec{y}/\vec{x}]).$$

Intuitively this makes sense since we can view contexts as 'binding' the variables they declare.

Definition 2.15 The syntactic category $\mathbb{C}^{\mathrm{reg}}_{\mathbb{T}}$ of a regular theory $\mathbb{T} = (\Sigma, A)$ is defined as follows.

- Objects are regular formulas-in-context $(\vec{x} \mid \alpha)$ over Σ.
- Morphisms from $(\vec{x} \mid \alpha)$ to $(\vec{y} \mid \beta)$ – where we assume that $\vec{x}$ and $\vec{y}$ are disjoint lists of variables, possibly after α-renaming – are formulas-in-context $(\vec{x}, \vec{y} \mid \phi)$ such that the judgments
  - $(\vec{x}, \vec{y} \mid \phi \vdash \alpha \wedge \beta)$
  - $(\vec{x}, \vec{y}, \vec{y'} \mid \phi, \phi[\vec{y'}/\vec{y}] \vdash \vec{y} = \vec{y'})$[8]
  - $(\vec{x} \mid \alpha \vdash \exists \vec{y} \,.\, \phi)$

  are derivable from A. We identify $(\vec{x}, \vec{y} \mid \phi)$ and $(\vec{x}, \vec{y} \mid \psi)$ as morphisms from $(\vec{x} \mid \alpha)$ to $(\vec{y} \mid \beta)$, if the judgments $(\vec{x}, \vec{y} \mid \phi \vdash \psi)$ and $(\vec{x}, \vec{y} \mid \psi \vdash \phi)$ are derivable from A.
- The compositions of morphisms

  $$(\vec{x} \mid \alpha) \xrightarrow{(\vec{x},\vec{y} \mid \phi)} (\vec{y} \mid \beta) \xrightarrow{(\vec{y},\vec{z} \mid \psi)} (\vec{z} \mid \gamma)$$

  is given by

  $$(\vec{x} \mid \alpha) \xrightarrow{(\vec{x},\vec{z} \mid \exists \vec{y} \,.\, \phi \wedge \psi)} (\vec{z} \mid \gamma).$$

[8] $\vec{y} = \vec{y'}$ is short for $y_1 = y'_1 \wedge \cdots \wedge y_n = y'_n$.

- The identity morphism of $(\vec{x} \mid \alpha)$ is given by

$$(\vec{x} \mid \alpha) \xrightarrow{(\vec{x},\vec{x'} \mid \alpha \wedge \vec{x}=\vec{x}')} (\vec{x'} \mid \alpha[\vec{x}'/\vec{x}]).$$

  Here we replaced formula-in-context $(\vec{x} \mid \alpha)$ by the α-equivalent formula $(\vec{x'} \mid \alpha[\vec{x'}/\vec{x}])$ in the codomain, to make the contexts disjoint. $\diamondsuit$

The following theorems – for whose proofs we refer to [1, Sec. D1.4] – state the central properties of $\mathbb{C}_{\mathbb{T}}^{\mathrm{reg}}$.

Theorem 2.16 $\mathbb{C}_{\mathbb{T}}^{\mathrm{reg}}$ *is a regular category.*

Theorem 2.17 (Completeness) $\mathbb{C}_{\mathbb{T}}^{\mathrm{reg}}$ *contains a 'generic model' M of $\mathbb{T}$ satisfying precisely the regular formulas that can be derived from A. Thus, formulas that are satisfied by all models of $\mathbb{T}$ are derivable from A.*

Theorem 2.18 *For every regular category $\mathbb{R}$ there is an equivalence of categories*

$$\mathbf{Reg}(\mathbb{C}_{\mathbb{T}}^{\mathrm{reg}}, \mathbb{R}) \simeq \mathbb{T}\text{-}\mathbf{Mod}(\mathbb{R}),$$

where $\mathbf{Reg}(\mathbb{C}_{\mathbb{T}}^{\mathrm{reg}}, \mathbb{R})$ is the category of regular functors $F : \mathbb{C}_{\mathbb{T}}^{\mathrm{reg}} \to \mathbb{R}$, and arbitrary natural transformations between them.

2.5 The internal language of a regular category

The 'internal language' of a regular category $\mathbb{R}$ is a many-sorted first order language which contains sort symbols for all objects of $\mathbb{R}$, function symbols for all morphisms of $\mathbb{R}$, and relation symbols for all subobjects in $\mathbb{R}$. Before defining it formally we have to introduce many-sorted signatures.

Definition 2.19 A *many-sorted first-order signature* is a triple

$$\Sigma \;=\; (\Sigma_0, \Sigma^{\mathrm{fun}}, \Sigma^{\mathrm{rel}}) \;=\; \big(\Sigma_0, (\Sigma^{\mathrm{fun}}_{\vec{S},T})_{\vec{S}\in\Sigma_0^*, T\in\Sigma_0}, (\Sigma^{\mathrm{rel}}_{\alpha})_{\vec{S}\in\Sigma_0^*}\big)$$

where

- elements of $\cdots \in \Sigma_0$ are called *sorts*,
- elements of $\Sigma^{\mathrm{fun}}_{S_1\ldots S_n,T}$ are called *function symbols of arity $S_1\times\cdots\times S_n \to T$*,
- elements of $\Sigma^{\mathrm{rel}}_{S_1\ldots S_n}$ are called *relation symbols of arity $S_1 \times \cdots \times S_n$*. $\diamondsuit$

To take the presence of multiple sorts into account, we have to adapt the definitions of contexts, terms, and formulas.

Definition 2.20 Let Σ be a many-sorted first-order signature.

1. A *variable-context* (over Σ) is a list $x_1 : S_1, \ldots, x_n : S_n$ where $S_i \in \Sigma_0$ for $1 \le i \le n$ (in practice we will often omit the sort declarations when they are irrelevant or easily inferred).

2. Terms-in-context over Σ are inductively defined by the rules

$$\frac{}{x_1 : S_1, \ldots, x_n : S_n \mid x_i : S_i} \qquad \frac{\vec{x} : \vec{S} \mid t_1 : T_1 \quad \ldots \quad \vec{x} : \vec{S} \mid t_k : T_k}{\vec{x} : \vec{S} \mid f(t_1 \ldots t_k) : U}$$

   where in the first rule we assume $1 \leq i \leq n$, and in the second rule $f \in \Sigma^{\mathrm{fun}}_{T_1 \ldots T_k, U}$.

3. The definition of formulas-in-context is as before, except that we have to check for sorts for atomic formulas, i.e.
   - if $(\vec{x} : \vec{S} \mid t_1 : T_1), \ldots, (\vec{x} : \vec{S} \mid t_n : T_n)$ are terms in context and $R \in \Sigma^{\mathrm{rel}}_{T_1 \ldots T_n}$ then $(\vec{x} : \vec{S} \mid R(t_1 \ldots t_n))$ is a formula in context, and
   - if $(\vec{x} : \vec{S} \mid t : T)$, $(\vec{x} : \vec{S} \mid u : T)$ are terms in context, then $(\vec{x} : \vec{S} \mid t = u)$ is a formula in context. ♢

Definition 2.21 Let Σ be a many-sorted first-order signature. A Σ-*structure* M in a finite-limit category $\mathbb{C}$ consists of

- objects $S_M \in \mathbb{C}$ for all sorts $S \in \Sigma_0$,
- morphisms $f_M : S_{1,M} \times \cdots \times S_{n,M} \to T_M$ for all function symbols $f \in \Sigma^{\mathrm{fun}}_{S_1 \ldots S_n, T}$, and
- subobjects $R_M \in \mathrm{Sub}(S_{1,M} \times \cdots \times S_{n,M})$ for all relation symbols $R \in \Sigma^{\mathrm{rel}}_{S_1 \ldots S_n}$. ♢

Adaptation of interpretation of formulas from the single-sorted to the many-sorted case is straightforward.

We can now define the signature of the internal language.

Definition 2.22 (Signature of the internal language) Let $\mathbb{C}$ be a finite-limit category. The many-sorted signature $\Sigma(\mathbb{C})$ is defined as follows.

- $\Sigma(\mathbb{C})_0 = \mathrm{obj}(\mathbb{C})$
- $\Sigma(\mathbb{C})^{\mathrm{fun}}_{A_1 \ldots A_n, B} = \hom(A_1 \times \cdots \times A_n, B)$
- $\Sigma(\mathbb{C})^{\mathrm{rel}}_{A_1 \ldots A_n} = \mathrm{Sub}(A_1 \times \cdots \times A_n)$ ♢

This signature comes with a canonical structure in $\mathbb{C}$:

Definition 2.23 Let $\mathbb{C}$ be a finite-limit category. The $\Sigma(\mathbb{C})$-structure $M(\mathbb{C})$ in $\mathbb{C}$ is given as follows.

- $A_{M(\mathbb{C})} = A$ for $A \in \Sigma(\mathbb{C})_0$
- $f_{M(\mathbb{C})} = f$ for $f \in \Sigma(\mathbb{C})^{\mathrm{fun}}_{A_1 \ldots A_n, B}$

- $R_{M(\mathbb{C})} = R$ for $R \in \Sigma(\mathbb{C})^{\mathrm{rel}}_{A_1 \dots A_n}$ $\diamondsuit$

The point about defining this signature and interpretation is that we can use logical reasoning in category theoretic proofs. To this end we introduce the following terminology.

Definition 2.24 Let $\mathbb{R}$ be a regular category. We say that a judgment $(\vec{x} \mid \Gamma \vdash \varphi)$ consisting of regular formulas over $\Sigma(\mathbb{R})$ *holds in* $\mathbb{R}$, if the structure $M(\mathbb{R})$ satisfies the judgment. $\diamondsuit$

We will demonstrate the use of the internal language in some examples.

Definition 2.25 Let $f : A \to B$ in a regular category $\mathbb{C}$. The *graph of* f is the subobject

$$\mathrm{Gr}_f = (A \xrightarrow{\langle \mathrm{id}, f \rangle} A \times B) \in \mathrm{Sub}(A \times B). \qquad \diamondsuit$$

Lemma 2.26 *Let* $f : A \to B$ *in a regular category* $\mathbb{R}$. *Then*

1. $(a, a' \mid fa = fa' \vdash a = a')$ *holds in* $\mathbb{R}$ *iff* f *is a monomorphism,*
2. $\mathrm{Gr}_f \cong [\![a, b \mid fa = b]\!]$
3. $(b \mid \vdash \exists a \,.\, fa = b)$ *holds in* $\mathbb{R}$ *iff* f *is a cover.*

Proof. For the first claim, note that we have $[\![a, a' \mid fa = fa']\!] = \ker(f)$ and $[\![a, a' \mid a = a']\!] = \delta_A$ It can be shown diagrammatically that a morphism is monic iff its kernel is contained in the diagonal.

The second claim follows since

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \mathrm{Gr}_f}\downarrow & & \downarrow{\scriptstyle \delta_B} \\ A \times B & \xrightarrow{f \times B} & B \times B \end{array}$$

is a pullback.

3rd: TODO ■

Lemma 2.27 *Let* $f : A \to B$ *in* $\mathbb{R}$. *The judgments* $(a, b, b' \mid \mathrm{Gr}_f(a, b), \mathrm{Gr}_f(a, b') \vdash b = b')$ *and* $(a \mid \vdash \exists b \,.\, \mathrm{Gr}_f(a, b))$ *hold in* $\mathbb{R}$

Proof. TODO ■

This shows that graphs are 'functional relations'. Conversely we have the following.

Lemma 2.28 *Let* $R \in \mathrm{Sub}(A \times B)$ *in a regular category* $\mathbb{R}$. *If the judgments* $(a, b, b' \mid R(a, b), R(a, b') \vdash b = b')$ *and* $(a \mid \vdash \exists b \,.\, R(a, b))$ *hold in* $\mathbb{R}$, *then there exists a morphism* $f : A \to B$ *with* $R \cong \mathrm{Gr}_f$.

Proof. TODO ■

2.6 Coherent logic and coherent categories

Coherent logic is the extension of regular logic by disjunction $\vee$ and falsity $\bot$, subject to the rules

$$\frac{\vec{x} \mid \Gamma, \varphi \vdash \theta \qquad \vec{x} \mid \Gamma, \psi \vdash \theta}{\vec{x} \mid \Gamma, \varphi \vee \psi \vdash \theta} \qquad \frac{\vec{x} \mid \Gamma, \varphi \vee \psi \vdash \theta}{\vec{x} \mid \Gamma, \varphi \vdash \theta}$$

$$\frac{}{\vec{x} \mid \Gamma, \bot \vdash \theta} \qquad \frac{\vec{x} \mid \Gamma, \varphi \vee \psi \vdash \theta}{\vec{x} \mid \Gamma, \psi \vdash \theta} \tag{2.1}$$

extending the rules of regular logic in Table 1.

Coherent logic can be modeled in *coherent categories*, which are defined as follows.

Definition 2.29 A *coherent category* is a regular category $\mathbb{R}$ in which all subobject lattices $\mathrm{Sub}(A)$ have finite joins (denoted by $\bot$ for the least element and $\vee$ for binary joins), and all inverse image maps $f^* : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$ for $f : B \to A$ preserve them. $\diamondsuit$

The interpretation of regular formulas (Def. 2.11) is extended to coherent formulas via the following clauses.

- $[\![\vec{x} \mid \bot]\!] = \bot$
- $[\![\vec{x} \mid \varphi \vee \psi]\!] = [\![\vec{x} \mid \varphi]\!] \vee [\![\vec{x} \mid \psi]\!]$

To show that this interpretation is sound w.r.t. the rules (2.1), we need the following lemma which says that subobject lattices in coherent categories are distributive.

Lemma 2.30 *We have* $m \wedge (n \vee u) \cong (m \wedge n) \vee (m \wedge u)$ *for* $m, n, u \in \mathrm{Sub}(A)$ *in a coherent category* $\mathbb{R}$.

Proof. We have

$$\begin{aligned}
& m \wedge (n \vee u) \\
\cong\ & m \circ m^*(n \vee u) \\
\cong\ & m \circ (m^*(n) \vee m^*(u)) && m^*(-) \text{ preserves } \vee \text{ by assumption} \\
\cong\ & m \circ m^*(n) \vee m \circ m^*(u) && m \circ (-) \text{ preserves } \vee \text{ as left adjoint to } m^*(-) \\
\cong\ & (m \wedge n) \vee (m \wedge u)
\end{aligned}$$

■

Using this lemma, we can show that the interpretation of coherent logic in coherent categories is sound w.r.t. the given rules, and we can define theories and models just as for regular logic. The construction of syntactic categories and internal language also carries over directly.

2.7 First order logic and Heyting categories

To obtain full first order logic we have to introduce the connectives $\Rightarrow$ and $\forall$, which come with the following rules.

$$\frac{\vec{x}, y \mid \Gamma \vdash \varphi}{\vec{x} \mid \Gamma \vdash \forall y\,.\,\varphi} \qquad \frac{\vec{x} \mid \Gamma, \varphi \vdash \psi}{\vec{x} \mid \Gamma \vdash \varphi \Rightarrow \psi}$$

These rules axiomatize *intuitionistic* first order logic – to obtain *classical* first order logic, we have to furthermore postulate either of the following rules (which are equivalent in presence of the other rules)

$$\frac{\vec{x} \mid \Gamma \vdash \neg\neg\varphi}{\vec{x} \mid \Gamma \vdash \varphi} \qquad \frac{\vec{x} \mid \Gamma, \neg\varphi \vdash \varphi}{\vec{x} \mid \Gamma \vdash \varphi} \qquad \frac{}{\vec{x} \mid \Gamma \vdash \varphi \vee \neg\varphi}$$

where $\neg\varphi \;\equiv\; \varphi \Rightarrow \bot$ is the usual encoding of negation in intuitionistic logic.

The categorical counterpart of intuitionistic first order logic is given by *Heyting categories*, defined as follows.

Definition 2.31 A *Heyting category* is a coherent category $\mathbb{H}$ in which for every $f : B \to A$, the reindexing map $f^* : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$ has a *right adjoint* $\forall_f : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$. ♢

We have the following lemmas.

Lemma 2.32 (Beck-Chevalley condition for $\forall$) *If*

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

is a pullback square in a Heyting category $\mathbb{H}$ and $m \in \mathrm{Sub}(A)$, then

$$g^*\forall_f m \cong \forall_k h^* m \quad \textit{in} \quad \mathrm{Sub}(B).$$

Proof. Using the Beck-Chevalley condition for $\exists$ (Lem. 2.9) we have

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{n \leq \forall_k h^* m}{k^* n \leq h^* m}}{\exists_h k^* n \leq m}}{f^* \exists_g n \leq m}}{\exists_g n \leq \forall_f m}}{n \leq g^* \forall_f m}$$

for arbitrary $n \in \mathrm{Sub}(B)$, from which the claim follows by the Yoneda lemma. ∎

Definition 2.33 A *Heyting algebra* is a preorder $(H, \leq)$ with finite meets and joins and a binary operation $(- \Rightarrow -) : H \times H \to H$ (called *Heyting application*) satisfying

$$a \wedge b \leq c \qquad \text{iff} \qquad a \leq b \Rightarrow c \qquad \text{for } a, b, c \in H.$$

(This condition determines the Heyting implication operation uniquely, and implies that $\Rightarrow$ is antimonotone in its first variable, and monotone in the second.) $\diamondsuit$

In other words, a Heyting algebra is a posetal cartesian closed category with coproducts.

Lemma 2.34 *The subobject lattices* $\mathrm{Sub}(A)$ *of a Heyting category* $\mathbb{H}$ *are Heyting algebras, and the reindexing maps preserve Heyting application.*

Proof. For $n, u \in \mathrm{Sub}(A)$ we set

$$n \Rightarrow u \quad := \quad \forall_n n^* u.$$

Then we have

$$\dfrac{\dfrac{\dfrac{m \wedge n \leq u}{n \circ n^* m \leq u}}{n^* m \leq n^* u}}{m \leq \forall_n n^* u}$$

for arbitary $m \in \mathrm{Sub}(A)$, which shows that $\mathrm{Sub}(A)$ is a Heyting algebra.

Now let $f : B \to A$ and consider the pullback square

$$\begin{array}{ccc} Q & \xrightarrow{g} & P \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ B & \xrightarrow{f} & A \end{array}$$

of f and n. We have

$$\begin{aligned} & f^*(n \Rightarrow u) \\ \cong\ & f^* \forall_n n^* u \\ \cong\ & \forall_m g^* n^* u \\ \cong\ & \forall_m m^* f^* u \\ \cong\ & m \Rightarrow f^* u \\ \cong\ & f^* n \Rightarrow f^* u \end{aligned}$$

which shows that Heyting implication is stable under reindexing. ■

Now the interpretation of formulas extends to full first order logic with the clauses

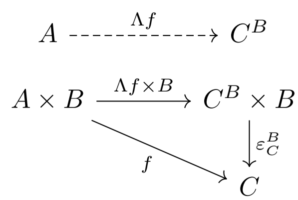
- $[\![\vec{x} \mid \varphi \Rightarrow \psi]\!] = [\![\vec{x} \mid \varphi]\!] \Rightarrow [\![\vec{x} \mid \psi]\!]$
- $[\![\vec{x} \mid \forall y \, . \, \varphi]\!] = \forall_p [\![\vec{x}, y \mid \varphi]\!]$

where in the second clause, p is the appropriate projection (just as for $\exists$). We define structures, theories, and models just as in the regular and coherent case and get analogous soundness and completeness results. The internal language also works the same way.

3 Cartesian closed categories and the λ-calculus

Definition 3.1 A category $\mathbb{C}$ with finite products is called *cartesian closed*, if either of the following equivalent conditions hold.

1. For all $B \in B$, the functor $(- \times B) : \mathbb{C} \to \mathbb{C}$ has a right adjoint.
2. For all $B, C \in \mathbb{C}$ there exists an object C^B and an arrow $\varepsilon_C^B : C^B \times B \to C$ such that for every $f : A \times B \to C$ there exists a unique $\Lambda f : A \to C^B$ with $\varepsilon_C^B \circ (\Lambda \times B) = f$.



$\diamondsuit$

Definition 3.2 A λ-*signature* is a pair $\Sigma = (\Sigma_0, \Sigma^{\mathsf{c}})$ where

- Σ_0 is a set of *base-types*, generating a set $\tau(\Sigma_0)$ of types via the grammar

$$A, B \ ::= \ X \mid 1 \mid A \times B \mid A \Rightarrow B \qquad\qquad (X \in \Sigma_0)$$

- $\Sigma^{\mathsf{c}} = (\Sigma^{\mathsf{c}}_A)_{A \in \tau(\Sigma_0)}$ is a family of sets of typed constants. $\diamondsuit$

Definition 3.3 Given a λ-signature Σ, a Σ-*structure* M in a cartesian closed category $\mathbb{C}$ consists is given by the following data:

- an assignment of objects $X_M \in \mathbb{C}$ to base types $X \in \Sigma_0$, extending to an interpretation of types via the inductive clauses
  - $[\![X]\!]_M = X_M$ for $X \in \Sigma_0$
  - $[\![1]\!]_M = 1$
  - $[\![A \times B]\!]_M = [\![A]\!]_M \times [\![B]\!]_M$
  - $[\![A \Rightarrow B]\!]_M = [\![B]\!]_M^{[\![A]\!]_M}$
- for each $A \in \tau(\Sigma_0)$ and $c \in \Sigma^{\mathsf{c}}_A$ an arrow $c_M : 1 \to [\![A]\!]_M$ $\diamondsuit$

The typing-rules in Table 2 generate the *well-typed terms-in-context* over a fixed λ-signature Σ. These terms-in-context (a.k.a. *typing judgments*) are of the form $\Delta \mid t : B$, where $\Delta \equiv x_1 : A_1 \cdots x_n : A_n$ is a context of typed variables as usual, and $A_1 \cdots A_n, B \in \tau(\Sigma_0)$.

Given a λ-signature Σ and a Σ-structure M in a cartesian closed category $\mathbb{C}$, we inductively define the interpretation $[\![\Delta]\!]_M$ of contexts $\Delta \equiv x_1 : A_1 \cdots x_n : A_n$ by

$$\frac{}{x_1 : A_1 \dots x_n : A_n \mid x_i : A_i} \qquad \frac{}{\Delta \mid \star : 1} \qquad \frac{}{\Delta \mid c : A}\ c \in \Sigma^{\mathsf{c}}_A$$

$$\frac{\Delta, x : A \mid t : B}{\Delta \mid \lambda x \,.\, t : A \Rightarrow B} \qquad \frac{\Delta \mid t : A \Rightarrow B \qquad \Delta \mid u : A}{\Delta \mid tu : B}$$

$$\frac{\Delta \mid t : A \qquad \Delta \mid u : B}{\Delta \mid (t, u) : A \times B} \qquad \frac{\Delta \mid t : A \times B}{\Delta \mid \mathrm{fst}(t) : A} \qquad \frac{\Delta \mid t : A \times B}{\Delta \mid \mathrm{snd}(t) : B}$$

Table 2: Typing rules for the simply typed λ-calculus.

- $[\![\diamond]\!]_M = 1$ (empty context)
- $[\![\Delta, x : A]\!]_M = [\![\Delta]\!]_M \times [\![A]\!]_M$

The interpretation

$$[\![\Delta \mid t : A]\!]_M : [\![\Delta]\!]_M \to [\![A]\!]_M$$

of well-typed terms $\Delta \mid t : A$ is then given by the following inductive clauses.

- $[\![x_1 : A_1 \dots x_n : A_n \mid x_i : A_i]\!]_M = p_i$
- $[\![\Delta \mid * : 1]\!]_M = ([\![\Delta]\!]_M \xrightarrow{!} 1)$
- $[\![\Delta \mid c : A]\!]_M = ([\![\Delta]\!]_M \xrightarrow{!} 1 \xrightarrow{c_M} [\![A]\!]_M)$ for $c \in \Sigma^{\mathsf{c}}_A$
- $[\![\Delta \mid \lambda x \,.\, t : A \Rightarrow B]\!]_M = \Lambda([\![\Delta, : A \mid t : B]\!]_M)$
- $[\![\Delta \mid tu : B]\!]_M = \varepsilon^{[\![A]\!]_M}_{[\![B]\!]_M} \circ \langle [\![\Delta \mid t : A \Rightarrow B]\!]_M, [\![\Delta \mid u : A]\!]_M \rangle$
- $[\![\Delta \mid (t, u) : A \times B]\!]_M = \langle [\![\Delta \mid t : A]\!]_M, [\![\Delta \mid u : B]\!]_M \rangle$
- $[\![\Delta \mid \mathrm{fst}(t) : A]\!]_M = p_1 \circ [\![\Delta \mid t : A \times B]\!]_M$
- $[\![\Delta \mid \mathrm{snd}(t) : B]\!]_M = p_2 \circ [\![\Delta \mid t : A \times B]\!]_M$

Lemma 3.4 (Substitution Lemma) *Given a λ-signature Σ and a Σ-structure M in a cartesian closed category $\mathbb{C}$, we have*

$$[\![\Delta \mid u[\vec{t}/\vec{x}] : B]\!]_M = [\![x_1 : A_1 \dots x_n : A_n \mid u]\!]_M \circ \langle [\![\Delta \mid t_i]\!]_M \mid 1 \leq i \leq n \rangle$$

for terms-in-context $\Delta \mid t_i : A_i$ $(1 \leq i \leq n)$ and $x_1 : A_1 \dots x_n : A_n \mid u : B$. ■

Definition 3.5 Given a λ-signature Σ, a Σ-structure M in a cartesian closed category $\mathbb{C}$, and terms-in-context $\Delta \mid t : A$ and $\Delta \mid u : A$, we write

$$\Delta \mid t =_M u : A$$

as a shorthand for $[\![\Delta \mid t : A]\!]_M = [\![\Delta \mid u : A]\!]_M$. ♢

Theorem 3.6 *Given a λ-signature Σ and a Σ-structure M in a cartesian closed category $\mathbb{C}$, the relation $=_M$ on well-typed terms is closed under the following rules (written without types and contexts):*

- $s =_M t \quad \Rightarrow \quad s[\vec{u}/\vec{x}] =_M t[\vec{u}/\vec{x}]$
- $s =_M t \quad \Rightarrow \quad \lambda x \,.\, s =_M \lambda x \,.\, t$
- $s =_M s', t =_M t' \quad \Rightarrow \quad st =_M s't'$
- $s =_M s', t =_M t' \quad \Rightarrow \quad (s, t) =_M (s', t')$
- $s =_M t \quad \Rightarrow \quad \mathrm{fst}(s) =_M \mathrm{fst}(t), \;\; \mathrm{snd}(s) =_M \mathrm{snd}(t)$
- $t =_M \star$
- $(\lambda x \,.\, t)x =_M t$
- $(\lambda x \,.\, tx) =_M t$ *($x \notin \mathrm{FV}(t)$)*
- $\mathrm{fst}(t, u) =_M t$
- $\mathrm{snd}(t, u) =_M u$
- $(fst(t), \mathrm{snd}(t)) =_M t$

Definition 3.7 A *λ-theory* is a pair $\mathbb{T} = (\Sigma, E)$ where Σ is a λ-signature, and E is a set of equations $\Delta \mid t = u : A$ in context.

A *model* of a λ-theory $\mathbb{T} = (\Sigma, E)$ in a cartesian closed category $\mathbb{C}$ is a Σ-structure M in $\mathbb{C}$ such that $\Delta \mid t =_M u : A$ for all equations $\Delta \mid t = u : A$ in E. $\diamondsuit$

3.1 The internal language of a cartesian closed category

As for regular categories (Section 2.5), the internal language of cartesian closed category $\mathbb{C}$ is given by a maximal choice of signature together with a canonical interpretation. More precisely, we have the following.

Definition 3.8 Let $\mathbb{C}$ be a cartesian closed category. The λ-signature $\Sigma(\mathbb{C}) = (\Sigma(\mathbb{C})_0, \Sigma(\mathbb{C})^{\mathfrak{c}})$ and the $\Sigma(\mathbb{C})$-structure $M(\mathbb{C})$ are mutually inductively defined as follows.

- $\Sigma(\mathbb{C})_0 = \mathrm{obj}(\mathbb{C})$
- $\Sigma(\mathbb{C})^{\mathfrak{c}}_A = \hom(1, [\![A]\!]_{M(\mathbb{C})})$ for $A \in \tau(\Sigma(\mathbb{C})_0)$
- $X_{M(\mathbb{C})} = X$ for all $X \in \Sigma(\mathbb{C})_0$
- $f_{M(\mathbb{C})} = f$ for all $A \in \tau(\Sigma(\mathbb{C})_0)$ and $f \in \Sigma(\mathbb{C})^{\mathfrak{c}}_A$ $\diamondsuit$

TODO: syntactic categories, completeness

4 Toposes and higher order logic

Definition 4.1 A *topos* is a category $\mathcal{E}$ which has

1. finite limits,
2. exponentials (i.e. $\mathcal{E}$ is cartesian closed), and
3. a *subobject classifier*, i.e. a pointed object $1 \xrightarrow{\mathsf{t}} \Omega$ such that for every object A and subobject $U \overset{m}{\rightarrowtail} A$ there exists a unique $\chi_m : A \to \Omega$ making

$$\begin{array}{ccc} U & \longrightarrow & 1 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{t}} \\ A & \xrightarrow{\chi_m} & \Omega \end{array}$$

a pullback. ♢

In condition 3 we call χ_m the *classifying map* of m. The condition on the subobject classifier says that for every A there is a bijection between isomorphism classes of subobjects of A and arrows $A \to \Omega$. This observation leads to an alternative characterization of toposes.

Lemma 4.2 *A category $\mathcal{E}$ with finite limits and exponentials is a topos, iff the functor*

$$|\mathrm{Sub}| : \mathcal{E}^{\mathsf{op}} \to \mathbf{Set},$$

which sends each $A \in \mathcal{E}$ to the set of isomorphism classes of subobjects[9] *of A, is representable.*

Proof. If $\mathcal{E}$ is a topos, then

$$\hat{\mathsf{t}} : \hom(-, \Omega) \to |\mathrm{Sub}|, \qquad \hat{\mathsf{t}}_A(f) = f^*\mathsf{t}$$

is a natural isomorphism.

Conversely, assume that $|\mathrm{Sub}|$ is representable, i.e. there exists an object Ω and a subobject $m : U \rightarrowtail \Omega$ such that the induced natural transformation (via Yoneda) $\hat{m} : |\mathrm{Sub}| \to \hom(-, \Omega)$ is a natural isomorphism. To show that $m : U \rightarrowtail \Omega$ constitutes a subobject classifier it suffices to check that U is terminal. To see this let $A \in \mathcal{E}$ and consider the composition

$$\hom(A, U) \overset{(-\circ m)}{\rightarrowtail} \hom(A, \Omega) \underset{\cong}{\xrightarrow{\hat{\mathsf{t}}_A}} |\mathrm{Sub}|(A).$$

An arrow $f : A \to \Omega$ factors through m precisely if $f^*\mathsf{t} \cong \mathrm{id}_A$, i.e. $\hat{\mathsf{t}}_A(f) = \mathrm{id}_A$, and by this condition f is uniquely determined since $\hat{\mathsf{t}}$ is a natural iso. ■

[9] Thus, $|\mathrm{Sub}|(A)$ is the set of isomorphism classes of the preorder $\mathrm{Sub}(A)$ of subobjects introduced in 2.2-3.

4.1 Presheaf toposes

We already know that presheaf categories $\widehat{\mathbb{C}}$ on small categories $\mathbb{C}$ have finite limits and exponentials. To show that they are toposes it remains to find a subobject classifier. Its definition will be derived via the Yoneda lemma, making use of the fact that we have canonical representatives of subobjects in presheaf toposes, given by *subfunctors*:

Definition 4.3 Let $\mathbb{C}$ be a small category, and $F\mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ a presheaf. A *subfunctor* of F is a family $(U(C) \subseteq F(C))_{C\in\mathbb{C}}$ of subsets of the values of F such that for all $f : C \to D$ in $\mathbb{C}$ and $x \in U(D)$ we have $F(f)(x) \in U(C)$. $\diamondsuit$

Any subfunctor $(U(C) \subseteq F(C))_{C\in\mathbb{C}}$ of a presheaf $F : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ gives rise to a presheaf $U : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ together with a monic natural transformation $\iota_U : U \rightarrowtail F$ where $U(f) = F(f)|_{U(D)}$ for $f : C \to D$ and the components of ι_U are subset inclusions. Conversely, given a subobject $\eta : G \to F$ of F, we can define a subfunctor U by setting

$$U(C) = \mathrm{im}(\eta_C) = \{\eta(x) \mid x \in G(C)\} \subseteq F(C)$$

for $C \in \mathbb{C}$. It is easy to see that these constructions establish a bijection between $|\mathrm{Sub}|(F)$ and the subfunctors of F, i.e. the equivalence classes in $|\mathrm{Sub}|(F)$ each contain a unique element whose components are subset inclusions.

Of special interest in this section are subfunctors of representable functors

$$YC = \hom(-, C) : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$$

for objects $C \in \mathbb{C}$. By definition, a subfunctor of YC is a family

$$(\ S_D \ \subseteq\ \hom(D, C)\ )_{D\in\mathbb{C}}$$

of sets of arrows satisfying $hk \in S_{D'}$ whenever $h \in S_D$ and $k : D \to D'$ in $\mathbb{C}$. By taking the union of the fibers of such a subfunctor, we arrive at the notion of a *sieve* on $\mathbb{C}$.

Definition 4.4 Given an object C in a small category $\mathbb{C}$, A *sieve* on C is a set $S \subseteq \mathrm{mor}(\mathbb{C})$ of morphisms such that $\mathrm{cod}(f) = C$ for all $f \in S$, and $hf \in S$ whenever $(f : D \to C) \in S$ and $h : D' \to D$ in $\mathbb{C}$.

We denote the set of sieves on C by $\mathrm{Siev}(C)$. $\diamondsuit$

Since sieves are just reformulations of subfunctors of YC, we do in particular have a bijection between $|\mathrm{Sub}|(YC)$ and $\mathrm{Siev}(C)$.

Putting things together, we derive the formula for the subobject classifier $\Omega : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ in $\widehat{\mathbb{C}}$ as follows.

$$\begin{aligned} \Omega(\mathbb{C}) &\cong \mathrm{Nat}(YC, \Omega) && \text{by the Yoneda lemma} \\ &\cong |\mathrm{Sub}|(YC) && \text{by the universal property of } \Omega \\ &\cong \mathrm{Siev}(C). \end{aligned}$$

This gives us the object part of Ω, and extending it to morphisms in the obvious way, we obtain the following description of the subobject classifier in $\widehat{\mathbb{C}}$.

Table 3: Rules of equational higher order logic

Theorem 4.5 *For small* $\mathbb{C}$*, the presheaf category* $\widehat{\mathbb{C}}$ *has a subobject classifier* $\Omega : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ *given by*

$$\Omega(C) = \mathrm{Siev}(C)$$
$$\Omega(f : C \to D)(S \in \mathrm{Siev}(D)) = \{h \in \mathrm{mor}(\mathbb{C}) \mid \mathrm{cod}(h) = C, fh \in S\}.$$

Thus $\widehat{\mathbb{C}}$ *is a topos.* ∎

4.2 Interpreting higher order logic in toposes

In the following we introduce a system of higher order logic as a kind of sequent calculus on certain typed λ-terms. These terms are defined over λ-signatures containing special symbols, called *higher order signatures*.

Definition 4.6 A *higher order signature* is a λ-signature $\Sigma = (\Sigma_0, \Sigma^{\mathsf{c}})$ (Def 3.2) where

- there is a special base type $\Omega \in \Sigma_0$, and
- for every $A \in \tau(\Sigma_0)$ there is a constant $(=_A) \in \Sigma^{\mathsf{c}}_{A\times A\Rightarrow\Omega}$. ♢

The semantic counterpart of higher order signatures are *higher order* Σ*-structures*. Before giving the precise definition we introduce a terminology and notation that will be handy in the following.

Definition 4.7 For $f : A \to B$ in a topos $\mathcal{E}$ we call morphism

$$\ulcorner f \urcorner \;=\; \Lambda(1 \times A \xrightarrow{p_2} A \xrightarrow{f} B) \;:\; 1 \to B^A$$

the *name* of f. ♢

Definition 4.8 If Σ is a higher order signature, a *higher order* Σ*-structure* in a topos $\mathcal{E}$ is a Σ-structure M in $\mathcal{E}$ such that

- $[\![\Omega]\!]_M = \Omega$
- for $A \in \tau(\Sigma_0)$ we have

$$[\![(=_A)]\!]_M = \ulcorner \mathsf{eq}_{[\![A]\!]_M} \urcorner : 1 \to \Omega^{[\![A]\!]_M \times [\![A]\!]_M},$$

  where $\mathsf{eq}_B : B \times B \to \Omega$ is the classifying map of $\delta_B : B \rightarrowtail B \times B$ for $B \in \mathbb{B}$. ♢

Grammatically, higher order logic can be understood as a kind of logic where, contrary to first order logic, terms and formulas are defined mutually inductively. While in first order logic terms are defined first, and formulas are then defined incorporating the terms, in higher order logic terms can be defined out of formulas, typically using a construct like 'set comprehension', which forms a term $\{x:A \mid \varphi\} : PA$ of power type out of a formula φ possibly containing a free variable $x:A$.

The 'equational higher order logic' which we use here is a simplified implementation of this idea which gets around two mutually inductively defined syntactic classes by *identifying* formulas with terms of a type Ω of propositions. Moreover, power types are not primitive but defined using function space and Ω as

$$PA \quad \equiv \quad A \to \Omega,$$

thus intuitively λ-abstractions $(\lambda x:A\,.\,\varphi)$ over terms φ of type Ω can be understood as set comprehensions.

Following the principle that terms of type Ω are formulas, the rules of equational higher order logic – given in Table 3 – are defined as derivation rules on judgments $(\vec{x} \mid \varphi_1 \ldots \varphi_n \vdash \psi)$ consisting of terms $(\vec{x} \mid \varphi_1 : \Omega), \ldots, (\vec{x} \mid \varphi_n : \Omega), (\vec{x} \mid \psi : \Omega)$.

5 Existence and choice in toposes

Definition 5.1 An object U in a finite-limit category $\mathbb{C}$ is called *subterminal*, if its terminal projection $U \to 1$ is a monomorphism. ♢

Definition 5.2 Let $\mathcal{E}$ be a topos.

1. We say that $\mathcal{E}$ satisfies the *axiom of choice (AC)*, if every epimorphism $e : B \to A$ splits (i.e. has a right inverse).

2. We say that $\mathcal{E}$ satisfies *'supports split' (SS)*, if all epis $B \to U$ split, where U is subterminal. ♢

References

[1] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2*, volume 44 of *Oxford Logic Guides.* The Clarendon Press Oxford University Press, Oxford, 2002.