Category Theory — 80-413/713 — Fall 2023
Due Thursday September 14, 11:59PM
Starred problems are for students enrolled in 80-713

Homework 2

1. Given a functor $F : \mathbb{C} \to \mathbb{D}$ between locally small categories and objects A, B in $\mathbb{C}$, we define a function

$$\begin{array}{rcl} F_{A,B} \; : \; \hom_{\mathbb{C}}(A,B) & \to & \hom_{\mathbb{D}}(F(A), F(B)) \\ f & \mapsto & F(f) \end{array}$$

which is well defined since functors commute with domain and codomain. The functor F is called

- *faithful*, if the function $F_{A,B}$ is injective for all $A, B \in \mathbb{C}_0$, and
- *full*, if the function $F_{A,B}$ is surjective for all $A, B \in \mathbb{C}_0$.

(a) Is the forgetful functor $U : \mathbf{Pos} \to \mathbf{Set}$ from the category of posets to the category of sets faithful? Is it full? (If it is, give a short explanation why, otherwise give a counterexample. Same for (b) and (c).)

(b) Is the inclusion functor $J : \mathbf{Pos} \to \mathbf{Preord}$ from posets to preorders faithful? Is it full?

(c) There are forgetful functors $U_0, U_1 : \mathbf{Cat} \to \mathbf{Set}$ which send a small category $\mathbb{C}$ to their sets $\mathbb{C}_0$ of objects and $\mathbb{C}_1$ of morphisms, respectively. Complete the definition the functors by giving their morphism parts. No verification of axioms necessary.

   i. Is U_0 faithful? Is it full?
   ii. Is U_1 faithful? Is it full?

2. Let $F : \mathbb{C} \to \mathbb{D}$ be a functor.

(a) Show that if F is faithful then it *reflects monomorphisms*, i.e. if $f : A \to B$ is an arrow in $\mathbb{C}$ such that $F(f)$ is a monomorphism in $\mathbb{D}$ then f is a monomorphism in $\mathbb{C}$.

(b) Show that if F is both full and faithful[1] then it *reflects isomorphisms*, i.e. whenever $f : A \to B$ is an arrow in $\mathbb{C}$ such that $F(f)$ is an isomorphism in $\mathbb{D}$, then f is an isomorphism in $\mathbb{C}$. (Isomorphism-reflecting functors are also called *conservative*.)

3. The following exercises ask to define functors with a given object part. In each case it's sufficient to give definitions of the morphism part, no verification of axioms necessary.

   (a) Define a functor $G : \mathbf{Rel} \to \mathbf{Pos}$ which sends each set to the poset $(PA, \subseteq)$, i.e. the power set ordered by inclusion.

   (b) Define a functor $P : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Pos}$ which sends every set to the power set ordered by inclusion.

   (c) Define a functor $H : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Rel}$ such that $G \circ H = P$.

4. $(*)$ Define an 'inclusion' functor $I : \mathbf{Preord} \to \mathbf{Cat}$ which sends every preorder to the associated category introduced in the lecture. Show that this functor has a left inverse. (No verification of functor axioms necessary.)

[1] in this case it is also called *fully faithful*