Lecture Notes on Pitch-Class Set Theory

Topic 3: Pitch, Interval, and Transposition

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Pitch

We have already discussed numbering pitch classes; pitches, too, can be numbered. Some pitch is chosen as 0; the fixed standard is C4=0, but it is also possible to choose some other 0 based on the piece, as a “moveable 0”.

Other pitches are then numbered in relation to 0, based on the number of semitones up or down. If C4=0, then C5=12, G5=19, Ab3=-4, etc.

Pitch Intervals

If pitches are represented as numbers, then intervals can also be represented as number, and very simply so: the interval between two pitches is simply the difference between the two numbers: ip=p1-p2.[1] Thus the interval between B4 and D4 is 11-2, which is 9, the number of semitones in a major sixth.

We represent intervals between pitches as ‘ip’, for pitch interval.

Just as we can either speak simply of a major sixth or else specify an ascending major sixth, we can either have directed intervals (which specify motion up or down) or undirected intervals (which don’t specify this).

For undirected intervals, we use parentheses around the number, and we always subtract the lower pitch from the higher; thus the undirected pitch interval between D4 and B4 is ip(9), because 11-2=9. Undirected pitch intervals are never negative.

For directed intervals, we used angled brackets around the number, and we subtract the first pitch from the second; thus the direct pitch interval from B4 to D4 is ip<-9>, because 2-11=-9. Directed pitch intervals are negative for descending intervals and positive for

[1] We already use numbers for intervals (second, third, etc.), but they work out very strangely. I call this math for musicians: 1+1=1; 3+3=5; 3+3+3=7. Can you see why it is that our usual numbering system for intervals doesn’t really make sense?

ascending intervals. For ascending intervals, used the '+' sign before the number; for example, the directed pitch interval from F3 to B4 is ip<+18>, because 11-(-5)=18.

Pitch Intervals mod 12

In discussing pitch classes and clocks, we have been talking about modular arithmetic. In modular arithmetic, a line is understood to loop around and around a spool – in the case of the clock, the line of time wraps around the circle of the clock, so that 1:50 is always the same position, whether am or pm, and whatever day it may occur in.

It is easy to think of numbers mod 12 in terms of adding or subtracting 12 as many times as necessary to get a number between 0 and 11; the result is the original number mod 12. For example, -4 mod 12 is 8, because -4+12=8. Similarly, 35 mod 12 is 11, because 35-2*12=11.

Undirected pitch intervals mod 12 are very simple to simple intervals used as shorthand for compound intervals; for example, the interval between E4 and F5 is ip(13); the undirected pitch interval mod 12 is ip mod 12 (1), because 13-12=1. This is similar to referring loosely to the minor ninth as a minor second.

We can also talk about ordered pitch intervals mod 12, using + or – to indicate the direction of motion.[2] Thus from B4 to F3 is ip mod 12 <-6>.

Pitch intervals mod 12 are helpful in many contexts, as they are equivalent to simple intervals as shorthand for intervals in general. Memorize the following chart as quickly as possible.

Pitch interval mod 12	**Simple interval**
0	P1
1	m2
2	M2
3	m3
4	M3
5	P4
6	A4/d5
7	P5
8	m6
9	M6
10	m7
11	M7

Pitch Transposition

[2] The negative sign in this case is a symbol that is separate from the number it precedes. Thus ip mod 12 <-4> means downward motion by ip mod 12 (4). Numbers mod 12 are between 0 and 11, so it makes no sense to say that some number mod 12 is -4.

We are already familiar with transposition of pitches; to transpose a pitch (or a chord, or a melody) is to shift all of the pitches in the same direction by the same interval. Thus we can transpose the major triad {C4 E4 G4} up a perfect fourth to {F4 A4 C5}. Using numbers, if C4=0, we would say that {0 4 7} has been transposed ip<+4> to {5 9 12}. We designate pitch transpositions as Tp, and we would call this transposition Tp +5. Note the nice relationship between the operation of transposition and the pitch numbers; to find Tp +5 of {0 4 7}, just add 5 to all the numbers. Similarly, if we wanted to transpose down by a major 7th, which we would call Tp -11, we would add -11 to each of the numbers, resulting in {-11 -7 -4}.

Pitch-Class Transposition

We have already seen in the notes on set classes that transposing pitch classes is equivalent to rotating them around the circle.

There are 11 possible pitch-class transpositions, because wherever a pitch starts out, it could be rotated to 11 other positions on the circle. To visualize this, think of transposing C, or 0, to the various other positions on the pitch circle. Because there are 11 transpositions, it makes sense to number them from 0 to 11 (where 0 is the transposition that leaves the pitches where they were, like transposing by a perfect unison).

Pitch-class transposition is additive, using modular arithmetic: T_n {p_1 p_2 p_3…} = { (p_1+n mod 12) (p_2+n mod 12) (p_3+n mod 12)…}. That is, take each pitch number, add the index of transposition, n, and take the result mod 12.

Let's work through an example. Suppose we want to take our C major triad {0 4 7} and transpose it "down by half step" (as we are dealing with pitch-classes and not pitches, this is a very imprecise way of speaking). Transposing "down by half step" corresponds to rotating the set one step counterclockwise on the circle – but counterclockwise motion on the circle would correspond to a negative index of transposition. Since we are using only indices between 0 and 11, we will get to the same place on the circle by rotating clockwise by 11 steps; this means we are using T_{11}. We then use the following calculations to find our new set:

(0+11) mod 12 = 11 mod 12 == 11

(4+11) mod 12 = 15 mod 12 = 3

(7+11) mod 12 = 18 mod 12 = 6

So T_{11} {0 4 7} = {11 3 6}, which corresponds perfectly to our intuitions about "transposing down by half step."

It is important to bear in mind that pitch-class transposition does not necessarily preserve registral ordering – this is because it is the pitch classes, not the pitches, that are transposed. In the previous section we observed that {F4 A4 C5} is a pitch transposition of {C4 E4 G4}.

Consider the set {F6 A2 C5}; this is a pitch-class transposition but not a pitch transposition of {C4 E4 G4}. The first note was transposed up by a perfect fourth plus two octaves, the second down down by a perfect fifth plus one octave, and the third note up by a perfect fourth.