Semidefinite programming and polynomials

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Tutorial on SDP, Uniandes/Externado

Plan

Semidefinite programming (SDP) and sums of squares (SOS)

Positive polynomials and SOS

Copositive matrices

Structured SDP

Beyond SDP

Semidefinite programming (SDP)

SDP: generalization of linear programming (LP).

$$\begin{array}{ll} \min & C \bullet X \\ \text{s.t.} & A_j \bullet X = b_j,\ j = 1, \ldots, m \\ & X \succeq 0. \end{array}$$

Here $C, A_j, X \in \mathbb{S}^n :=$ symmetric $n \times n$ matrices.

$M \succeq 0 \Leftrightarrow M \in \mathbb{S}^n_+ := \{M \in \mathbb{S}^n : x^{\mathrm{T}} M x \geq 0 \ \ \forall x \in \mathbb{R}^n\}$

$X \bullet S := \mathrm{trace}(XS) = \sum_{ij} X_{ij} S_{ij}$.

Nesterov & Nemirovskii, Alizadeh 1980s & 1990s:
Interior-point methods (ipm) for SDP.

Applications: control, combinatorial optimization,...

Natural connection SDP/sums of squares (SOS):

$M \succeq 0 \Leftrightarrow M = LL^{\mathrm{T}}$ for some $L \in \mathbb{R}^{n \times n}$ (Cholesky factorization)
Can rephrase $M = LL^{\mathrm{T}}$ as

$$x^{\mathrm{T}} M x = \sum (l_i^{\mathrm{T}} x)^2,$$

where $L = \begin{bmatrix} l_1 & \cdots & l_n \end{bmatrix}$.

In other words, $M \succeq 0 \Leftrightarrow x^{\mathrm{T}} M x$ is a sum of squares of linear forms.

Positive polynomials and sums of squares (SOS)

Suppose p, q_i are polynomials in n variables.
Observe: If $p = \sum_i q_i^2$ then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Theorem (Hilbert 1800s)

Converse is true if and only if

$\deg(p) = 2$,

or $n = 1$,

or $n = 2$ *&* $deg(p) = 4$.

Say that p is a sum of squares (SOS) if $p = \sum_i q_i^2$ for some polynomials q_i.

SOS can be naturally phrased in SDP terms.

Given a polynomial q can write $q(x) = c(q)^{\mathrm{T}} v(x)$
where $c(q)$: vector of coefficients of $q(x)$
$v(x)$: vector of monomials in x of degree $\leq \deg(q)$.

Observe

$$p(x) = q(x)^2 \Leftrightarrow p(x) = v(x)^{\mathrm{T}} \left(c(q) c(q)^{\mathrm{T}} \right) v(x)$$

Notice: $c(q)c(q)^{\mathrm{T}} \succeq 0$.

In general, p is SOS if and only if there exists $M \succeq 0$ such that

$$p(x) = v(x)^{\mathrm{T}} M v(x)$$

SDP/SOS/Positive polynomials

Suppose we are interested in

$$p^* = \min p(x)$$

Observe that

$$p^* = \max\{\lambda : p(x) - \lambda \geq 0 \ \ \forall x \in \mathbb{R}^n\}$$

Shor's method: approximate p^* with

$$p_{SOS} := \max\{\lambda : p(x) - \lambda \text{ is SOS}\}$$

Latter can be formulated as an SDP:

$$\begin{array}{ll} \max\limits_{\lambda, M} & \lambda \\ \text{s.t.} & p(x) - \lambda = v(x)^{\mathrm{T}} M v(x) \\ & M \succeq 0 \end{array}$$

The cone of copositive matrices

$$\mathcal{C}_n := \{M \in \mathbb{S}^n : x^{\mathrm{T}} M x \geq 0 \text{ for all } x \geq 0\}$$

Copositive matrices appear in:

complementarity problems

moment problems

combinatorial problems

Example: (Motzkin & Strauss, de Klerk & Pasechnik)
Assume $G = (V, E)$ loopless undirected graph and let $A(G)$ be its incidence matrix. Then

$$\alpha(G) = \min\{\lambda : \lambda(I + A(G)) - ee^{\mathrm{T}} \in \mathcal{C}_n\}$$

Murty & Kabadi: Checking membership in $\mathcal{C}_n$ is NP-hard.

Approximating the copositive cone via SOS

Theorem (Pólya 1928)

If $M \in int(\mathcal{C}_n)$ then for $r \in \mathbb{N}$ sufficiently large

$$\left(\sum_{i=1}^{n} x_i\right)^r x^{\mathrm{T}} M x = \sum_{|\beta|=r+2} c_\beta x^\beta$$

where all c_β are positive.

Parrilo 2001: Based on Pólya's Theorem, construct (via SOS) cones $\mathcal{K}_n^r \subseteq \mathcal{C}_n$ with $\mathcal{K}_n^r \uparrow \mathcal{C}_n$.

Theorem (Zuluaga, Vera, P. 2003)

$M \in \mathcal{K}_n^r$ if and only if

$$\left(\sum_{i=1}^{n} x_i\right)^r x^{\mathrm{T}} M x = \sum_{|\beta| \leq r+2} q_\beta(x) x^\beta$$

where each q_β is SOS.

Zuluaga, Vera, P. 2003: General version of the above for approximating

$$\{p : p(x) \geq 0 \text{ for all } x \in \{x : g_i(x) \geq 0\}\}$$

given $g_1, \ldots, g_k$.

Unifies a number of related results by Lasserre, Kojima, Parrilo,...

SDP descriptions of $\mathcal{K}_n^r$

$$M \in \mathcal{K}_n^r \Leftrightarrow \mathcal{L}(\vec{P}) \leq \mathcal{B}(M) \text{ for some } \vec{P} \succeq 0.$$

$r = 0$: $\vec{P} = P \in \mathbb{S}^n$

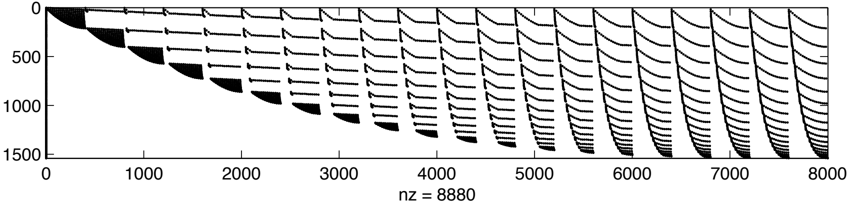
$r = 1$: need n matrices in $\mathbb{S}^n$.

$r = 2$: need $\binom{n}{2}$ matrices in $\mathbb{S}^n$ and one in $\mathbb{S}^{\binom{n}{2}}$.

$$\vdots$$

Structured SDPs

Matrix $\mathcal{L}$ for $r = 1,\ n = 20$



Both $\mathcal{L}$ and $\mathcal{B}$ have very special structure.
Exploit such structure to simplify the SDPs.

Example: Recall

$$\alpha(G) := \min\{\lambda : \lambda(I + A(G)) - ee^{\mathrm{T}} \in \mathcal{C}_n\}$$

De Klerk & Pasechnik: Approximate $\alpha(G)$ by

$$\vartheta^{(r)}(G) := \min\{\lambda : \lambda(I + A(G)) - ee^{\mathrm{T}} \in \mathcal{K}_n^r\}$$

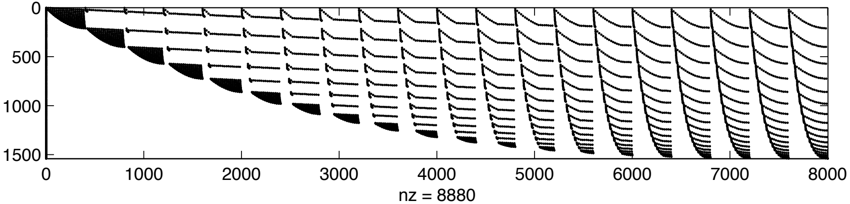
Since $\mathcal{K}_n^r \uparrow \mathcal{C}_n$, we have $\vartheta^{(r)}(G) \downarrow \alpha(G)$.

Structured SDPs: $\vartheta^{(1)}(G)$ for vertex-transitive graphs

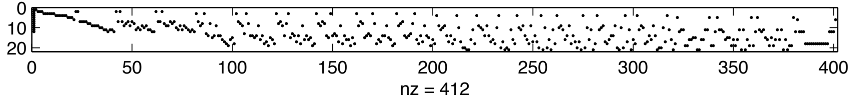
Theorem (Vera & P. 2006)

If G is vertex-transitive under an abelian automorphism group, then $\vartheta^{(1)}(G)$ can be formulated as a reduced SDP in one matrix variable P (instead of n matrix variables).

E.g., for $n = 20$ can reduce from an SDP with constraint matrix



to one with constraint matrix



$\vartheta^{(1)}(G)$ for vertex-transitive G

Want smallest λ such that

$$\sum_{i=1}^{n} x_i x^{\mathrm{T}} \left(\lambda(I + A(G)) - ee^{\mathrm{T}}\right) x = \sum_{i=1}^{n} x_i p_i(x)$$

for $p_i(x) = x^{\mathrm{T}}(P_i + N_i)x, \ P_i \succeq 0, \ N_i \geq 0$

If G vertex transitive can take $p_i(x) = p(\Pi_i x)$.
Π_i: permutation that sends $1 \mapsto i$.
For one single $p(x)$.

Some examples

G	$\alpha(G)$	$\vartheta^{(1)}(G)$	$\vartheta^{(0)}(G)$
C_{2m+1}	m	m	$m + 1/2 - O(1/m)$
C_5^3	10	10.935	11.180
C_7^2	10	10.269	11.006
C_7^3	33	35.341	36.517
C_9^2	17	18.000	19.010
$C_5 \times C_7$	7	7.000	7.4185
$C_5 \times C_7^2$	22	23.853	24.612

Work in progress (Vera & P):

Use these types of reductions to estimate $\alpha(C_7^4), \ldots$

Structured SDPs

SOS approaches often yield SDPs with invariant properties. Get

$$\begin{array}{ll} \min & C \bullet X \\ \text{s.t.} & A_i \bullet X = b_i,\ i = 1, \ldots, m \\ & X \succeq 0. \end{array}$$

Such that for a large multiplicative group G of permutation matrices

$$C = P^{\mathrm{T}} C P,\ A_i = P^{\mathrm{T}} A_i P, \text{ for all } P \in G \text{ and all } i$$

In this case the SDP has a solution in the "centralizer" of G:

$$C(G) := \{X : P^{\mathrm{T}} X P = X\}.$$

Theorem (De Klerk, Pasechnik, Schrijver 2005)

Construct $D_j \in \mathbb{S}^n$, $L_j \in \mathbb{S}^d$, $j = 1, \ldots, d$ *with* $d \leq n$ *such that*

$$X \in C(G),\, X \succeq 0 \Leftrightarrow X = \sum_{j=1}^{d} x_j D_j \text{ and } \sum_{j=1}^{d} x_j L_j \succeq 0$$

Here d: number of G-orbits in $\mathbb{S}^n$.
If G is large then $d << n$.

Limitation:
G has to be large (relative to n).
In particular, it does not apply to $\vartheta^{(1)}(G)$

Structured SDPs

Consider

$$\begin{array}{rl} \min & C \bullet X \\ \text{s.t.} & \mathcal{A}X = b \\ & X \succeq 0. \end{array}$$

Here $\mathcal{A}X := \begin{bmatrix} A_1 \bullet X & \cdots & A_m \bullet X \end{bmatrix}^{\mathrm{T}}$.
Assume $(\mathcal{A}, b, C)$ has the following invariant property:

$$\mathcal{A} = M_g \mathcal{A} N_g^{-1},\ b = M_g b,\ C = N_g^{\mathrm{T}} C$$

for some linear representations $\{M_g : g \in G\}$ and $\{N_g : g \in G\}$ of a finite group G.

Theorem (Vera & P. 2005)

Under the above assumptions can reduce the SDP to a smaller conic program

$$\begin{array}{rl} \min & \langle \bar{c}, x \rangle \\ s.t. & \bar{A}x = \bar{b} \\ & x \in K. \end{array}$$

Reduction is constructive (via numerical linear algebra) and can be incorporated within ipm algorithms.

Example:

$$\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} \succeq 0 \Leftrightarrow a \geq b \text{ and } a \geq -2b.$$

Can reduce SDP constraint

$$\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} \succeq 0$$

to an LP constraint

$$\begin{bmatrix} a - b \\ a + 2b \end{bmatrix} \geq 0$$

De Klerk-Pasechnik-Schrijver's approach does not identify this reduction.

Interior-point methods beyond SDP

Heart of ipm: "barrier function".

(LP) Barrier function for $\mathbb{R}^n_{++}$: $-\sum_{i=1}^n \log x_i = -\log x_1 \cdots x_n$

(SDP) Barrier function for $\mathbb{S}^n_{++}$: $-\log\det X$

Consider

$$\begin{array}{rl} \min & \langle c, x \rangle \\ \text{s.t.} & Ax = b \\ & x \in K \end{array}$$

where K is a convex cone.

Can solve via ipm as long as a barrier for $\mathrm{int}(K)$ is available.

Amount of work depends on how complicated the barrier function is.

Symmetric cones

Definition: Assume $K \subseteq \mathbb{R}^n$ is a convex cone.

$$\mathrm{Aut}(K) := \{g \in GL(\mathbb{R}^n) : gK = K\}.$$

Definition: Assume K is a convex cone. K is homogeneous if for all $x, y \in K$ there exists $g \in \mathrm{Aut}(K)$ such that $y = gx$.

Definition: Assume K is a convex cone. K is symmetric if it is homogeneous and $K = K^*$.

Nesterov & Todd 1997: Generalized ipm from SDP to conic programming over symmetric cones. (Currently implemented in `SeDuMi`, `SDPT3`.)

Symmetric cones

Examples of symmetric cones:

(i) Psd $n \times n$ symmetric matrices.

(ii) Lorentz cone: $\{x \in \mathbb{R}^n : x_1 \geq \|(x_2, \dots, x_n)\|\}$.

(iii) Psd $n \times n$ Hermitian matrices with entries in $\mathbb{C}$.

(iv) Psd $n \times n$ Hermitian matrices with entries in $\mathbb{H}$.

(v) Psd 3×3 Hermitian matrices with entries in $\mathbb{O}$.

Theorem

Every symmetric cone is a product of cones of types (i)–(v).

Symmetric cones are slices of $\mathbb{S}^n_+$

Theorem (Hauser & P, 2004)

- *Cones of types (ii) – (v) can be written as* $\{x : Lx \in \mathbb{S}^d_+\}$ *for some explicit L. (Thus have barrier* $x \mapsto -\log\det Lx$*.)*
- *In each case get* $\det(Lx) = p(x)q(x)$ *where factor* $q(x)$ *is "reduntant"*
  Hence barrier $x \mapsto -\log\det Lx$ *can be simplified to* $x \mapsto -\log p(x)$*.*

Example

$$\{x \in \mathbb{R}^n : x_1 \geq \|(x_2, \dots, x_n)\|\} = \left\{ x : \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & x_1 & 0 & \cdots & 0 \\ x_3 & 0 & x_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & 0 & \cdots & x_1 \end{bmatrix} \succeq 0 \right\}.$$

$$\det \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & x_1 & 0 & \cdots & 0 \\ x_3 & 0 & x_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & 0 & \cdots & x_1 \end{bmatrix} = x_1^{n-1}(x_1^2 - x_2^2 - \cdots - x_n^2)$$

Work in progress...

Hauser, Vera, P: Consider $\{x : Lx \in \mathbb{S}^d_+\}$ for a given L. When can we factor $\det(Lx) = p(x)q(x)$ so that $q(x)$ is redundant?

Special cases:

- Symmetric cones
- Subsets of $\mathbb{S}^d_+$ with a large symmetry group
- More...?

Redundant factor $q(x)$ has to do with some invariant property of the cone $\{x : Lx \in \mathbb{S}^d_+\}$

The polynomial $\det(Lx)$ as well as its factors $p(x), q(x)$ are "hyperbolic" polynomials.

Hyperbolic polynomials

Definition: A polynomial $p(x)$ of degree d is hyperbolic if

- $p(tx) = t^d p(x)$, i.e., p is homogeneous
- there exists e such that for every x the roots of

$$t \mapsto p(x + te)$$

are real.

Examples:

- $x_1 \cdots x_n$, $e = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$
- $\det X$, $e = I$,
- $x_1^2 - (x_2^2 + \cdots + x_n^2)$, $e = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}$.

SDP/Hyperbolic polynomials

Theorem (Gårding, 1959)

Assume p is hyperbolic. Then each connected component of $\{x : p(x) > 0\}$ is an open convex cone.

Hyperbolicity cone: a component of $\{x : p(x) > 0\}$ for hyperbolic polynomial p.

Theorem (Güler, 1996)

Assume p is hyperbolic. Then $-\log p(x)$ is a barrier function for each component of $\{x : p(x) > 0\}$

LP, SOCP and SDP are special cases of hyperbolicity cones:

> $\mathbb{R}^n_{++}$: component of $\{x : x_1 \cdots x_n > 0\}$ that contains $\begin{bmatrix}1 & \cdots & 1\end{bmatrix}$.
> $\mathbb{S}^n_{++}$: component of $\{X : \det X > 0\}$ that contains I.
> $\mathrm{int}(\mathcal{Q}_n)$: component of $\{x : x_1^2 - (x_2^2 + \cdots + x_n^2) > 0\}$ that contains $\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^{\mathrm{T}}$.

Lax conjecture, 1958

Original version:

> *A polynomial $p(x, y, z)$ is hyperbolic if and only if there exist $A, B, C \in \mathbb{S}^d$ such that*
>
> $$p(x, y, z) = \det(xA + yB + zC)$$

Theorem (Helton & Vinnikov, 2002)

Lax conjecture is true.

Lewis, Parrilo, Ramana 2003: Observed that Helton & Vinnikov proved Lax conjecture.

General version (still open):

> *Any hyperbolicity cone is a slice of $\mathbb{S}^n_+$.*

Homogeneous cones

Recall: A convex cone K is homogeneous if for all $x, y \in K$ there exists $g \in \text{Aut}(K)$ such that $y = gx$.

Theorem (Güler 1996)

If K homogeneous then K is a hyperbolicity cone.

Theorem (Chua, Faybusovich 2003)

Every homogeneous cone is a slice of $\mathbb{S}^n_+$.

Conjecture (Hauser, P.): Given a hyperbolic polynomial $p(x)$ can construct $q(x)$ and L such that

$$p(x)q(x) = \det Lx$$

for some appropriate L.

Concluding remarks

- SDP approach to problems involving polynomials via SOS. Resulting SDPs are large but highly structured
- Conversely, polynomials suggest extensions of SDP.
- Slices of $\mathbb{S}_+^n$ play a special role: homogeneous & symmetric cones are slices of $\mathbb{S}_+^n$.
- Nice interaction of ideas from optimization/algebra/analysis