The effective theory of an initial state

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references

scalar field: hep-th/0501158, hep-th/0507081

gravity/back-reaction: hep-th/0605107, hep-th/0609002

power spectrum: *in progress*

collaborator: Rich Holman (*Carnegie Mellon University*)

Natural UV Cutoffs in Expanding Space-times — September 7, 2006

Overview

(1) Inflation & structure formation

(2) The trans-Planckian problem

(3) An effective description of a state

(4) Propagation & evolution

(5) Renormalization of the energy-momentum tensor & back-reaction

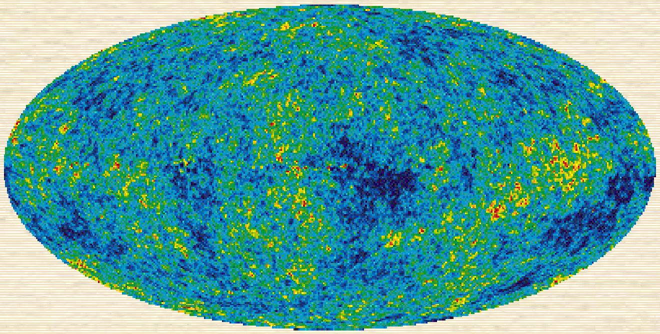
(6) Observability

Inflation and structure formation

- Inflation was originally proposed as a means for explaining several observed peculiarities of the universe
  - horizon problem: the extreme similarity of regions that would not have been in contact had the universe been solely radiation & matter dominated
  - it was later realized that it not only explained the extreme smoothness, but the tiny inhomogenieties as well

- Inflation provides a very elegant and economical explanation for the observed structure, relying upon two essential ingredients:
  - quantum fluctuations
  - rapid expansion

From the WMAP Science Team



With only a matter-radiation evolution, the size of a causal patch would be only about one pixel across

Quantum fields are always fluctuating,

$$\langle 0|\varphi(x)\varphi(y)|0\rangle \neq 0$$

We also need some rapid expansion,

$$ds^2 = a^2(\eta)\left[d\eta^2 - d\mathrm{x}\cdot d\mathrm{x}\right]$$

with $\frac{d^2 a}{dt^2} > 0$

Inflation and structure formation

- How do these ingredients conspire to produce the structure?

- Inflation is usually implemented with one or more scalar fields
  - the classical part drives the expansion
  - and since it is a quantum field, it also fluctuates (+ scalar part of the metric)

- To accelerate, the kinetic energy of the field must be smaller than its potential
  - 'slow-roll' consistency conditions
  - thus the mass & couplings must be small

- So, many different inflationary models share a common set of predictions

Divide the scalar field into a zero mode and a small fluctuation,

$$\Phi(t,\mathrm{x}) = \phi(t) + \varphi(t,\mathrm{x})$$

From the field equations,

$$\frac{1}{a}\frac{d^2 a}{dt^2} = \frac{1}{6}\frac{1}{M_{\mathrm{pl}}^2}\left[V(\phi) - \left[\frac{d\phi}{dt}\right]^2\right]$$

Slow-roll conditions:

$$\left|V''(\phi)\right| << 9H^2$$

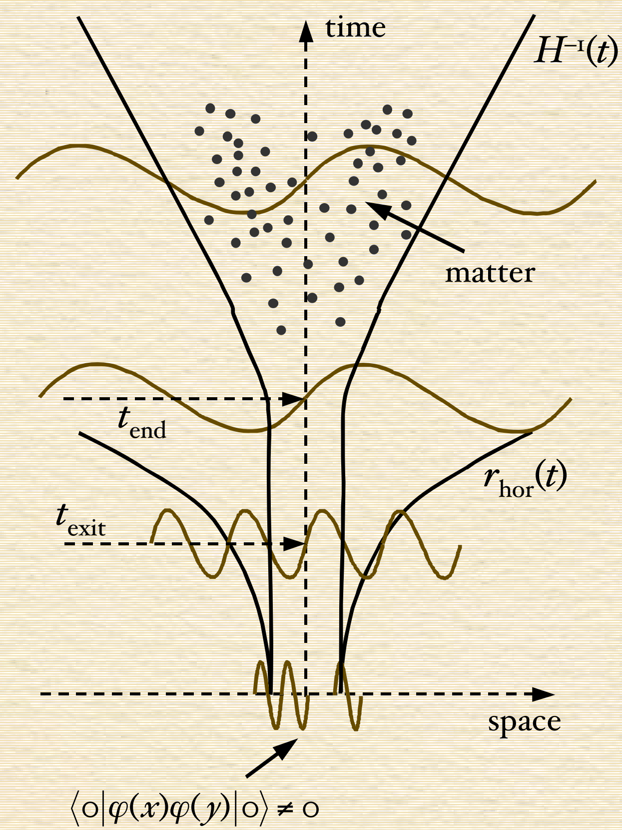
$$\left|V'(\phi)/V(\phi)\right| << (48\pi)^{1/2}/M_{\mathrm{pl}}$$

The primordial perturbations

- How inflation makes structures:
  - quantum fluctuations occur at some early stage during inflation
  - the expansion stretches them until they are larger than the Hubble horizon
  - at this stage, they are essentially frozen into the background forming a set of primordial perturbations
  - after inflation ends, they re-enter the horizon, influencing any material present

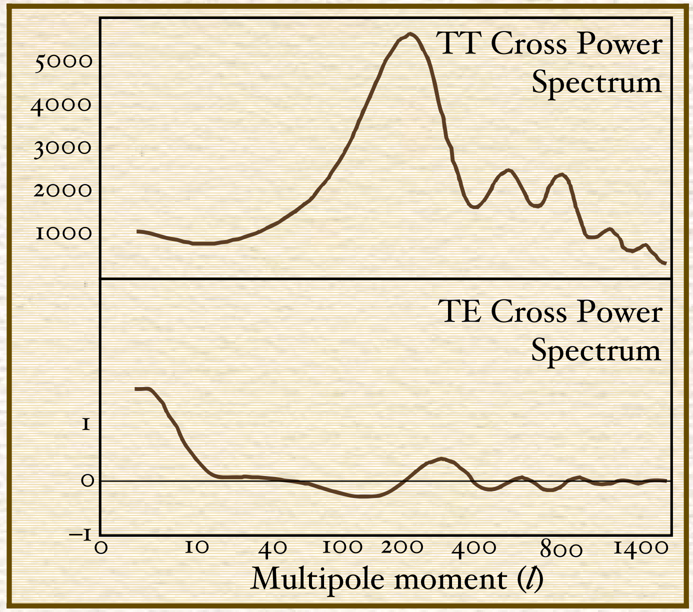
- Inflation makes a set of predictions shared across many particular models
  - nearly flat primordial power spectrum
  - small non-Gaussianities
  - correlations of features (TT-TE) on super-horizon* scales
  - phase synchronization

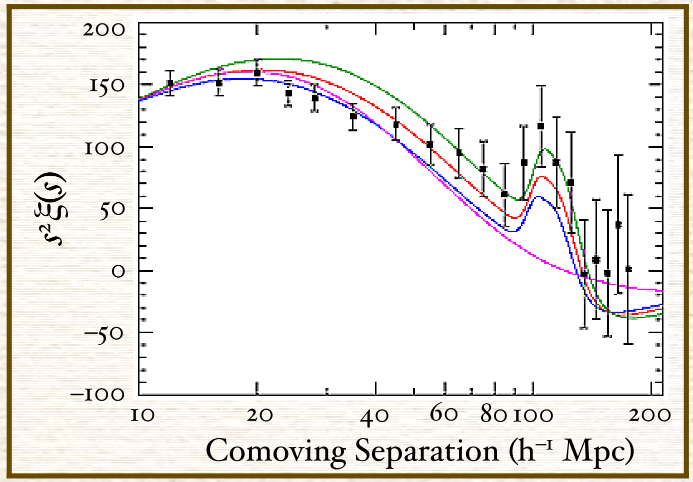
Let us follow a particular mode,



Observations

- In fact, just these oscillations are seen in the cosmic microwave background
  - WMAP, Acbar, Boomerang, CBI …
- Recently, the first baryon peak has also been observed in the distribution of luminous red galaxies
  - SDSS (astro-ph/0501171)
  - this approach promises the best accuracies eventually
- So far, what is seen is suggestive but not conclusive; all that is really required is
  - Gaussian white noise
  - phase synchronization
  - structure extending beyond the 'horizon'





Problems with inflation

- Despite these successes, this picture is not without several serious problems

Brandenberger

  - the amplitude problem
  - the cosmological constant problem
  - the inflaton problem (scalar fields)
  - the singularity problem
  - the trans-Planckian problem

- Of these problems, perhaps the most serious is the trans-Planckian problem

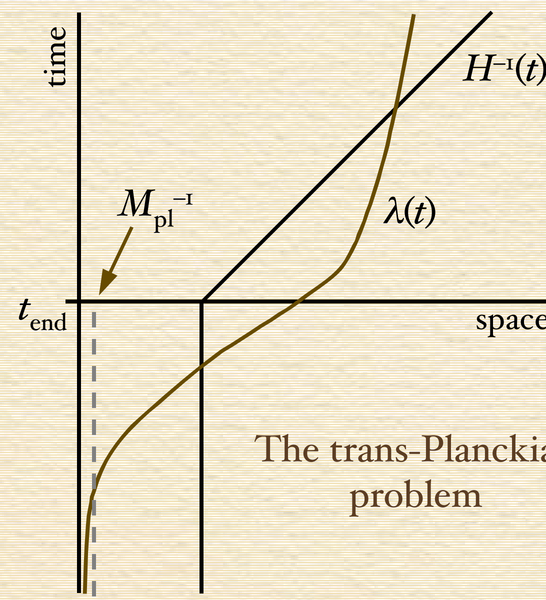
- It arises from exactly those two basic ingredients needed to produce inflation
  - it is based on a quantum theory
  - the continual expansion of space-time

The amplitude problem,

$$V(\phi)/\Delta\phi^4 \leq 10^{-12}$$

The cosmological constant problem,

$$L = 2M_{\rm pl}^4 \frac{\Lambda}{M_{\rm pl}^4} + M_{\rm pl}^2 R$$



The trans-Planckian problem

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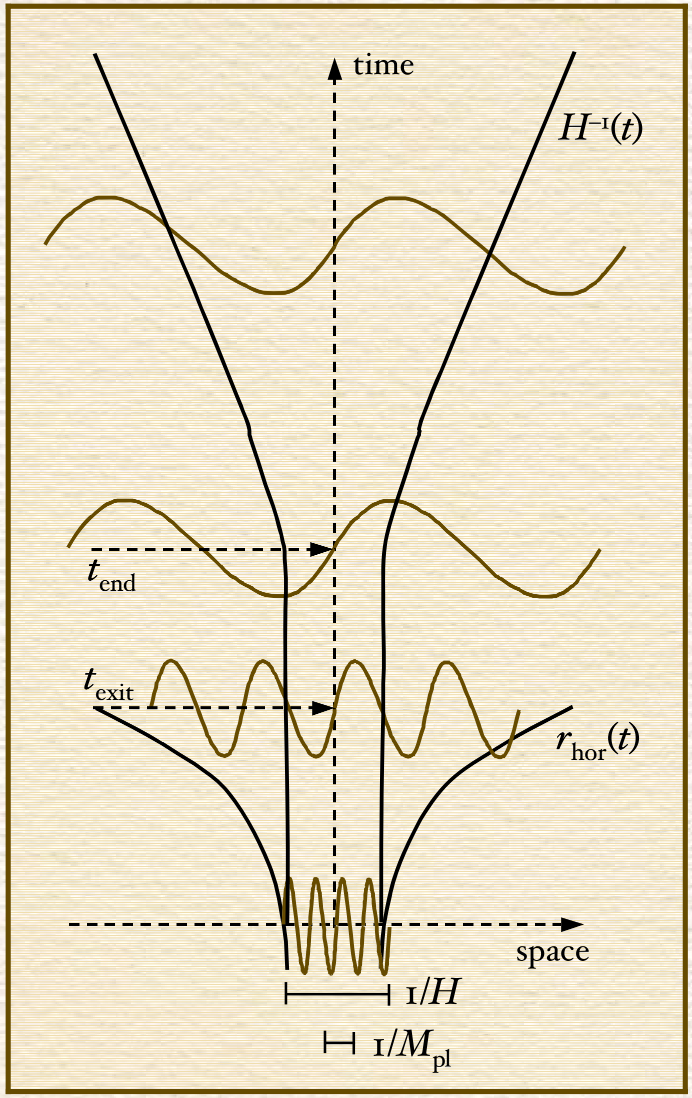
The trans-Planckian problem of inflation

- One problem in particular arises directly from the ingredients that make inflation so appealing
  - 60–70 *e*-folds of expansion solves the horizon problem
  - but most models have much more
- A little thought-experiment:

  consider some feature in the primordial power spectrum at the end of inflation that later will cause a feature in the microwave background radiation
  - starts outside horizon (t_{end})
  - going backwards: left the horizon (t_{exit})
  - still earlier: it was smaller than $1/M_{pl}$
- Do we need to understand the subtleties at the Planck scale to predict the CMB?



Approaches to the trans-Planckian problem

- Because of the importance of this problem, there have been many attempts to address it
- The first attempts chose a particular model for what happens near the Planck scale
  - modified dispersion relations

    Brandenberger, Martin
  - stringy uncertainty relation

    Easther, Greene, Kinney, Shiu
  - shortest distance prescription

    Kempf, Niemeyer
  - invariant de Sitter states (α-states)

    Danielsson; Collins, Holman, M. Martin
  - couplings to excited fields

    Burgess, Cline, Lemieux, Holman
- This last case emphasizes that the scale of new physics does not need to be $M_{\rm pl}$

Some ideas

Consider a dispersion relation that is modified at the Planck scale:

$$k_{\rm eff}^2(k,\eta) = k^2 - k^2 \frac{|b_m|}{a(\eta)} \frac{k}{M_{\rm pl}}$$

A modified uncertainty relation at short distances,

$$[x, p] = i\hbar(1 + \beta\, p \cdot p + \cdots)$$

Long-distance given by an α-state,

$$U_k^\alpha = N_k\left[U_k^{BD} + e^\alpha U_k^{BD*}\right]$$

An inflaton (φ) coupled to to a heavy, excited field (χ),

$$L = \sqrt{g}\Big\{\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi + \tfrac{1}{2}m^2\varphi^2 + \lambda(\chi^2 - v^2)^2 + \tfrac{1}{2}g\chi^2\varphi^2 + \gamma\varphi^4\Big\}$$

The general implications from many pictures

- Although each of these models proceeded from very different physical principles, they all reached similar conclusions about the size of the trans-Planckian effect

- The typical effect was suppressed by *H/M*
  - *H* is the Hubble scale during inflation
  - *M* is the scale for the new physics

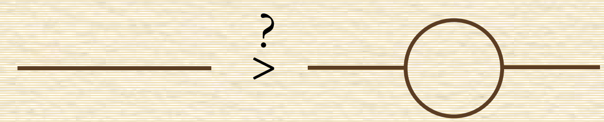
- Some questions:
  - is this estimate truly universal?
  - to what extent can these different models be distinguished?
  - is a particular model renormalizable?
  - we should not impose a crude cutoff; in field theories, we integrate well beyond the scale of applicability of our theory

A fairly typical estimate of the trans-Planckian correction to the primordial power spectrum is

$$P_\varphi = \frac{H^2}{4\pi^2}\left(1 - \frac{H}{M}\sin\frac{2M}{H}\right)$$

Danielsson, PRD 66, 023511

Most trans-Planckian calculations are tree-level calculation of the corrections to the power spectrum



Do these theories make sense peturbatively?
Are loop effects genuinely small?
How large is their effect on the background space-time (back-reaction)?

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Effective field theory

- There is another approach more in keeping with the principles of effective field theory

- In an effective field theory, we divide phenomena according whether they are higher or lower energy than a scale M
  - the fields and symmetries of the low energy ($< M$) degrees of freedom fix the renormalizable operators
  - the effects of new physics are included as non-renormalizable operators, suppressed by powers of $1/M$

- The theory is consistent since, for a measurement made at energy $E < M$, the effects of the new physics are always suppressed by

$$E^n / M^n$$

Consider a scalar field with an $\varphi \leftrightarrow -\varphi$ invariance,

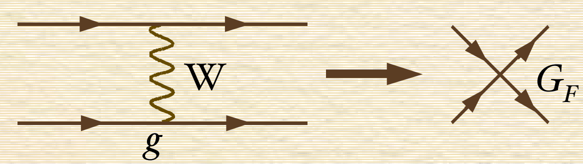
$$L = \tfrac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \tfrac{1}{2} m^2 \varphi^2 - \tfrac{1}{24} \lambda \varphi^4$$

$$+ \sum_{n=1}^{\infty} c_n \frac{\varphi^{2n+4}}{M^{2n}}$$

Another example is Feynman & Gell-Mann's theory with dimension 6 operators,

$$L = -\frac{G_F}{\sqrt{2}} \mathcal{J}^{\dagger}_{\lambda}(x)\, \mathcal{J}^{\lambda}(x) + \text{h.c.}$$

These arise by integrating out a W exchange,



so that $G_F/\sqrt{2} = g^2/M_W^2$

The effective description of a state

- Effective field theory strictly applies to the *evolution* of a state
- So we need to apply the same principle to the structures of the state, dividing them according to whether they vary significantly or not over a distance 1/*M*
  - long-distance features we set empirically
  - but short-distance structures are left general
- Measurements made at long-distances should not be greatly affected by the short-distances structure
  - inflation naturally provides a scale, *H*
  - the 'trans-Planckian' signature should be suppressed by powers of *H/M*

There are potentially two objects to which we could apply an effective description—the state and its evolution:

$$|0_{\rm eff}(\eta)\rangle = U_I(\eta,\eta_0)|0_{\rm eff}(\eta_0)\rangle$$

Expand the field in spatial eigenmodes,

$$\varphi = \int \frac{d^3k}{(2\pi)^3}\left[\varphi_k e^{i\vec{k}\cdot\vec{x}} a_{\vec{k}} + \varphi_k^* e^{-i\vec{k}\cdot\vec{x}} a_{\vec{k}}^\dagger\right]$$

In solving for the modes, we must make some assumption at arbitrarily large *k*,

$$\varphi_{k<M}(\eta_0) \to \text{understood}$$

$$\varphi_{k>M}(\eta_0) \to ?$$

Add some general structure that goes away for *k* << *M*

$$\varphi_k(\eta_0) - \varphi_k^{BD}(\eta_0) \to \sum d_n \frac{k^n}{M^n}$$

The effective description of a state — the idea

- The idea behind an effective state is that whenever we choose a 'vacuum' we are implicitly extrapolating our understanding of nature at low energies into arbitrarily short-distance regimes

- To avoid the trans-Planckian problem, we start our theory at an initial time, $\eta_\circ$
  - sufficiently early that the main features in the CMB are in the Hubble horizon
  - sufficiently late that none of them are smaller than the Planck length

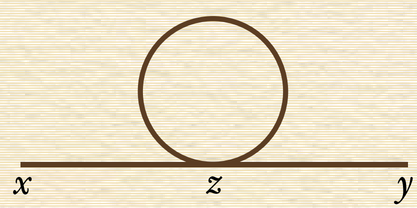
- Of course, processes that sum over all scales will sum over the structure we added to describe the effect of new physics
  - this produces new divergences which we shall show how to renormalize

We need some method for describing how the actual state differs from the nominal vacuum (or other state),

$$\varphi_k(\eta_\circ) - \varphi_k^{BD}(\eta_\circ) \to \sum d_n \frac{k^n}{M^n}$$

Note: We must apply a formalism that never evaluates any quantity before the initial time $\eta = \eta_\circ$; before this time, the effective description might not be perturbatively controlled

A loop sums over all scales,



The mode functions of an effective state

- We begin with a free scalar field theory propagating in a curved background
  - to keep the calculation simpler, we shall not include an $R\varphi^2$ term

- We usually specify the state by first restricting to the maximally symmetric solutions of the Klein-Gordon equation

$$\varphi_k'' + 2aH\varphi_k' + (k^2 + a^2 m^2)\varphi_k = 0$$

with

$$\varphi(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\varphi_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}a_{\mathbf{k}} + \varphi_{\mathbf{k}}^*(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}a_{\mathbf{k}}^{\dagger}\right]$$

and then impose a condition on the asymptotic behavior of the modes

The gravitational and scalar action,

$$S = S_G + S_\varphi$$

Scalar field part,

$$S_\varphi = \int d^4x\sqrt{g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}m^2\varphi^2\right]$$

The universe at large scales and early times appears to be homogeneous, isotropic and spatially flat,

$$ds^2 = a^2(\eta)\left[d\eta^2 - d\mathbf{x}\cdot d\mathbf{x}\right]$$

The Hubble scale is given by

$$H(\eta) = \frac{a'}{a^2} = \frac{1}{a^2}\frac{da}{d\eta}$$

The mode functions of an effective state

- Standard prescription: choose the state that at short distances ($k >> H$) matches with the flat-space vacuum
  - note that H can be just a few orders of magnitude below $M_{\rm pl}$
  - this defines the 'Bunch-Davies' state
- Since the Bunch-Davies state gives a good description of the long-distance features of the CMB
  - the effective state modes ($\varphi_k(\eta)$) are described in terms of its modes ($U_k(\eta)$)
  - f_k is a 'structure function' for the state
- Define the state through an initial condition on the mode functions,

$$\partial_\eta \varphi_k(\eta)\Big|_{\eta=\eta_0} = -i\varpi_k \varphi_k(\eta_0)$$

Define a general mode with respect to the Bunch-Davies modes, $U_k(\eta)$,

$$\varphi_k(\eta) = \frac{U_k(\eta) + f_k U_k^*(\eta)}{\sqrt{1 - f_k f_k^*}}$$

In principle we would like to define the trans-Planckian part of the state by

$$f_k \approx \sum_{n=1}^{\infty} d_n \frac{k^n}{a^n M^n}$$

But in practice we shall use, including a 'relevant' part of the state as well,

$$f_k = \sum_{n=1}^{\infty} c_n \frac{a^n m^n}{\omega_k^n} + \sum_{n=1}^{\infty} d_n \frac{\omega_k^n}{a^n M^n}$$

where

$$\omega_k^2 = k^2 + a^2 m^2$$

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Propagation

- We shall denote the effective state by $|0_{\rm eff}\rangle$
- The propagator should also be compatible with the initial boundary condition
  - in terms of the propagator modes:

$$-iG(x,x') = \langle 0_{\rm eff}|T(\varphi(x)\varphi(x'))|0_{\rm eff}\rangle$$
$$= -i\int \frac{d^3{\rm k}}{(2\pi)^3}\, e^{i{\rm k}\cdot({\rm x}-{\rm x}')}\, G_k(\eta,\eta')$$

- There is a subtlety since both the modes and the time-ordering have a time dependence
  - if we demand that the loop corrections are free of pinched singularities, then this ambiguity is removed

$$G_k(\eta,\eta') = \Theta(\eta-\eta')\,U_k(\eta)U_k^*(\eta') + \Theta(\eta'-\eta)\,U_k^*(\eta)U_k(\eta') + f_k^*\,U_k(\eta)U_k(\eta')$$

Apply the boundary condition to the propagator too,

$$\partial_\eta G_k(\eta,\eta')\Big|_{\eta=\eta_0} = i\varpi_k^* G_k(\eta_0,\eta')$$
$$\partial_{\eta'} G_k(\eta,\eta')\Big|_{\eta'=\eta_0} = i\varpi_k^* G_k(\eta,\eta_0)$$

ϖ_k is related to the structure f_k by

$$f_k = -\frac{\varpi_k U_k(\eta_0) - iU'_k(\eta_0)}{\varpi_k U_k^*(\eta_0) - iU'^*_k(\eta_0)}$$

The same propagator appears in Schalm, Shiu & van der Shaar,

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik\cdot(x-x')} + f_k^*(k_0)e^{-ik\cdot(x_I-x')}}{k^2-m^2+i\varepsilon}$$

For illustration, we have shown the flat space-time propagator, $x_I = (2t_0 - t, {\rm x})$

A comparison to other effective approaches

- Several other groups have also proposed using the principles of effective theories to study the trans-Planckian problem

- Kaloper, Kleban, Lawrence, Shenker & Susskind
  hep-th/0201158, hep-th/0209231
  - standard effective field theory (evolution)
  - the state was still a Bunch-Davies state

- Greene, Schalm, Shiu & van der Schaar
  hep-th/0401164, hep-th0411217
  - closest in spirit to our approach (propagator)
  - defined the state with a boundary action

- Anderson, Molina-París & Mottola
  hep-th/0504134
  - no modification to the propagator
  - states defined asymptotically ($\eta_0 \to -\infty$)
  - restricts to 4^{th} order adiabatic states

Write a general fully covariant action,

$$S_{\text{eff}}[\varphi] = \int d^4p\, \varphi(p)\varphi(-p)\Big\{\tfrac{1}{2}p^2 + \tfrac{1}{2}H^2 + \big[c_0 H^2 + c_1 p^2\big]\big(H^2/M^2\big) + c_2\, p^4/M^2 + \cdots\Big\}$$

$$\int \frac{d^4k}{(2\pi)^4}\, \frac{e^{-ik\cdot(x-x')} + f_k^*(k_0)e^{-ik\cdot(x_I-x')}}{k^2 - m^2 + i\varepsilon}$$

$$S_{\text{bnd}}[\varphi] = -\frac{1}{2M}\int d^3x\Big\{\beta_1\partial^i\varphi\partial_i\varphi + \beta_2\partial_n\varphi\partial_n\varphi + \beta_3\varphi\partial_n^2\varphi + \beta_4\varphi^4\Big\}$$

Modified states still rapidly approach the Bunch-Davies vacuum as $k \to \infty$

$$\varphi_k = A_k U_k + B_k U_k^*$$

$$\lim_{k\to\infty} k^{4+p}|B_k| = 0$$

Evolution

- Because of the trans-Planckian problem, it is not appropriate to evaluate matrix elements before the 'initial time' $\eta_\circ$
  - we should not use an *S*-matrix approach
  - apply a Schwinger-Keldysh evolution

- The time evolution is given by the interaction picture
  - operators evolve using the free part of the Hamiltonian
  - states evolve using the interacting part, H_I

- In the Schwinger-Keldysh formalism, both the state and its dual evolve

$$\langle 0_{\mathrm{eff}}(\eta)|\mathcal{O}(\eta)|0_{\mathrm{eff}}(\eta)\rangle$$
$$= \langle 0_{\mathrm{eff}}|U_I^{\dagger}(\eta,\eta_\circ)\,\mathcal{O}(\eta)\,U_I(\eta,\eta_\circ)|0_{\mathrm{eff}}\rangle$$

How does a matrix element evolve?

$$\langle 0_{\mathrm{eff}}(\eta)|\mathcal{O}(\eta)|0_{\mathrm{eff}}(\eta)\rangle$$

Define the time-evolution operator,

$$|0_{\mathrm{eff}}(\eta)\rangle = U_I(\eta,\eta_\circ)|0_{\mathrm{eff}}\rangle$$

where

$$|0_{\mathrm{eff}}\rangle \equiv |0_{\mathrm{eff}}(\eta_\circ)\rangle$$

U_I is given by Dyson's formula,

$$U_I(\eta,\eta_\circ) = Te^{-i\int_{\eta_\circ}^{\eta} d\eta'\, H_I(\eta')}$$

Evolution — field doubling

- To write the matrix element more compactly we shall formally double the fields

- First, insert a factor of 1 = $U_I^\dagger(0,\eta)\, U_I(0,\eta)$

$$\langle 0_{\rm eff}|\underbrace{U_I^\dagger(0,\eta_0)}_{-}\underbrace{U_I(0,\eta)O(\eta)U_I(\eta,\eta_0)}_{+}|0_{\rm eff}\rangle$$

- Next, label the arguments of the right three operators with a '+' and label those of the $U_I^\dagger$ with a '–' so that we can group everything within one time-ordering

$$\langle 0_{\rm eff}|T\left(O(\eta^+)e^{-i\int_{\eta_0}^{0} d\eta'^+\, H_I(\eta'^+)}\, e^{-i\int_{0}^{\eta_0} d\eta'^-\, H_I(\eta'^-)}\right)|0_{\rm eff}\rangle$$

- Finally, let us introduce '±' fields, $\varphi^\pm$ whose arguments are implicitly $\eta^\pm$

$$\langle 0_{\rm eff}|T\left(O^+(\eta)e^{-i\int_{\eta_0}^{0} d\eta'\left[H_I^+(\eta')-H_I^-(\eta')\right]}\right)|0_{\rm eff}\rangle$$

The time-evolution operator,

$$U_I(\eta,\eta_0) = Te^{-i\int_{\eta_0}^{\eta} d\eta'\, H_I(\eta')}$$

Important Note:

'–' times occur after and in the opposite order as '+' times

Another Note:

since the '±' on the time coordinates is now redundant, we shall drop their labels

Wick contractions

- When the interactions are weak, then we can evaluate a matrix element perturbatively
- However, in taking the Wick contractions of the fields, we have four basic possibilities
  - depending upon the '±' labels of the two fields being contracted

$$-iG_k^{++}(\eta,\eta') = \Theta(\eta-\eta')U_k(\eta)U_k^*(\eta') + \Theta(\eta'-\eta)U_k^*(\eta)U_k(\eta') + f_k^* U_k(\eta)U_k(\eta)$$

$$-iG_k^{+-}(\eta,\eta') = U_k^*(\eta)U_k(\eta') + f_k^* U_k(\eta)U_k(\eta)$$

$$-iG_k^{-+}(\eta,\eta') = U_k(\eta)U_k^*(\eta') + f_k^* U_k(\eta)U_k(\eta)$$

$$-iG_k^{--}(\eta,\eta') = \Theta(\eta'-\eta)U_k(\eta)U_k^*(\eta') + \Theta(\eta-\eta')U_k^*(\eta)U_k(\eta') + f_k^* U_k(\eta)U_k(\eta)$$

- For illustration, we have used the scalar field, but the same analysis applies to the graviton, $h_{\mu\nu}(x)$, as well

The time-evolution of a matrix element,

$$\langle 0_{\rm eff}(\eta)|O(\eta)|0_{\rm eff}(\eta)\rangle = \langle 0_{\rm eff}| T\left(O^+(\eta) e^{-i\int_{\eta_0}^{0} d\eta' \left[H_I^+(\eta') - H_I^-(\eta')\right]}\right)|0_{\rm eff}\rangle$$

There are four possible contractions,

$$-iG^{\pm\pm}(x,x') = \langle 0_{\rm eff}| T\left(\varphi^\pm(x)\varphi^\pm(x')\right)|0_{\rm eff}\rangle$$

where

$$G^{\pm\pm}(x,x') = \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} G_k^{\pm\pm}(\eta,\eta')$$

We can look at the graviton in the same way,

$$g_{\mu\nu}(x) = a^2(\eta)\left[\eta_{\mu\nu} + h_{\mu\nu}(x)\right]$$

Divergences and renormalization

- Thus we have a formalism for including short-distance structure in the state that differ from the Bunch-Davies vacuum
  - our description is in terms of the modes
  - the difference grows at shorter and shorter distances

- We should next show that this description is sensible perturbatively—processes that sum over all scales will be sensitive to this new structure we have added
  - the energy-momentum tensor
  - loop corrections from interactions

- So, in principle, we should expect to meet new divergences and we must provide a prescription for their renormalization

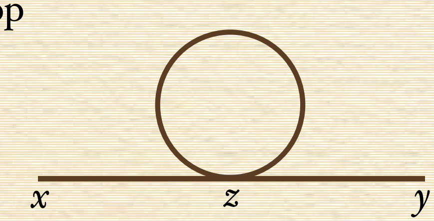
The initial state structure function,

$$f_k = \sum_{n=1}^{\infty} c_n \frac{d^n m^n}{\omega_k^n} + \sum_{n=1}^{\infty} d_n \frac{\omega_k^n}{d^n M^n}$$

The expectation value of the energy-momentum tensor,

$$\langle 0_{\mathrm{eff}}(\eta) | T_{\mu\nu}(x) | 0_{\mathrm{eff}}(\eta) \rangle$$

A loop



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The energy-momentum tensor & its divergences

hep-th/0605107
hep-th/0609002

- As an illustration, we shall examine the expectation value of the energy-momentum tensor for a scalar field
  - its renormalization
  - the size of the back-reaction
- The standard approach is to reduce the energy-momentum tensor to a classical function
- However, since the boundary effects are genuine quantum effects, it is more consistent to treat both gravity and the scalar field quantum mechanically
- The gravitational equations of motion are fixed by a renormalization condition
  - the vanishing of the graviton tadpole

The back-reaction is the renormalized trans-Planckian contribution to the energy density

The vacuum energy driving inflation is

$$\rho_{\rm vac} \approx H^2 M_{\rm pl}^{\ 2}$$

Treat both the field and the metric as fluctuations about classical backgrounds,

$$g_{\mu\nu}(x) = a^2(\eta)\left[\eta_{\mu\nu} + h_{\mu\nu}(x)\right]$$

$$\Phi(x) = \phi(\eta) + \varphi(x)$$

A renormalization condition,

$$\langle 0_{\rm eff}(\eta) | h_{\mu\nu}(x) | 0_{\rm eff}(\eta) \rangle = 0$$

The action

- Consider the gravitational action as an effective theory, in a derivative-expansion
- The divergences naturally form two classes:
  - state-independent or 'bulk' divergences
  - state-dependent or 'boundary' divergences
- The former occur for <u>any</u> state, and they require the renormalization of the parameters of the 4d gravitational action
  - Λ, $M_{\rm pl}$, α, ...
  - for simplicity, we shall not show the renormalization associated with the R^2 term; it can be found in the references
- Here we shall evaluate the part that depends on the initial state we chose, $S_{\rm ct}$

The full action,

$$S = S_\Phi + S_G + S_{\rm ct} + S_{GH}$$

The 'bulk' gravitational action,

$$S_G = \int d^4x \sqrt{g} \Big[2\Lambda + M_{\rm pl}^2 R + \alpha R^2 + \cdots \Big]$$

A free, minimally coupled scalar field,

$$S_\Phi = \int d^4x \sqrt{g} \Big[\tfrac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \tfrac{1}{2} m^2 \Phi^2 \Big]$$

The boundary counterterms,

$$S_{\rm ct} = \int_{\eta_0} d^3{\rm x} \sqrt{g_3}\, L_{\rm ct}$$

Gravity in the weak field limit

- To evaluate the graviton tadpole to leading order, we must determine the interaction Hamiltonian, $H_I(\eta)$
- Therefore, we expand the action to linear order in $h_{\mu\nu}$
  - from S_G, $h_{\mu\nu}$ couples to the Einstein tensor
  - from S_Φ, $h_{\mu\nu}$ couples to the energy-momentum tensors for ϕ and for φ
  - for now, we shall neglect the graviton loop
- A few comments:
  - there is a φ-ϕ cross-term in L_Φ, but it gives no contribution because of $\langle 0_{\text{eff}}|\,\varphi(x)\,|0_{\text{eff}}\rangle = 0$
  - $D_G^{(1)}$ is a total derivative, associated with the standard Gibbons-Hawking term
  - the leading interaction Hamiltonian is just

$$H_I = \int d^3x\, \tfrac{1}{2} a^2 h^{\mu\nu}\left[-2G_{\mu\nu} + T^{\text{cl}}_{\mu\nu} + T_{\mu\nu}\right]$$

Add small fluctuations,

$$g_{\mu\nu}(x) = a^2(\eta)\left[\eta_{\mu\nu} + h_{\mu\nu}(x)\right]$$

$$\Phi(x) = \phi(\eta) + \varphi(x)$$

Look for the linear terms in $h_{\mu\nu}$

$$L_G = \sqrt{g}\left[2\Lambda + M_{\text{pl}}^2 R + \cdots\right]$$

$$= L_G^{\text{cl}} + a^2 h^{\mu\nu} G_{\mu\nu} + D_G^{(1)} + \cdots$$

$$L_\Phi = \tfrac{1}{2}\sqrt{g}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - m^2\Phi^2\right]$$

$$= \cdots + \tfrac{1}{2} a^2 h^{\mu\nu}\left[T^{\text{cl}}_{\mu\nu} + T_{\mu\nu}\right] + \cdots$$

The energy-momentum operator,

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}\eta_{\mu\nu}\partial_\lambda\varphi\partial^\lambda\varphi + \tfrac{1}{2}\eta_{\mu\nu} a^2 m^2 \varphi^2$$

The expectation value of $T_{\mu\nu}$ for the effective state

- To account for the effect of the initial state structure, we must properly include it in the expectation value, $T_{\mu\nu}$

- The propagator provides a paradigm for how to include this structure, so let us write the expectation value in terms of $G(x,x')$

$$T_{\mu\nu} = -i \lim \left[\partial_\mu \partial'_\nu - \tfrac{1}{2}\eta_{\mu\nu}\left(\partial_\lambda \partial'^\lambda + a^2 m^2 \right) \right] G(x,x')$$

- In terms of the Bunch-Davies modes, U_k, and the structure function, f_k, we find

$$\rho = \frac{1}{2}\frac{1}{a^2}\int \frac{d^3k}{(2\pi)^3}\Big\{ U'_k U'^*_k + (k^2 + a^2 m^2) U_k U^*_k + f^*_k \Big[U'_k U'_k + (k^2 + a^2 m^2) U_k U_k \Big] \Big\}$$

$$p = -\rho + \frac{1}{a^2}\int \frac{d^3k}{(2\pi)^3}\Big\{ U'_k U'^*_k + \tfrac{1}{3}k^2 U_k U^*_k + f^*_k \Big[U'_k U'_k + \tfrac{1}{3}k^2 U_k U_k \Big] \Big\}$$

Define the expectation value as

$$T_{\mu\nu}(\eta) = \left\langle 0_{\rm eff} \left| T_{\mu\nu}(x) \right| 0_{\rm eff} \right\rangle$$

Standard trick:
- evaluate the fields at separate points
- extract the derivatives
- take the expectation value
- return to the limit, $x' \to x$

The η-derivatives do not act on the Θ-functions

$T_{\mu\nu}$ has the same symmetries as the background space-time,

$$T_{00}(\eta) = a^2(\eta)\rho(\eta)$$

$$T_{ij}(\eta) = a^2(\eta)p(\eta)\delta_{ij}$$

The graviton tadpole to leading order

- The gravitational equations of motion are a consequence of the vanishing of the tadpole,

$$\langle 0_{\rm eff}(\eta)|h^{+}_{\lambda\sigma}(\eta,x)|0_{\rm eff}(\eta)\rangle$$
$$= \tfrac{1}{2}\int_{\eta_0}^{\eta} d\eta'\, a^{-2}(\eta')\Gamma_{\lambda\sigma,\mu\nu}(\eta,\eta')$$
$$\times\left\{2G^{\mu\nu}(\eta') - T^{{\rm cl}\,\mu\nu}(\eta') - T^{\mu\nu}(\eta') + \cdots\right\}$$

- The tadpole vanishes if its integrand vanishes—in particular, the terms within the braces
  - yields the Einstein equations
  - note that we have included Λ in $G_{\mu\nu}$

- Here, $\Gamma_{\lambda\sigma,\mu\nu}$ represents the graviton leg

$$\Gamma_{\lambda\sigma,\mu\nu}(\eta,\eta')$$
$$= a^4(\eta')\left[\Pi^{>}_{\lambda\sigma,\mu\nu}(\eta,\eta';0) - \Pi^{<}_{\lambda\sigma,\mu\nu}(\eta,\eta';0)\right]$$

The leading interaction Hamiltonian is

$$H_I = \int d^3x\, \tfrac{1}{2} a^2 h^{\mu\nu}\left[-2G_{\mu\nu} + T^{\rm cl}_{\mu\nu} + T_{\mu\nu}\right]$$

At an arbitrary intermediate time we have the Einstein equation,

$$2G_{\mu\nu} = T_{\mu\nu} + T^{\rm cl}_{\mu\nu}$$

where $T_{\mu\nu}$ is the expecation value of the energy-momentum operator for $\varphi(x)$

$$T_{\mu\nu}(\eta) = \langle 0_{\rm eff}|T_{\mu\nu}(x)|0_{\rm eff}\rangle$$

Note that $\Pi^{>}_{\lambda\sigma,\mu\nu}$ and $\Pi^{<}_{\lambda\sigma,\mu\nu}$ are the Wightman functions for the graviton

Bulk divergences in the energy-momentum

- As we mentioned, the divergences can be characterized according to whether they occur at an arbitrary time along the evolution or only occur at the initial time, η_0
- We shall call the former 'bulk' divergences; they occur in the part independent of the state (f_k)
- For example, $\rho_{\rm bulk}$ contains quartic, quadratic and logarithmic divergences that require the renormalization of Λ, $M_{\rm pl}$ and α
  - expand the integrand in powers of ω_k
  - dimensionally regularize the integrals
  - rescale the gravitational parameters to render the theory finite
- This also cancels the divergences in $p_{\rm bulk}$

Define the bulk part of ρ to be

$$\rho_{\rm bulk} = \frac{1}{2}\frac{1}{a^2}\int\frac{d^3k}{(2\pi)^3}\Big\{U_k' U_k'^* + (k^2 + a^2 m^2) U_k U_k^*\Big\}$$

Expand in $\omega_k^2 = k^2 + a^2m^2$,

$$\rho_{\rm bulk} = \frac{1}{2}\frac{1}{a^2}\int\frac{d^3k}{(2\pi)^3}\,\omega_k + \frac{1}{4}\frac{1}{a^4}\left(\frac{a'}{a}\right)^2\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_k} + \frac{a^2m^2}{\omega_k^3}\right] + \cdots$$

Renormalize Λ and $M_{\rm pl}^2$,

$$\Lambda_R = \Lambda + \frac{m^4}{64\pi^2}\frac{1}{\varepsilon}$$

$$M_{{\rm pl},R}^2 = M_{\rm pl}^2 - \frac{m^2}{192\pi^2}\frac{1}{\varepsilon}$$

Boundary divergences in the energy-momentum

- The second class of divergences depends on the details of the initial state we have chosen

- They occur in the following parts of the energy-momentum tensor,

$$\rho_{\rm surf} = \frac{1}{2}\frac{1}{a^2}\int \frac{d^3{\rm k}}{(2\pi)^3} f_k^*\left[U_k' U_k' + (k^2 + a^2 m^2) U_k U_k\right]$$

$$p_{\rm surf} = -\rho_{\rm surf} + \frac{1}{a^2}\int \frac{d^3{\rm k}}{(2\pi)^3} f_k^*\left[U_k' U_k' + \tfrac{1}{3} k^2 U_k U_k\right]$$

- The gravitational field will have a similar behavior, since $\rho_{\rm surf}$ and $p_{\rm surf}$ are the sources for the background curvature

(in progress)

- Here, however, we shall focus on the types of divergences in $\rho_{\rm surf}$ and $p_{\rm surf}$, outlining briefly how they are renormalized

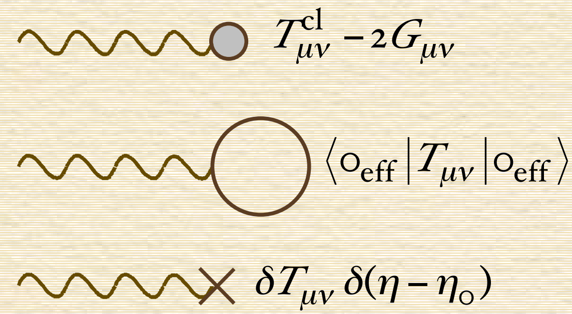
To one-loop order, the graviton tadpole is given by

$$\langle 0_{\rm eff}(\eta)|h^+_{\mu\nu}(\eta,{\rm x})|0_{\rm eff}(\eta)\rangle$$

$$= \tfrac{1}{2}\int d\eta'\, a^{-2}(\eta')\Gamma_{\lambda\sigma,}{}^{\mu\nu}(\eta,\eta')$$

$$\times\left\{2G_{\mu\nu}(\eta') - T^{\rm cl}_{\mu\nu}(\eta') - T_{\mu\nu}(\eta') + \cdots\right\}$$

Diagrammatically, we have



$+\cdots$

Boundary renormalization prescription

- The prescription for renormalizing the boundary divergences is as follows:

  hep-th/0609002

  - consider a particular moment in f_k
  - apply an adiabatic expansion of the integrands of $\rho_{\rm surf}$ and $p_{\rm surf}$ to isolate the divergent terms
  - note: because of the $d\eta'$ integral, logarithms are integrable
  - integrate by parts until the integrand is finite
  - in the process, we obtain terms evaluated at the η_0 boundary that are divergent
  - dimensionally regularize the divergences
  - cancel with an appropriate boundary action

The initial state structure is given by

$$f_k = \sum_{n=0}^{\infty} c_n \frac{a^n m^n}{\omega_k^n} + \sum_{n=1}^{\infty} d_n \frac{\omega_k^n}{a^n M^n}$$

The adiabatic approximation effectively expands Ω_k in powers of ω_k

$$U_k(\eta) = \frac{e^{-i\int_{\eta_0}^{\eta} d\eta' \Omega_k(\eta')}}{a(\eta)\sqrt{2\Omega_k(\eta)}}$$

Recall that there is a $d\eta'$ integration

$$\langle 0_{\rm eff}(\eta)| h^+_{\mu\nu}(\eta, {\rm x}) | 0_{\rm eff}(\eta)\rangle$$
$$= -\tfrac{1}{2}\int_{\eta_0}^{\eta} d\eta' \Big\{ \Gamma_{\mu\nu,}^{\ \ 00}(\eta,\eta')\,\rho_{\rm surf}(\eta')$$
$$+ \Gamma_{\mu\nu,}^{\ \ ij}(\eta,\eta')\delta_{ij}\, p_{\rm surf}(\eta') \Big\} + \cdots$$

Add counterterms at the boundary:

$$H_I^{\rm ct} = -\int d^3{\rm x} \left\{ a^3 h^{\mu\nu}\, \delta T_{\mu\nu}\, \delta(\eta - \eta_0) \right\}$$

A very simple example: a surface tension

- For illustration, a simple case should suffice
- The first divergence actually occurs in the 'long-distance' part of the structure
- In this case only the pressure diverges
- The divergence has exactly the same structure as the action for a surface tension
- The next order terms require successively less relevant boundary operators

initial state		counterterm dimension	
$c_2\, m^2/\omega_k^{\,2}$	$\rightarrow$	dim 1	(relevant)
$c_1\, m/\omega_k$	$\rightarrow$	dim 2	(relevant)
c_0	$\rightarrow$	dim 3	(marginal)
$d_1\, \omega_k/M$	$\rightarrow$	dim 4	(irrelevant)

Consider an initial state with

$$f_k = c_3 \frac{a^3(\eta_0) m^3}{\omega_k^3(\eta_0)}$$

The leading divergence in $p_{\rm surf}$ is

$$-\frac{c_3^*}{3}\frac{a^3(\eta_0)}{a^4(\eta')}\int \frac{d^3k}{(2\pi)^3}\frac{\omega_k(\eta') e^{-2i\int_{\eta_0}^{\eta'} d\eta''\,\Omega_k(\eta'')}}{\omega_k^3(\eta_0)}$$

Since only the spatial part (T_{ij}) diverges,

$$S_\sigma = \int d^3x\,\sqrt{g_3}\,\sigma = \int d^3x\, a^3 h^{ij}\left\{\tfrac{1}{2}\sigma\delta_{ij}\right\}$$
$$= \int_{\eta_0}^{0} d\eta \int d^3x\, a^3 h^{ij}\left\{\tfrac{1}{2}\sigma\delta_{ij}\right\}\delta(\eta-\eta_0)$$
$$+\cdots$$

Recall that the initial state structure is

$$f_k = \sum_{n=1}^{\infty} c_n \frac{a^n m^n}{\omega_k^n} + \sum_{n=1}^{\infty} d_n \frac{\omega_k^n}{a^n M^n}$$

Back-reaction

- After renormalizing the divergences in the energy-momentum tensor, and away from the $\eta = \eta_0$ boundary, how large is what remains?

Greene, Schalm, van der Schaar & Shiu
Porrati, Nitti & Rombouts
Collins & Holman: hep-th/0605107

- Let us calculate how the density & pressure scale in the divergences
  - look at a generic part of the trans-Planckian structure of the state

$$f_k = d_n \frac{\omega_k^n(\eta_0)}{a^n(\eta_0) M^n}$$

  - after renormalization we should have a similar term multiplying $\ln(\mu^2/m^2)$

To compare, the vacuum energy that sustains the inflationary expansion is

$$\rho_{\rm vac} \approx -p_{\rm vac} \approx M_{\rm pl}^2 H^2$$

The leading divergence for a trans-Planckian initial state is

$$\rho_{\rm surf}(\eta_0), p_{\rm surf}(\eta_0) \approx \frac{H^4}{16\pi^2} \frac{H^n}{M^n} \frac{d_n^*}{\varepsilon} \left[1 + \mathcal{O}\left(\frac{m^2}{H^2}\right) + \mathcal{O}\left(\frac{H'}{H^2}\right) \right]$$

After renormalization,

$$\rho_{\rm surf}^R, p_{\rm surf}^{\rm R} \approx \frac{H^4}{16\pi^2} \frac{H^n}{M^n} d_n^*$$

Thus the back-reaction is suppressed by

$$\frac{\rho_{\rm surf}^R}{\rho_{\rm vac}} \approx \frac{1}{16\pi^2} \frac{H^2}{M_{\rm pl}^2} \frac{H^n}{M^n}$$

Overview

(1) Inflation & structure formation

(2) The trans-Planckian problem

(3) An effective description of a state

(4) Propagation & evolution

(5) Renormalization of the energy-momentum tensor & back-reaction

(6) Observability

Observability of a trans-Planckian signal in the CMB

- Before concluding, let us consider whether a trans-Planckian signal can be observed

- The natural place to look for a trans-Planckian signal is in the CMB (Martin's talk)
  - it adds a modulation about the standard, flat primordial power spectrum
  - typical size of the signal scales as *H/M*
  - depends on an amplitude and phase, but the frequency is not independent (unlike a signal from the inflaton potential)
  - a source for non-Gaussianities (?)

- Some works have already tried to look for a trans-Planckian signal in the CMB

Martin & Ringeval, astro-ph/0310382
Easther, Kinney & Peiris, astro-ph/0505426
Easther, Kinney & Peiris, astro-ph/0412613

From Spergel's talk at string cosmology 5

$$\frac{P(k)}{\sigma_P(k)} = \frac{l_{max}}{\sqrt{1 + n_l/c_l}}$$

so we can improve our errors by looking at higher mulitpoles
(here c_l is the cosmic variance}

<u>Experimental upper limits</u>:

WMAP (1 yr)	$l_{max} \approx 300$
WMAP (6 yr)	$l_{max} \approx 600$
Planck	$l_{max} \approx 1500$
Ideal limit	$l_{max} \approx 2000$

beyond l = 2000 all CMB is affected by physics along the line of sight

The measurement of the CMB fix the power spectrum to about 1 part in 100

With Planck, we should be able to measure it to 1 part in 1000

Observability of a trans-Planckian signal in the LSS

- A better place to look for a trans-Planckian signal—at least eventually—will be in the distribution of the large scale structure
- Matter is also affected by the primordial perturbations
  - we need to look at large enough scales so that the non-linearities of the gravitational collapse have set in
  - in fact, the acoustic oscillations have already been seen in the Sloan Digital Sky Survey

astro-ph/0501171

- Over the next 10-20 years, several surveys should look at the distribution of galaxies on very large volumes
  - Square kilometre array (furthest along)
  - 21 cm high redshift gas (Loeb & Zaldarriaga)
  - cosmic inflation probe (look out to $z = 2$)

More from Spergel's ISCAP talk:

$$\frac{P(k)}{\sigma_P(k)} = \sqrt{\frac{k_{\max}^3 V}{1 + P(k)V / N_{\text{galaxies}}}}$$

so we can improve our errors by looking at larger volumes

Current/future surveys:

SDSS	$V \approx 10^8$ Mpc^3
SKA	$V \approx 10^9$ Mpc^3

the challenge is to resolve the initial conditions on small (Mpc) scales

The planned galaxy surveys should fix power spectrum to 1 part in 10,000

But future surveys should be able to measure it to 1 part in 100,000

Conclusions

- State-dependent effects can have a larger effect on the primordial power spectrum than sub-leading effects in the evolution
- An effective initial state provides a fairly generic method for following the effects of features in the state that differ from the Bunch-Davies vacuum at short-distances
  - provides a renormalizable framework
  - there is no strict cut-off, we integrate over all scales, but most higher-order structures have a negligible effect at lower energies ($H << M$)
  - so far, the back-reaction seems to be small
- Experimentally, over the next few years we can detect a 0.1% signal; by the end of the next decade, this should improve to 0.001%
  - CMB: WMAP (6 yr), Planck, ...
  - LSS: SKA, 21 cm gas, CIP, ...

The basic effective state idea:

$$\varphi_k(\eta_0) - \varphi_k^{BD}(\eta_0) \to \sum d_n \frac{k^n}{M^n}$$

How do specific models translate?

- modified uncertainty
- quantum-deformed symmetries
- composite inflaton/excited states

How do we correctly choose the state in inflation? (η_0, M)