Hiding a cosmological constant in a warped extra dimension

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Abstract. We present a scenario in which extra dimensions are used to address the cosmological constant problem. For a theory of gravity in $4+1$ dimensions whose dynamics are governed by an effective action that includes quadratic terms in the curvature and a compact scalar field, the field equations admit solutions that are compact in one direction and Poincaré invariant in the remaining directions. These solutions do not require any fine-tuning of the parameters in the action—including the cosmological constant—only that they should satisfy some mild inequalities. We further discuss several features of this picture, including an example of a metric that localizes gravity to a hypersurface without including a brane as well as how to combine this approach with the Randall-Sundrum model. [1]

The old idea that the universe might contain more than the observed four space-time dimensions has re-emerged recently in novel attempts to explain the weakness of gravity compared to the other forces [1] and the hierarchy problem [2], but it was realized earlier [3] that such theories might be able to address the cosmological constant problem [4]. The hope is that with extra dimensions, the metric might be able, through a non-trivial dependence on the extra coordinates, both to accommodate an arbitrary value for the cosmological constant and to maintain Poincaré invariance in $3+1$ of the directions. If this warping is accomplished with a metric that is both smooth and periodic in the extra dimensions, the period provides a natural compactifaction size for the extra dimensions.

An explicit realization of this idea occurs in $4+1$ dimensions [5] for an action with a generic set of curvature invariants with up to four derivatives of the metric and a compact scalar field. The 4D cosmological constant is determined by both the 5D cosmological constant and the geometry of the extra dimension. Therefore, we can achieve $3+1$ dimensional Poincaré invariance even when the 5D cosmological constant is not zero by choosing the solution to the field equations with the appropriate behavior in the extra dimension. Yet while no fine-tuning of the action is required, some further mechanism is still required to explain why this particular solution should be preferred.

This approach can also be adapted to eliminate the fine-tuning present in the Randall-Sundrum scenario [6]. Starting with an effective action for gravity in six dimensions as well as two parallel 4-branes, with some mild bounds on the parameters in the action, upon integrating out the compact sixth dimension we recover the action originally considered by Randall and Sundrum [2].

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A WARPED KALUZA-KLEIN MODEL

At energies approaching the Planck scale, corrections to the standard Einstein-Hilbert action can play an important role in determining the geometry of space-time. Treating gravity as an effective theory by expanded in powers of derivatives, an action that includes a generic set of terms with up to four derivatives of the metric is

$$S_{\rm gravity} = M_5^3 \int d^4x dy \sqrt{-g} \left(-2\Lambda + R - a R^2 - b R_{ab} R^{ab} - c R_{abcd} R^{abcd} + \cdots \right), \quad (1)$$

We also include a scalar field whose dynamics are determined by[2]

$$S_\phi = M_5^3 \int d^4x dy \sqrt{-g} \left(-\tfrac{1}{2} \nabla_a \phi \nabla^a \phi - \tfrac{1}{4} k (\nabla_a \phi \nabla^a \phi)^2 + \cdots \right). \quad (2)$$

Here Λ and M_5 are respectively the cosmological constant and the five dimensional Planck constant. g_{ab} is the metric for the space-time. We denote the coordinates that correspond to the usual space-time dimensions by x^μ, where $\mu, \nu, \cdots = 0, 1, 2, 3$, and the fifth coordinate by y, with $a, b, c, \ldots = 0, 1, 2, 3, y$. We shall often set $M_5 = 1$.

To produce a universe that resembles a flat, $3+1$ dimensional universe at lengths scales that have been observed, we consider a space-time metric with a warped Kaluza-Klein form,

$$ds^2 = g_{ab} dx^a dx^b = e^{A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2. \quad (3)$$

Since the extra dimension is to be small, we search for solutions in which $A(y)$ is smooth, periodic and non-singular. Unlike the usual Kaluza-Klein compactification, the metric depends strongly on the fifth coordinate y. We find [5]–[7] that the parameters of the higher-derivative terms in (1) determine a unique period for the extra dimension and this picture, provided such periodic functions $A(y)$ exist, does not suffer from any radius stabilization problem. The metric (3) is conformally flat so we can parameterize the effects of the R^2 terms by

$$\mu \equiv 16a + 5b + 4c \qquad\qquad \lambda \equiv 5a + b + \tfrac{1}{2} c. \quad (4)$$

The parameter λ, in particular, represents the coefficient of the Gauss-Bonnet term.

Varying the action, we obtain for a warped Kaluza-Klein geometry

$$\begin{aligned} 3(A')^2 + 3A'' + \lambda \left[2A''(A')^2 + (A')^4 \right] & \qquad\qquad (5) \\ + \mu \left[A'''' + 4A'A''' + 3(A'')^2 + 4A''(A')^2 \right] &= -2\Lambda - \tfrac{1}{2}(\phi')^2 - \tfrac{1}{4}k(\phi')^4 \\ 3(A')^2 + \lambda (A')^4 + \mu \left[2A'A''' - (A'')^2 + 4A''(A')^2 \right] &= -2\Lambda + \tfrac{1}{2}(\phi')^2 + \tfrac{3}{4}k(\phi')^4. \end{aligned}$$

Here the prime denotes a y-derivative. We can solve for

$$\begin{aligned} (\phi'(y))^2 &= (3k)^{-1} \Big[-1 \pm \{ 1 + 12k \left[2\Lambda + 3(A')^2 + \lambda (A')^4 \right] \qquad (6) \\ &\qquad + 12k\mu \left(2A'A''' - (A'')^2 + 4A''(A')^2 \right) \}^{1/2} \Big] \end{aligned}$$

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[2] This choice for the scalar action is not necessary for the existence of periodic solutions. We also found periodic metrics when we include a free scalar field and the effects of asymmetric Casimir effect [6].

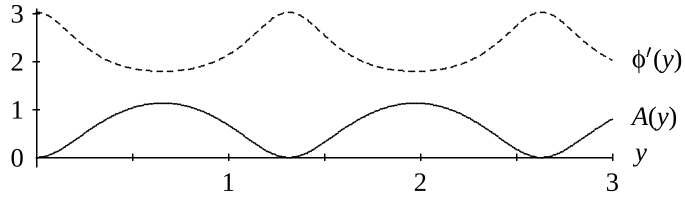

FIGURE 1. A periodic warp function $A(y)$ (solid line) and $\phi'(y)$ (dashed line) for $\Lambda = 1$, $\lambda = 0$, $\mu = 0.1$, and $k = -0.25$. The initial condition is $A''(0) = 23.77364592$.

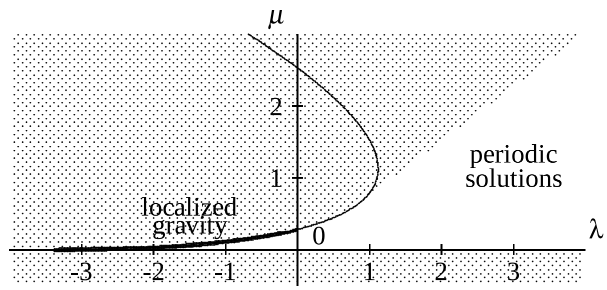

FIGURE 2. A plot of the parameter space $\{\lambda, \mu\}$ when $\Lambda = 1$. In the unshaded region, we have found periodic numerical solutions for $A(y)$ for arbitrarily chosen points. The curve depicts a set exact solutions discussed in [5] while the darker part of the curve shows the location of the solutions in (7).

and substitute the result into (5) to obtain a differential equation for $A(y)$.

A periodic solution for a generic set of values of Λ, μ, λ, and k is found by numerically integrating the resulting differential equation for $A(y)$. The coordinate y does not explicitly appear in the equations (5)–(6) which moreover only depend on the warp function $A(y)$ through its derivatives. Thus, we can choose $A(0) = A'(0) = 0$ without any loss of generality. We also chose $A'''(0) = 0$ for simplicity. The subsequent evolution of the warp function away from $y = 0$ then depended solely upon the initial value of the second derivative, $A''(0)$.

Numerically we find that there exists a precise value of $A''(0)$ that produces a periodic solution for each arbitrarily chosen set of parameters $\{\Lambda, \mu, \lambda, k\}$ within a region of the parameter space with a non-zero volume. This result demonstrates the existence of periodic solutions without finely tuning any of the parameters in the action. An example of such a solutions has been sketched in Fig. 1 for $\Lambda = 1$, $\lambda = 0$, $\mu = 0.1$, and $k = -0.25$ and choosing the minus root in (6). Many further examples appear in [5] and [6].

A slice of the region in parameter space for which periodic metrics exist is shown in Fig. 2, with $\Lambda = 1$ and $k \to 0$. The choice of the former is always possible by the rescaling, $y \to \sigma y$, $\Lambda \to \sigma^{-2}\Lambda$, $\mu \to \sigma^2 \mu$, $\lambda \to \sigma^2 \lambda$ and $k \to \sigma^2 k$, where σ is a real constant, which leaves (5) unchanged. We have found numerical solutions for arbitrarily chosen points throughout the unshaded portion of Fig. 2.

Using gravity to localize gravity

The purely gravitational part of the action (1) can also generate a warp function that is localized along a $3+1$ dimensional hypersurface. The profile of the function $A(y)$ for these solutions superficially resembles that appearing in models in which a domain wall is used to localize gravity,

$$A(y) = -\frac{2}{\kappa l} \ln\left[2\cosh(\kappa y)\right]. \qquad [\phi(y) = 0] \tag{7}$$

Here the width, κ^{-1}, and the asymptotic AdS_5 length, l, are respectively

$$\kappa = \left(\frac{3-4\sqrt{2\Lambda\mu}}{2\mu}\right)^{1/2} \qquad l = \left(\frac{3-4\sqrt{2\Lambda\mu}}{\Lambda}\right)^{1/2}, \tag{8}$$

with $0 \leq \Lambda\mu \leq \frac{9}{32}$, $\Lambda > 0$ and $\mu \geq 0$. In this configuration the R^2 terms are in no sense negligible. This metric does require one fine-tuning among Λ, λ and μ, given in [5] by

$$\Lambda\lambda = -\left(3-4\sqrt{2\Lambda\mu}\right)\left(\tfrac{9}{8}-\tfrac{1}{2}\sqrt{2\Lambda\mu}\right), \tag{9}$$

which can presumably be effected by adding a sixth dimension with an appropriately warped compactification.

THE RANDALL-SUNDRUM SCENARIO AS AN EFFECTIVE THEORY

Randall and Sundrum [2] proposed that if the universe were to consist of two 3-branes bounding a bulk region of five dimensional anti-de Sitter (AdS_5) space-time,

$$ds^2 = g_{ab}\,dx^a dx^b = e^{-2|r|/l}\,\eta_{\mu\nu}\,dx^\mu dx^\nu + dr^2, \tag{10}$$

then the redshift induced by the bulk metric at one of the branes could generate an exponential hierarchy between the Planck scale and the scale of electroweak symmetry breaking. The bulk Einstein equations determine $\Lambda = -6l^{-2}$ and the specific choice of $\sigma = \pm 6l^{-1}$ for the brane tensions is necessary for the low energy four dimensional theory to be free of a cosmological constant.

As the cosmological constant and the surface tension appear in the action in [2], they represent fundamental parameters of the theory and we have no reason *a priori* that the fine-tuning condition is satisfied. If instead these quantities arise from some more fundamental theory, then it might be possible for a dynamical mechanism to exist that favors solutions in which the low energy, four dimensional theory is nearly flat.

We can adapt the picture developed above without branes to one which resembles the Randall-Sundrum construction but where the AdS_5 length, l, is not uniquely determined by the higher dimensional cosmological constant. The structure for such a model would include *two* extra dimensions—one small periodic dimension to avoid fine-tuning the

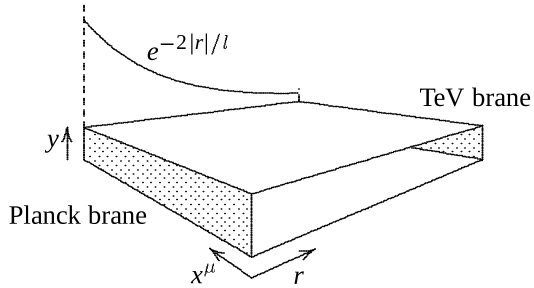

FIGURE 3. The geometry of a six dimensional model with two 4-branes. The small periodic coordinate is y. The direction orthogonal to the 4-branes, r, becomes the extra coordinate of the Randall-Sundrum model when we integrate out the y dimension. The model assumes an orbifold geometry about $r = 0$.

cosmological constant as before and a second to generate the electroweak-Planck hierarchy, as shown in Fig. 3.

By generalizing the four derivative action of (1)–(2) to six dimensions and adding two 4-branes at $r = 0$ and $r = r_c$, we shall show that after integrating out the y-dependence, we can recover the action of the Randall-Sundrum scenario [2]. The important new feature is that the 6D cosmological constant, $\tilde{\Lambda}$, no longer needs to be finely tuned with respect to the tensions on the branes, which we write as $\tilde{\sigma}^{(0)}$ and $\tilde{\sigma}^{(r_c)}$ respectively.

We begin with a metric of the form

$$ds_6^2 = \tilde{g}_{MN}(x^\lambda, r, y)\, dx^M dx^N = e^{A(y)} g_{ab}(x^\lambda, r)\, dx^a dx^b + dy^2 \tag{11}$$

with the AdS_5 metric (10) for the (x^λ, r)-subspace. As in the 5D case earlier, when $A(y)$ is a periodic function of y, we can obtain a compact extra dimension with a very non-trivial y-dependence without any singularities. However, unlike the previous example, the (x^λ, r) subspace is not flat. The shape of $A(y)$ determines the effective cosmological constant of the g_{ab} metric. The metric (11) with an AdS_5 subspace is still conformally flat, but the definition of the coefficients of the linear combinations of $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ in the 6D version of (1) orthogonal to the Weyl squared term depend on the number of dimensions. In $5+1$ dimensions we have $\tilde{\mu} \equiv 20\tilde{a} + 6\tilde{b} + 4\tilde{c}$ and $\tilde{\lambda} \equiv 15\tilde{a} + \frac{5}{2}\tilde{b} + \tilde{c}$.

The 6D curvature tensors $\tilde{R}$, $\tilde{R}_{MN}$ and $\tilde{R}^L{}_{MNP}$ are related to their counterparts, R, R_{ab} and $R^a{}_{bcd}$ derived from the 5D metric g_{ab} in (11) and derivatives of the warp function. We thus derive a 5D effective action by integrating out the small y dimension in this background,

$$\begin{aligned} S_{\text{eff}} &= M_5^3 \int d^4x dr \sqrt{-g} \left(-2\Lambda + R - aR^2 - bR_{ab}R^{ab} - cR_{abcd}R^{abcd} \right) \\ &\quad + M_5^3 \int_{r=0} d^4x \sqrt{-h} \left[-2\sigma^{(0)} \right] + M_5^3 \int_{r=r_c} d^4x \sqrt{-h} \left[-2\sigma^{(r_c)} \right] + \cdots . \end{aligned} \tag{12}$$

h_{ab} represents the metric induced on the branes by the metric g_{ab}. The new parameters that appear in this effective action depend partially upon the "fundamental" parameters of the original action but also upon the behavior of the warp function. Explicitly, the parameters in the low energy 5D theory are (M_6 is the 6D Planck mass)

$$\begin{aligned} M_5^3\Lambda &= M_6^4\int_0^{y_c}dy\, e^{\frac{5}{2}A(y)}\left[\tilde{\Lambda}+\tfrac{1}{4}(\phi')^2+\tfrac{1}{8}k(\phi')^4-\tfrac{5}{2}(A')^2+\tfrac{5}{8}\tilde{\mu}(A'')^2-\tfrac{5}{24}\tilde{\lambda}(A')^4\right] \\ M_5^3 &= M_6^4\int_0^{y_c}dy\, e^{\frac{3}{2}A(y)}\left[1+\tfrac{1}{8}(3\tilde{\mu}-4\tilde{\lambda})(A')^2\right] \\ M_5^3 a &= M_6^4\tilde{a}\int_0^{y_c}dy\, e^{\frac{1}{2}A(y)} \\ M_5^3\sigma^{(0)} &= M_6^4\tilde{\sigma}^{(0)}\int_0^{y_c}dy\, e^{2A(y)}. \end{aligned} \tag{13}$$

In the weak 5D gravity limit, $M_5 l \gg 1$, the R^2 terms become negligible and the leading behavior is governed by the Einstein-Hilbert terms in (12). Since $\Lambda \sim l^{-2}$, we require the effective 5D cosmological constant to be small which can easily occur when the contribution from the bulk cosmological constant is partially cancelled by effects from the warp function in (13). Thus, we can recover the Randall-Sundrum action.

The fine-tunings of the effective tensions on the two branes in [2] are

$$\sqrt{-6\Lambda} = \sigma^{(0)} = -\sigma^{(r_c)}. \tag{14}$$

Numerically, we find solutions [6] periodic in the y-direction provided that the effective cosmological constant is of the same order or smaller than the full cosmological constant, $|\Lambda| \lesssim O(\tilde{\Lambda})$. Using the desired value of the 5D Λ from (14) and applying (13), we can thus find solutions that are periodic in y and satisfy (14) without finely tuning any of the fundamental parameters when $(\tilde{\sigma}^{(0)})^2 \lesssim O(\tilde{\Lambda})$.

CONCLUDING REMARKS

A theory with an extra compact dimension and an action with a generic set of R^2 terms and a compact scalar field contains sufficient freedom to admit periodic metrics with a $3+1$ dimensional Poincaré invariance without the need for finely tuning the action. Yet for each choice of parameters that allows such a solution, a family of other periodic solutions exists whose elements are specified by the value of the effective low energy 4D cosmological constant. We should further investigate whether a universe in a generic initial state can relax into one in which the effective 4D theory is nearly flat.

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