Brane Cosmologies without Orbifolds

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Abstract. We examine the evolution of a 3-brane that separates two regions of a $4+1$ dimensional bulk space-time with potentially different cosmological constants. We have included a term in the brane action for the scalar curvature. When one or both of the regions has a zero cosmological constant, we find solutions for which the cosmology on the brane has the same time-dependence as a standard radiation-dominated universe, but which depends on a mass parameter in the bulk metric rather than the radiation density on the brane. When a non-zero cosmological constant is present in both bulk regions, it is possible to obtain a standard Robertson-Walker evolution. We then extract the effective theory of gravity in the weak-field limit seen by an observer on the brane in each of these scenarios.

I INTRODUCTION.

A remarkable feature of certain theories with more than the observed $3+1$ dimensions is that while these extra dimensions can extend infinitely, the geometry of the bulk space-time is nevertheless able to confine gravity to a three dimensional surface within the larger space. Randall and Sundrum [3] first showed that by attaching two semi-infinite slices of $4+1$ dimensional anti-de Sitter space (AdS_5) along a three dimensional hypersurface, or '3-brane', with orbifold conditions about this 3-brane, gravity behaves as though it is confined to its vicinity. This 3-brane is identified with our universe. In addition to reproducing ordinary Newtonian gravity, any successful model should also be able to produce a realistic cosmological evolution for the 3-brane. The dynamical evolution of the brane is determined by Einstein's equations for the combined bulk and brane system, but these equations might not produce the familiar Robertson-Walker cosmology along the brane. Viewed locally, near the brane the surrounding bulk introduces a new element into the field equations for gravity on the brane through a term for the change in the extrinsic curvature across the brane, as originally derived by Israel [4]. While generalizations of the original Randall-Sundrum orbifold [3] have been shown to admit

[1] This talk represents work done in collaboration with Bob Holdom.

the usual open, flat and closed Robertson-Walker cosmologies [5], we shall examine more asymmetric geometries for which the AdS curvature lengths on opposite sides of the brane are not necessarily equal.

In this talk, we shall show that it is possible to obtain a standard cosmological evolution on the brane even when the cosmological constants in the adjoining bulk regions are very different. For example, when one of the regions has a vanishing cosmological constant, a mass parameter in the bulk AdS-Schwarzschild metric can produce an evolution with the same time dependence as a radiation-dominated universe. When the cosmological constant in one or both of the bulk regions is sufficiently small, deviations from the classical Newton's law and Einstein's equation appear at unacceptably large distances. This difficulty can be removed when a curvature term is included in the brane action, and we shall study its role in the classical theory of gravity on the brane. This term has been typically omitted in previous studies [5–7] and represents the next natural term in an effective action [10] on the brane ordered by powers of derivatives.

The material in this talk is described more fully in [1,2].

II THE ACTION FOR ADS$_5$ WITH A SURFACE.

We would like to derive the form of Einstein's equations for a $(3+1)$-dimensional hypersurface embedded in a $(4+1)$-dimensional bulk space-time. To be general, we shall treat the bulk space-time as two regions, $\mathcal{M}_1$ and $\mathcal{M}_2$, that share a common boundary given by the hypersurface, $\mathcal{B}$. Note that these bulk regions do not need to have the same metric on either side of the brane but only need to satisfy the Israel conditions derived below. Since the boundary corresponds to the observed universe, we include an action on the brane containing, in addition to a surface tension term, a term for the scalar curvature on the brane plus the contributions from matter and gauge fields confined to the brane. At each point on the brane, we define a space-like unit normal to the surface, $N_a = N_a(x^\mu)$, that satisfies $g^{ab}N_aN_b = 1$. g_{ab} is the bulk metric. We shall denote the coordinates along the brane by x^μ, where $\mu, \nu, \cdots = 0, 1, 2, 3$; we use r to denote the fifth coordinate and let $a, b, c, \ldots = 0, 1, 2, 3, r$. The bulk metric induces a metric on the brane,

$$h_{ab} = g_{ab} - N_aN_b; \tag{1}$$

while the bulk metric can be discontinuous across the brane, the induced metric on the brane must be the same when calculated with the bulk metric for either region.

Combining all of these ingredients, the total action is the sum of the actions for the two bulk regions,[2]

$$S_{\rm bulk} = \frac{1}{16\pi G}\sum_{i=1,2}\left\{\int_{\mathcal{M}_i} d^5x\,\sqrt{-g}\,\left(R + \frac{12}{\ell_i^2}\right) - 2\int_{\mathcal{B}} d^4x\,\sqrt{-h}\,K^{(i)}\right\}, \tag{2}$$

[2] Our convention for the signature of the metric is $(-, +, \cdots, +)$ while the Riemann tensor is chosen with the sign $-R^a{}_{bcd} \equiv \partial_d\Gamma^a_{bc} - \partial_c\Gamma^a_{bd} + \Gamma^a_{ed}\Gamma^e_{bc} - \Gamma^a_{ec}\Gamma^e_{bd}$.

and that of the boundary,

$$S_{\rm surf} = \frac{1}{16\pi G}\int_{\mathcal{B}} d^4x\,\sqrt{-h}\,\left(-\frac{12}{\ell}\frac{\sigma}{\sigma_c} + \frac{1}{2}b\ell\mathcal{R} + 16\pi G\,\mathcal{L}_{\rm fields} + \cdots\right). \tag{3}$$

Here G is the bulk Newton's constant and K is the trace of the extrinsic curvature K_{ab}, defined by

$$K_{ab} = h_a^{\ c}\nabla_c N_b. \tag{4}$$

σ, $\mathcal{R}$ and $\mathcal{L}_{\rm fields}$ represent the brane tension, the scalar curvature of the *induced* metric and the Lagrangian of fields confined to the brane. We normalize the brane tension with respect to a critical tension, $\sigma_c = 3/4\pi G\ell$, as will be useful later, and we allow the two bulk regions to have potentially different curvature lengths, ℓ_1 and ℓ_2. This action is a generalization of that appearing in [5] and [9].

Varying the total action yields the usual Einstein equations in the bulk,

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{6}{\ell_{1,2}^2}g_{ab}, \tag{5}$$

where the appropriate AdS length is chosen for each region, plus the following equation for the surface,

$$\Delta K_{ab} = \frac{2}{\ell}\frac{\sigma}{\sigma_c}h_{ab} - \frac{1}{2}b\ell\left[\mathcal{R}_{ab} - \frac{1}{6}\mathcal{R}h_{ab}\right] + 8\pi G\left[t_{ab} - \frac{1}{3}t_c^c h_{ab}\right] + \cdots \tag{6}$$

where $\Delta K_{ab} \equiv K_{ab}^{(2)} - K_{ab}^{(1)}$, t_{ab} is the energy-momentum tensor for the fields confined to the brane,

$$t_{ab} \equiv h_{ab}\mathcal{L}_{\rm fields} + 2\frac{\delta\mathcal{L}_{\rm fields}}{\delta h^{ab}}, \tag{7}$$

and $\mathcal{R}_{ab}$ is the Ricci tensor for the induced metric. This Israel condition (6) describes the effect of the bulk space-time on the brane Einstein equations through the appearance of the extrinsic curvature term.

III COSMOLOGY ON THE SURFACE.

We shall now examine some specific solutions of the field equations for gravity on an 3-brane between two $4+1$ dimensional regions with negative cosmological constants. The metrics for the interior $r < R(\tau)$ and exterior $r > R(\tau)$ regions with respect to the brane can be written in the AdS$_5$-Schwarzschild form [11].

$$\begin{aligned} ds^2|_{\rm int} &= -u(r)\,dt^2 + \frac{dr^2}{u(r)} + r^2\,d\Omega_3^2 \qquad & u(r) &= \frac{r^2}{\ell_1^2} + k - \frac{m_1}{r^2} \\ ds^2|_{\rm ext} &= -v(r)\,dt^2 + \frac{dr^2}{v(r)} + r^2\,d\Omega_3^2 \qquad & v(r) &= \frac{r^2}{\ell_2^2} + k - \frac{m_2}{r^2}. \end{aligned} \tag{8}$$

An AdS$_5$ bulk corresponds to setting $m_1 = m_2 = 0$, but we have included the $-m_{1,2}/r^2$ terms in the metric since they can have an important effect on the brane cosmology. Their presence leads to black-hole horizons at some distance into the bulk whose masses are determined by m_1 and m_2 [11,12].

The type of cosmology observed on the brane depends upon the value of k; when $k = 1, 0, -1$ the resulting evolution is respectively that of a closed, flat or open Robertson-Walker universe:

$$d\Omega_3^2 \equiv \begin{cases} d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2) & k = 1 \\ \ell^{-2}(dx^2 + dy^2 + dz^2) & k = 0 \\ d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2) & k = -1 \end{cases} \tag{9}$$

Later, when extracting the effective theory of gravity on the brane, we shall set $k = 0$ for convenience.

The location of the brane will generally evolve in the bulk, so we let

$$(t, r, \chi, \theta, \phi) = (T(\tau), R(\tau), \chi, \theta, \phi) \tag{10}$$

where τ is the proper time for an observer at rest with respect to the brane. The normal to the brane is then

$$N_a = \left(-\dot{R}, \dot{T}, 0, 0, 0\right) \qquad \dot{T} = \frac{(\dot{R}^2 + u(R(\tau)))^{1/2}}{u(R(\tau))} \tag{11}$$

with a dot denoting differentiation with respect to τ. We have used $g^{ab}N_aN_b = 1$ to express $\dot{T}$ in terms of $\dot{R}$ in the interior. In the exterior region, $u(R)$ is replaced by $v(R)$. With the normal in this form we find that the induced metric on the brane is already in the standard Robertson-Walker form:

$$ds^2 \equiv h_{\mu\nu}\, dx^\mu dx^\nu = -d\tau^2 + R^2(\tau)\, d\Omega_3^2 \tag{12}$$

where μ, ν run over the coordinates on the brane. Both the interior and exterior regions produce exactly the same induced metric.

In terms of the coordinate system defined by (12), the interior contribution to the extrinsic curvature is

$$K^{(1)}_{\mu\nu}\, dx^\mu dx^\nu = -\frac{1}{u(R(\tau))\dot{T}}\left[\ddot{R} + \frac{1}{2}\frac{\partial u}{\partial R}\right] d\tau^2 + u(R(\tau))\dot{T}R\, d\Omega_3^2; \tag{13}$$

the exterior region contributes an analogous expression with $v(R(\tau))$ replacing $u(R(\tau))$.

When the matter on the brane is distributed as an isotropic perfect fluid, of density ρ and pressure p so that $t_\mu^\nu = \mathrm{diag}(-\rho, p, p, p)$, the spatial components of the Israel condition (6) yield

$$\sqrt{\dot{R}^2 + u(R)} \pm \sqrt{\dot{R}^2 + v(R)} = \frac{2R}{\ell}\frac{\sigma}{\sigma_c} - \frac{b\ell}{2R}\left[\dot{R}^2 + k\right] + \frac{8\pi G}{3}R\rho + \cdots. \tag{14}$$

The temporal component of (6) does not give an independent equation once we have imposed the conservation of energy on the brane [5,6] which demands that $\frac{d}{d\tau}\left(\rho R^3\right) = -p\frac{d}{d\tau}R^3$. The choice of the relative sign between the extrinsic curvature terms in (14) depends on the geometry of the bulk AdS_5 space that surrounds the brane. In the original Randall-Sundrum universe, the orbifold is made of two slices of AdS_5 attached so that the warp factor—the r^2/ℓ^2 in the AdS metric (8)—decreases as we move further from the brane in either direction. Thus for the orbifold geometry, the plus sign is chosen. When the warp factor behaves differently on opposite sides of the brane, as for a brane simply embedded in a single bulk AdS_5 space, the minus sign is used.

For no scalar curvature term, $b = 0$, the Israel condition (14) can be rewritten so that the evolution of $R(\tau)$ is determined by a potential,

$$\tfrac{1}{2}\dot{R}^2 + V(R) = -\tfrac{1}{2}k, \tag{15}$$

where

$$\begin{aligned} V(R) = &-\frac{1}{2}\frac{R^2}{\ell^2}\left\{\frac{(\sigma+\rho)^2}{\sigma_c^2} - \frac{1}{2}\left(\frac{\ell^2}{\ell_1^2}+\frac{\ell^2}{\ell_2^2}\right) + \frac{1}{16}\frac{\sigma_c^2}{(\sigma+\rho)^2}\left(\frac{\ell^2}{\ell_1^2}-\frac{\ell^2}{\ell_2^2}\right)^2\right\} \\ &-\frac{1}{4}\frac{1}{R^2}\left\{m_1+m_2-\frac{1}{4}\frac{\sigma_c^2}{(\sigma+\rho)^2}(m_1-m_2)\left(\frac{\ell^2}{\ell_1^2}-\frac{\ell^2}{\ell_2^2}\right)\right\} \\ &-\frac{1}{32}\frac{\sigma_c^2}{(\sigma+\rho)^2}\frac{\ell^2(m_1-m_2)^2}{R^6}. \end{aligned} \tag{16}$$

A similar result is implicit in [5]. For $R \gg \ell$ and for a generic tension, this potential does not produce a standard Robertson-Walker cosmology. However, when the brane tension is tuned to

$$\frac{\sigma}{\ell} = \pm\frac{\sigma_c}{2}\left|\frac{1}{\ell_1} \pm \frac{1}{\ell_2}\right| \tag{17}$$

the leading R^2/ℓ^2, ρ-independent term drops out of the potential. The appropriate signs in (17) again depend on the behavior of the AdS space to either side of the brane.

IV THE EVOLUTION OF A VACUUM BUBBLE.

When a bubble nucleates in a region having a vacuum energy higher than that in the bubble's interior, the bubble will expand or contract depending upon the surface tension of the bubble and the difference in the bulk vacuum energies. A simple example of this behavior occurs when a bubble of AdS_5 is surrounded by an asymptotically flat region. The 3-brane here is the surface of this bubble. The purpose of this section is to introduce this bubble as an example of an acceptable brane cosmology that is driven by one of the mass parameters in the bulk metric in a relatively simple setting.

For a bubble surrounded by a region with zero cosmological constant, the metrics for the interior and exterior regions are then given by (8) as one of the AdS lengths becomes infinite. Let us choose $\ell_2 \to \infty$, set $m_2 = m$, $m_1 = 0$ and fine-tune $\sigma = \sigma_c/2$ for simplicity. We can also set $\ell_1 = \ell$ without loss of generality. Although the Israel equation (14) with the brane scalar curvature contribution becomes a quartic polynomial in $(\dot{R}^2 + k)$, we still can extract the leading behavior in the $\rho \ll \sigma_c$, $\ell \ll R$ limit:

$$\begin{aligned} V(R) \approx & -\frac{1}{2}\frac{m}{R^2} - \frac{1}{8}\frac{\ell^2}{R^2}\frac{m^2}{R^4}(b+1)^2 + \frac{m}{R^2}\frac{\rho}{\sigma_c}(b+1) + \cdots \\ & -\frac{2\rho^2}{\sigma_c^2}\left(\frac{R^2}{\ell^2} + \frac{3}{2}\frac{m}{R^2}(b+1)^2 + \cdots\right) + \cdots . \end{aligned} \tag{18}$$

The potential for the vacuum bubble (18) does not contain a ρR^2 term since the same fine-tuning that removes the cosmological constant from the brane also eliminates such a term. If we consider sufficiently late times when $\frac{\rho}{\sigma_c} \ll \frac{\ell}{R}$ is satisfied so that the $\rho^2 R^2$ term is small, then the leading term that determines the cosmology on the brane is

$$\dot{R}^2 + k = \frac{m_2}{R^2} + \cdots . \tag{19}$$

Although this equation seems quite different from the evolution in a Robertson-Walker universe (22), the time dependence of its solution is exactly the same as for a radiation-dominated universe in which $\rho \propto R^{-4}$. If the b-dependent terms are not to overwhelm the m/R^2 term we must also have $b\rho/\sigma_c \ll 1$. Comparing with the condition already imposed by the requirement that the m/R^2 term and not the $\rho^2 R^2$ term should drive the cosmology, we see that we can accommodate a $b\ell$ up to cosmological scales without imposing any new constraint.

As a more realistic variation, consider a vacuum bubble that expands into another AdS$_5$-Schwarzschild region, rather than a flat bulk. Unlike the standard Randall-Sundrum picture we shall let the second AdS length, ℓ_2, have a large macroscopic size but which is yet much smaller than the length associated with the brane curvature: $\ell_1, \ell \ll \ell_2 \ll b\ell$. The leading behavior of the cosmology (14) for this universe is then governed by

$$\begin{aligned} V(R) = & -\frac{1}{2}\frac{1}{b}\frac{\ell_2}{\ell_1}\frac{m_2}{R^2} - \frac{1}{2}\frac{1}{b}\frac{m_1}{R^2} + \cdots - \frac{2\rho}{\sigma_c}\left(\frac{1}{b}\frac{R^2}{\ell^2} - \frac{1}{b^2}\frac{\ell_2}{\ell}\frac{R^2}{\ell^2} + \cdots\right) \\ & + \frac{4\rho^2}{\sigma_c^2}\left(\frac{3}{2}\frac{1}{b}\frac{\ell_1}{\ell}\frac{R^2}{\ell^2} + \cdots\right) + \cdots . \end{aligned} \tag{20}$$

Unlike (18), the ρR^2 term is present:

$$\dot{R}^2 + k = \frac{8\pi G}{3}\frac{2}{b\ell}\rho R^2 + \frac{1}{b}\frac{\ell_2}{\ell_1}\frac{m_2}{R^2} + \cdots. \tag{21}$$

For the standard Robertson-Walker universe, the equation that determines $R(\tau)$ is

$$\dot{R}^2 + k = \frac{8\pi G_4}{3}\rho R^2 \tag{22}$$

where G_4 is the $3+1$ dimensional Newton's constant. Thus, provided m_2 is not too large, we recover a standard Robertson-Walker cosmology with an effective $4d$ Newton's constant, $G_4 = 2G/b\ell$. What has happened for this bubble is that above the AdS lengths we expect that the bulk space produces an effectively $4d$ theory of gravity [13]. Since we have assumed that $\ell_1, \ell_2 \ll b\ell$, when we probe distances below ℓ_1, ℓ_2 we do not observe the extra dimensions of the bulk space since we are in the regime in which the effect of the brane curvature term dominates. This argument is borne out in next section where it is shown that the effective theory of gravity is governed by a $4d$ Einstein equation at all scales when $\ell_1, \ell_2 \ll b\ell$.

V GRAVITY ON THE SURFACE.

Thus far, we have studied the evolution of the bubble in the bulk space-time to determine the resulting cosmology witnessed by an observer on its surface. However, in addition to admitting an evolution that mimics the standard Robertson-Walker cosmology, the form of gravity in the weak-field limit should approach that of an effectively $3+1$-dimensional theory. For the standard Randall-Sundrum scenario in which the AdS lengths are equal, $\ell_1 = \ell_2 = \ell$, the weak-field behavior of the Newtonian potential about a stationary mass M confined to the brane receives a correction [3]

$$U(\vec{x}) = -\frac{G_4 M}{|\vec{x}|}\left(1 + \frac{\ell^2}{|\vec{x}|^2} + \cdots\right) \tag{23}$$

which becomes negligible at distances $|\vec{x}|$ along the brane that are large compared to ℓ. In the case of the vacuum bubble, in which at least one of the AdS lengths is larger than that millimeter limit to which classical gravity has been tested, the presence of the scalar curvature term provides a source for a four dimensional theory of gravity. Without such a term, the fact that the bubble borders either a flat region or a region with a large AdS length, as in the two cases considered in the previous section, the effective Newtonian potential is either $1/|\vec{x}|^2$ or receives a large $1/|\vec{x}|^2$ correction respectively, revealing the presence of the $5d$ space-time.

In this section, we shall examine the behavior of gravity on the brane in the weak-field limit. For simplicity, we shall restrict to the case of a flat $k=0$ (9) brane whose position is fixed at $r=0$. The bulk space will be purely AdS_5 ($m_1 = m_2 = 0$) although as before, the cosmological constants in the regions to either side of the brane to not need to be equal.

One of the difficulties in studying linearized gravity about a brane is that the presence of matter on the brane generically distorts the apparent position of the brane in the bulk coordinates [15,16]. Since it is important to express precisely the boundary conditions at the brane, it is advantageous to work in a gauge in which this 'brane-bending' does not occur. The problem of finding such a gauge in order to study linearized gravity was addressed in [17]. There, the metric tensor is written using the time-slicing formalism [18] as

$$g_{ab} = \begin{pmatrix} \hat{g}_{\mu\nu} & n_\mu \\ n_\nu & n^2 + n_\lambda n^\lambda \end{pmatrix} \qquad\qquad g^{ab} = \frac{1}{n^2} \begin{pmatrix} \hat{g}^{\mu\nu} & -\hat{g}^{\mu\lambda} n_\lambda \\ -\hat{g}^{\nu\lambda} n_\lambda & 1 \end{pmatrix} \tag{24}$$

where

$$\hat{g}_{\mu\nu} = f(r)(\eta_{\mu\nu} + \gamma_{\mu\nu}). \tag{25}$$

n and n_μ are respectively called the lapse function and the shift vector. Here we have written a more general warp factor, $f(r)$, to allow configurations in which the brane separates two regions of AdS_5 with different cosmological constants. In the limit in which the perturbations about the background are small, we consider $\gamma_{\mu\nu}$, n_μ and $\phi \equiv n^2 - 1$ all to be small quantities of roughly the same size.

The advantage of this metric is that it admits a foliation of the bulk space by hypersurfaces where r is constant. The normal vector to any of these hypersurfaces can be written as

$$N_a = (0,0,0,0,n) \qquad N^a = \frac{1}{n}(-\hat{g}^{\mu\nu} n_\nu, 1) \tag{26}$$

and the induced metric along the surface is given by

$$h_{ab} = g_{ab} - N_a N_b = \begin{pmatrix} \hat{g}_{\mu\nu} & n_\mu \\ n_\nu & n_\lambda n^\lambda \end{pmatrix} \qquad h^{ab} = \frac{1}{n^2} \begin{pmatrix} \hat{g}^{\mu\nu} - \hat{g}^{\mu\lambda}\hat{g}^{\nu\rho} n_\lambda n_\rho & 0 \\ 0 & 0 \end{pmatrix} \tag{27}$$

To leading order in the perturbations, the extrinsic curvature is then

$$K_{\mu\nu} = \frac{1}{2n}\left[\partial_r \hat{g}_{\mu\nu} - \partial_\mu n_\nu - \partial_\nu n_\mu\right] \qquad K_{\mu r} = \frac{1}{2}\frac{f'}{f} n_\mu \qquad K_{rr} = 0. \tag{28}$$

The prime denotes differentiation with respect to r.

The behavior of gravity in the bulk is governed by the usual Einstein equation (5), which we write in the Gauss-Codacci form by decomposing it into the transverse and orthogonal components with respect to an arbitrary $r =$ constant hypersurface,

$$\hat{R} + K^\mu_\nu K^\nu_\mu - K^2 = \frac{12}{\ell^2_{1,2}} \qquad \partial_\mu K - \hat{\nabla}_\nu K^\nu_\mu = 0 \qquad R_{\mu\nu} = \frac{4}{\ell^2_{1,2}} \hat{g}_{\mu\nu}. \tag{29}$$

Here $\hat{R} = \partial^\mu\partial^\nu\gamma_{\mu\nu} - \Box\gamma + \cdots$ is the scalar curvature associated with $\hat{g}_{\mu\nu}$. Away from the brane, the zeroth order terms from (29) determine the warp factor which appears in the unperturbed metric,

$$\frac{f'^2}{f^2} = -\frac{4}{\ell_{1,2}^2} \qquad \frac{f''}{f} = -\frac{4}{\ell_{1,2}^2} \qquad (r \neq 0). \tag{30}$$

The terms linear in the perturbations for the first two Gauss-Codacci equations (29) impose constraints on the metric:

$$\frac{1}{f}\left[\partial^\mu\partial^\nu\gamma_{\mu\nu} - \Box\gamma + 3\frac{f'}{f}\partial^\lambda n_\lambda\right] = -3\phi\frac{f'^2}{f^2} + \frac{3}{2}\frac{f'}{f}\partial_r\gamma \tag{31}$$
$$\frac{1}{f}\left[\Box n_\mu - \partial_\mu\partial^\nu n_\nu\right] = \frac{3}{2}\frac{f'}{f}\partial_\mu\phi + \partial_r\left[\partial^\nu\gamma_{\mu\nu} - \partial_\mu\gamma\right].$$

The important feature of these expressions is the appearance of the f^{-1} factors on their left sides. For regions of anti-de Sitter space, f^{-1} typically grows exponentially as we move farther into the bulk so that at some distance away from the brane, the assumption that $\gamma_{\mu\nu}$ and ϕ are small breaks down, unless we use gauge freedom to set the right sides of both expressions to zero. Therefore, in the bulk we choose, similarly to [17],

$$\phi = \frac{1}{2}\frac{f}{f'}\partial_\mu\gamma \qquad \partial^\mu\tilde{\gamma}_{\mu\nu} = 0 \tag{32}$$

where $\tilde{\gamma}_{\mu\nu}$ denotes the traceless part of $\gamma_{\mu\nu}$. For these equations, (31) constrains the shift vector to be of the following form:

$$n_\mu = \frac{1}{4}\frac{f}{f'}\partial_\mu\gamma + A_\mu. \tag{33}$$

Here A_μ is a free vector field ($\Box A_\mu = 0$) with vanishing 4-divergence ($\partial^\mu A_\mu = 0$). We shall henceforth use the gauge freedom to choose $A_\mu = 0$ [2].

For these choices of the gauge then, the behavior of $\tilde{\gamma}_{\mu\nu}$ in the bulk is determined by the transverse components of the Einstein equation (29),

$$\partial_r(f\partial_r\tilde{\gamma}_{\mu\nu}) + f'\partial_r\tilde{\gamma}_{\mu\nu} + \Box\tilde{\gamma}_{\mu\nu} = 0 \tag{34}$$

where we have substituted the equations determining the warp factor (30).

At the brane, the behavior of quantities intrinsic to the brane are related to the change in the extrinsic curvature across the brane by the Israel condition (6). For simplicity, we set $\sigma = \sigma_c$. The unperturbed piece of the Israel condition requires the usual fine-tuning of the brane tension,

$$f'|_{r=0-} - f'|_{r=0+} \equiv \Delta[f'] = \frac{4}{\ell}f. \tag{35}$$

The energy-momentum $t_{\mu\nu}$ of the fields on the brane determines the behavior of $\gamma_{\mu\nu}$ so it should also be treated as a small quantity. The transverse components of the Israel condition specify the boundary conditions at the brane,

$$f\Delta\left[\partial_r\tilde{\gamma}_{\mu\nu}\right]-\frac{1}{2}\Delta\left[\frac{f}{f'}\right]\partial_\mu\partial_\nu\gamma=16\pi\left[t_{\mu\nu}-\frac{1}{3}\eta_{\mu\nu}t\right]+\frac{1}{2}b\ell\left[\Box\tilde{\gamma}_{\mu\nu}+\frac{1}{2}\partial_\mu\partial_\nu\gamma\right]. \quad (36)$$

where we have imposed (35).

As in [17], we can eliminate the γ from (36) by taking the trace of this expression and solving for γ by inverting the operator $\Box$ to write formally

$$f\partial_r\tilde{\gamma}_{\mu\nu}|_{r=0^+}-f\partial_r\tilde{\gamma}_{\mu\nu}|_{r=0^-}=\frac{1}{2}b\ell\,\Box\tilde{\gamma}_{\mu\nu}-16\pi\left[t_{\mu\nu}-\frac{1}{3}\left(\eta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)t\right]. \quad (37)$$

For a universe with two regions of AdS_5 separated by a 3-brane at $r=0$, the value of the Fourier transform of the trace γ at the brane is given by

$$\gamma=-\frac{32\pi}{3p^2}\frac{2}{\ell_1+\ell_2+b\ell}\,t. \quad (38)$$

The solution to the bulk equation (34) that satisfies the new boundary condition (37) has the form

$$\tilde{\gamma}_{\mu\nu}(p,r)=\frac{16\pi}{|p|}\left[t_{\mu\nu}-\frac{1}{3}\left[\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right]t\right] \quad (39)$$
$$\times\frac{e^{2r/\ell_1}K_2(\ell_2|p|)K_2(\ell_1|p|e^{r/\ell_1})}{K_1(\ell_1|p|)K_2(\ell_2|p|)+K_1(\ell_2|p|)K_2(\ell_1|p|)+\frac{1}{2}b\ell|p|\,K_2(\ell_1|p|)K_2(\ell_2|p|)}$$

for $r>0$. The solution for the $r<0$ region is found by substituting $r\to -r$ and $\ell_1\leftrightarrow\ell_2$. $K_1(z)$ and $K_2(z)$ are Bessel functions.

A possible source for a discrepancy between a purely four dimensional theory of gravity and the effective low energy theory of gravity for an inhabitant of the 3-brane is in the appearance of deviations from the Einstein equation. The intrinsic Einstein tensor is given by [17]

$$\hat{R}_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\hat{R}=-\tfrac{1}{2}\Box\tilde{\gamma}_{\mu\nu}-\tfrac{1}{4}\left(\partial_\mu\partial_\nu\gamma-\eta_{\mu\nu}\Box\gamma\right). \quad (40)$$

As a first case, we look in the limit in which both of the lengths associated with the bulk AdS_5 regions are small compared to the typical length scale probed on the brane, $\ell_1|p|,\ell_2|p|\ll 1$. Using $K_2(z)\approx 2z^{-2}$ and $K_1(z)\approx z^{-1}$ for small z, we have

$$\tilde{\gamma}_{\mu\nu}(p,0)=\frac{16\pi}{p^2}\left[t_{\mu\nu}-\frac{1}{3}\left[\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right]t\right]\frac{2}{\ell_1+\ell_2+b\ell}+\cdots; \quad (41)$$

combining this with the trace (38), we recover the $4d$ Einstein equation:

$$\hat{R}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\hat{R}=\frac{16\pi}{\ell_1+\ell_2+b\ell}t_{\mu\nu}+\cdots. \quad (42)$$

So at sufficiently large distances, we always recover a standard theory of gravity.

We also examine whether a stationary point mass M on the brane produces a $1/|\vec{x}|$ Newtonian potential. For a point source at rest on the brane, the energy-momentum tensor has only one non-vanishing component: $t_{00}(x) = G\,M\,\delta^3(\vec{x})$. When the velocity of a test mass is small, the gravitational field is stationary and weakly perturbed away from the background metric, we can extract the Newtonian potential by following the motion of a test mass along a geodesic. The leading piece of the geodesic equation in this Newtonian limit is $\frac{d^2x^i}{dt^2} = \hat{\Gamma}^i_{00} = -\frac{1}{2}\partial_i\gamma_{00}$ from which we can extract the ordinary Newtonian potential, $\frac{d^2}{dt^2}x^i = \partial_i U$, which we have written as $U(\vec{x})$ to avoid confusion with $V(R)$ defined earlier (15).

The presence of the $4d$ curvature term in the action allows us to consider other corners of the ℓ_1, ℓ_2 parameter space as well. If we rewrite the expression for $\tilde{\gamma}_{\mu\nu}$ on the brane as

$$\tilde{\gamma}_{\mu\nu}(p,0) = \frac{16\pi}{|p|}\left[t_{\mu\nu} - \frac{1}{3}\left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right]t\right]\left[\frac{1}{2}b\ell|p| + \frac{K_1(\ell_1|p|)}{K_2(\ell_1|p|)} + \frac{K_1(\ell_2|p|)}{K_2(\ell_2|p|)}\right]^{-1} \quad (43)$$

we find that since $0 \le K_1(z)/K_2(z) < 1$ for $z \ge 0$, all that is needed to recover a $1/|\vec{x}|$ Newtonian potential is to have $b\ell|p| \gg 1$. In this limit, both $\tilde{\gamma}_{\mu\nu}$ and γ have a leading $1/p^2$ behavior which automatically leads to a $1/|\vec{x}|$ potential:

$$U(|\vec{x}|) = -\frac{GM}{|\vec{x}|}\frac{2}{3b\ell}\left[\frac{4(\ell_1+\ell_2)+3b\ell}{\ell_1+\ell_2+b\ell}\right] + \cdots. \quad (44)$$

This equation is valid regardless of the scale of ℓ_1 and ℓ_2, compared with either the coefficient of the $4d$ curvature, $b\ell$, or $|\vec{x}|$. However, when evaluating the Einstein equation, we find that for arbitrary values of ℓ_1 and ℓ_2 that a term involving the trace of the energy-momentum tensor for the brane fields appears:

$$\hat{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\hat{R} = \frac{16\pi}{b\ell}t_{\mu\nu} - \frac{16\pi}{3}\frac{1}{b\ell}\left[\frac{\ell_1+\ell_2}{b\ell+\ell_1+\ell_2}\right]\left[\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right]t + \cdots. \quad (45)$$

If $b\ell \gg \ell_1, \ell_2$, then the extra term can be neglected and the leading behavior is that of a standard theory of four dimensional gravity, in agreement with (42).

We can also study what happens when one or both of the AdS lengths becomes infinite, while keeping $b\ell|p| \gg 1$. These limits correspond respectively to universes in which the brane is between a flat and an AdS region, as in section IV, or is simply embedded in a flat bulk space-time. In either case, the effective theory is

$$\hat{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\hat{R} = \frac{16\pi}{b\ell}\left[t_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)t\right] + \cdots. \quad (46)$$

The case of a brane embedded in a flat bulk was also investigated in [14] where the effective theory on the brane was shown to contain a scalar graviton. From (46) it would appear that a scalar graviton is a generic feature of branes embedded in (partially) flat space-times.

In the remaining region of the ℓ_1, ℓ_2 parameter space, where $b\ell|p| \le 1$ and $\ell_1|p|, \ell_2|p| \gg 1$, the dominant contribution from (43) leads to a five dimensional Newtonian potential.

VI CONCLUSIONS.

In this talk, we have examined the cosmology seen on a brane embedded in a bulk space-time with different cosmological constants in the two regions separated by the brane. When one of these cosmological constants vanishes, it is possible to obtain a cosmological evolution with the same time dependence as a radiation-dominated universe. When a scalar curvature term is included in the brane action, a $4d$ Newton's law emerges in the effective theory of gravity on the brane; however, the theory does contain a scalar graviton. This brane scalar curvature term also allows us to consider scenarios in which one or both of the cosmological constants is substantially smaller than that considered in the original Randall-Sundrum picture. Such a brane universe admits a standard Robertson-Walker cosmology (21) and has a $4d$ Newton's law (44). The effective Einstein equation in this theory still contains a non-standard correction (45) but which can be made to vanish with a mild tuning of the coefficient of the curvature term in the brane action.

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