Scalar Fields without a Scale

Fields without scales

All of the particles discovered so far, or at least those that are associated with the matter, have a definite scale associated with them. A quark or a lepton—even a neutrino—has its own fixed physical mass. Yet, it is possible, at least in principle, for a field to have no such special scale. For example, if we look at a free, massless fermion, ψ,

$$S = \int d^4x\, i\bar{\psi}\partial\!\!\!/\psi, \tag{1}$$

its action remains entirely unaltered if we simultaneously rescale the coordinates by a number λ and the field by $\lambda^{-3/2}$,

$$x^\mu \to \lambda x^\mu \qquad \psi \to \frac{\psi}{\lambda^{3/2}}. \tag{2}$$

Although no scalar particles have yet been discovered in nature that are not themselves built from bound sets of fermions, a scalar field often useful to study as a simpler illustration of a phenomenon when the spin of the field is not a crucial element. As a second example, the action of a simple, free scalar field φ,

$$S = \int d^4x \left\{ \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \right\}, \tag{3}$$

is similarly unaffected if we rescale the coordinates and the field according to

$$x^\mu \to \lambda x^\mu \qquad \varphi \to \frac{\varphi}{\lambda}. \tag{4}$$

In both of these examples, the scaling dimension of the field is set by its kinetic term.

In these notes, we shall examine fields with other scaling dimensions. Writing such a field as σ to distinguish it from the massive field φ, we shall assume that its action is unchanged when we simultaneously rescale by

$$x^\mu \to \lambda x^\mu \qquad \sigma \to \frac{\sigma}{\lambda^d}. \tag{5}$$

The number d is the *scaling dimension* of the field σ, and for now we shall let it be an arbitrary real number. From what we have already seen, this field cannot have a standard kinetic energy. Since the kinetic term is usually taken as the starting point for deriving the propagator, it might seem that it would be

quite difficult to calculate the scattering properties of this field or its interactions with the particles of the standard model in any reasonably tractable form. Fortunately, however, this fear is not realized. The constraint imposed by the scale invariance of the theory is a very powerful one and alone it is enough to determine how these peculiar fields propagate though space and time. But in order to do so, we must start from a sufficiently general perspective. We shall illustrate this approach first for an ordinary massive scalar field φ, since the basic structure generalizes directly to the scaleless scalar field σ too.

The Lehmann-Källen relation

The usual method for describing how particles propagate is only an approximate one; we first treat a field as though it were perfectly free of the influences of everything else and then hope that these influences can be treated as small corrections that—depending on how much work we are will to do—can be used to describe better and better what is really happening in nature. A second approach starts instead with the symmetries that we have observed in some setting. With a few general assumptions about how the field theory behaves in that setting, we can then derive a correspondingly general form for its propagator. As we have not specified the kinetic term for the case of a general scale-invariant field σ, this second approach is far better suited to it.

Let us begin with by considering an ordinary massive scalar field φ in Minkowski space. We shall allow this field to interact both with itself and with any other fields around, but we shall assume that it is at least lighter than twice the mass of any other particles, just so that it does not spontaneously decay when it is at rest. Moreover, in addition to the invariant vacuum state, which we write as $|0\rangle$, we shall also assume that there exists a complete set of Fock states, which we denote by

$$\left\{ |0\rangle, |X,\vec{k}\rangle \right\}. \tag{6}$$

The X refers to an arbitrary assemblage of fields, both the field φ and any other stuff, which has a definite total momentum $\vec{k}$ and an associated energy

$$\omega_{X,k} = (|\vec{k}|^2 + M_X^2)^{1/2}, \tag{7}$$

M_X being the rest mass of the state X. The completeness of this set means that summing over everything should give the identity operator,

$$|0\rangle\langle 0| + \sum_X \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{X,k}} |X,\vec{k}\rangle\langle X,\vec{k}| = \mathbf{1}. \tag{8}$$

Since we are adding up everything, we must sum—or rather integrate—over all possible momenta for a given set of fields 'X'. This sum has been weighted

with a factor $(2\omega_{X,k})^{-1}$ so that it remains invariant under an arbitrary Poincaré transformation. We shall now use these three basic ingredients—

1. the Poincaré invariance of the background,
2. the existence of a unique vacuum state, and
3. the existence of a complete set of states which are eigenstates of the Poincaré transformations

—to derive a general expression for the scalar propagator,

$$\langle 0|\, T\big(\varphi(x)\varphi(y)\big)\,|0\rangle, \tag{9}$$

that is, the *full* propagator of the interacting theory and not just its *free* part.

We begin by inserting this complete set of states into the propagator between the fields. To neglect the time-ordering, we first choose the case $t > t'$, where $x = (t,\vec{x})$ and $y = (t',\vec{y})$,

$$\begin{aligned}\langle 0|\,\varphi(x)\varphi(y)\,|0\rangle &= \langle 0|\varphi(x)|0\rangle\langle 0|\varphi(y)|0\rangle \\ &\quad + \sum_X \int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega_{X,k}}\langle 0|\varphi(x)|X,\vec{k}\rangle\langle X,\vec{k}|\varphi(y)|0\rangle.\end{aligned}$$

One of the standard renormalization conditions is that the expectation value of a quantum field is zero,

A) $\langle 0|\varphi(x)|0\rangle = 0$,

from which follows,

$$\langle 0|\,\varphi(x)\varphi(y)\,|0\rangle = \sum_X \int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega_{X,k}}\langle 0|\varphi(x)|X,\vec{k}\rangle\langle X,\vec{k}|\varphi(y)|0\rangle. \tag{10}$$

Next, if we denote the 4-momentum *operator* by $\hat{K}^\mu$, then the fact that the field $\varphi(x)$ is a scalar representation of the Poincaré group implies that it transforms under a space-time translation as

$$\varphi(x) = e^{i\hat{K}\cdot x}\varphi(0)e^{-i\hat{K}\cdot x}. \tag{11}$$

The states are eigenvectors of these translation operators; the vacuum is left invariant, $e^{i\hat{K}\cdot x}|0\rangle = |0\rangle$, while the other state just yields the value of its momentum,

$$e^{-i\hat{K}\cdot x}|X,\vec{k}\rangle = e^{-ik\cdot x}|X,\vec{k}\rangle. \tag{12}$$

Assembling these ingredients together, we find that

$$\begin{aligned} \langle 0|\varphi(x)|X,\vec{k}\rangle &= \langle 0|e^{i\hat{K}\cdot x}\varphi(0)e^{-i\hat{K}\cdot x}|X,\vec{k}\rangle \\ &= \langle 0|\varphi(0)|X,\vec{k}\rangle\, e^{-ik\cdot x}\big|_{k_0=\omega_{X,k}} \\ &= \langle 0|\varphi(0)|X,\vec{0}\rangle\, e^{-ik\cdot x}\big|_{k_0=\omega_{X,k}}. \end{aligned} \tag{13}$$

The last step follows since the momentum eigenstates are orthogonal—the only overlap between $\langle 0|\varphi(0)$ and $|X,\vec{k}\rangle$ is when $\vec{k}=\vec{0}$. Thus we have

$$\langle 0|\,\varphi(x)\varphi(y)\,|0\rangle = \sum_X \big|\langle 0|\varphi(0)|X,\vec{0}\rangle\big|^2 \int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega_{X,k}} e^{-ik\cdot(x-y)}\big|_{k_0=\omega_{X,k}} \tag{14}$$

Notice that the quantity $\big|\langle 0|\varphi(0)|X,\vec{0}\rangle\big|^2$ no longer depends on the momentum, only on the rest mass M_X of the state. We can therefore convert the integral into one over the full 4-momentum by applying the Cauchy residue formula,

$$\int dk_0\,\frac{e^{-ik_0(t-t')}}{k^2-M_X^2+i\varepsilon} = -2\pi i\frac{e^{-i\omega_{X,k}(t-t')}}{2\omega_{X,k}}. \tag{15}$$

Since we are examining the case where $t>t'$, it is necessary to close the contour in the lower complex k_0-plane. The contour has a clockwise orientation which explains the origin of the minus sign.

The $t<t'$ case is exactly analogous to this last one, simply encircling the opposite pole of the k_0 integral. Combining both cases yields an expression where the Poincaré invariance of the propagator is made a little more obvious than before,

$$\langle 0|\,T\big(\varphi(x)\varphi(y)\big)\,|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\sum_X \frac{i\big|\langle 0|\varphi(0)|X,\vec{0}\rangle\big|^2}{k^2-M_X^2+i\varepsilon}. \tag{16}$$

As one last step, let us simplify the expression by defining a *spectral density function* $\rho(M^2)$,

$$\rho(M^2) = 2\pi\sum_X \delta\left(M^2-M_X^2\right)\big|\langle 0|\varphi(0)|X,\vec{0}\rangle\big|^2 \tag{17}$$

which results in a very general form for the propagator for the full, interacting theory,

$$\langle 0|\,T\big(\varphi(x)\varphi(y)\big)\,|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\int_0^\infty \frac{dM^2}{2\pi}\,\frac{i\rho(M^2)}{k^2-M^2+i\varepsilon}. \tag{18}$$

This expression is the *Lehmann-Källen spectral representation* of the propagator.

So far, although we have been speaking of φ as though it were a massive scalar field, what we have said applies equally well to the scale invariant field σ too. What differentiates one case from another arises from the physical considerations that we apply to it, which result in a more definite structure for the spectral density function, $\rho(M^2)$. For the massive field, for example, we have two more renormalization conditions,

B) the field has a pole at the *physical mass of the field*, m, and

C) the residue there is i.

These further conditions translate what we mean physically by a massive field into a more precise mathematical language. These conditions require $\rho(M^2)$ to have the form

$$\rho(M^2) = 2\pi Z\,\delta(M^2 - m^2) + \text{bound states} + \text{branch cut}; \tag{19}$$

the branch cut appears once we encounter multi-particle states. Notice that if we wish to satisfy the third renormalization condition (C), we must rescale the field itself to absorb the factor,

$$Z \equiv \left|\langle 0|\varphi(0)|X,\vec{0}\rangle\right|^2. \tag{20}$$

This requirement is the origin of the wavefunction renormalization.

The scale-invariant field requires its own conditions, which in turn fix its spectral density function. Obviously, such a field cannot have any isolated mass poles as in the previous case, since they would ruin the scale-invariance. Instead we shall impose an appropriate scaling condition,

B′) the propagator must scale as λ^{-2d} when we scale the coordinates by $x^\mu \to \lambda x^\mu$.

This transformation has the inverse effect on the momentum k_μ and the spectral mass M, so that

$$\begin{aligned}&\lambda^{-2d}\langle 0|\,T\big(\sigma(x)\sigma(y)\big)\,|0\rangle \\ &\quad = \frac{1}{\lambda^4}\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\frac{\lambda^2}{\lambda^2}\int_0^\infty\frac{dM^2}{2\pi}\,\frac{i\rho(M^2/\lambda^2)}{k^2-M^2+i\varepsilon}.\end{aligned} \tag{21}$$

If we are to satisfy the condition, then the spectral density must scale according to

$$\frac{\lambda^{2d}}{\lambda^4}\rho(M^2/\lambda^2) = \rho(M^2), \tag{22}$$

which in turn requires that

$$\rho(M^2) = A_d\,(M^2)^{d-2}, \tag{23}$$

where A_d is simply a constant.

We need one further condition to fix the normalization of the field σ. One convention is to choose the normalization so that the expectation value of two fields assumes some standard form in position space, which we shall describe a little later. Another convention is to choose A_d so that

C$'$) the phase space of the σ fields has the same form as it would for d massless particles.

If we apply this latter convention here, the value for the constant A_d is

$$A_d = \frac{d-1}{(16\pi^2)^{d-1}}\frac{2\pi}{\Gamma^2(d)}. \tag{24}$$

With the normalization of the field thus set, we obtain a final expression for the propagator of this scale-invariant field,

$$\langle 0|\,T\big(\sigma(x)\sigma(y)\big)\,|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{A_d}{2\pi}\int_0^\infty dM^2\,\frac{i(M^2)^{d-2}}{k^2-M^2+i\varepsilon}. \tag{25}$$

One of the advantages of this form is that the propagator still has the form of a simple rational function, which in a loop calculation makes it a simple matter to introduce Feynman parameters. One the other hand, the expression does contain an additional integral, which may or may not be something of a nuisance, depending on our purpose for it. By integrating over the spectral parameter, M^2, we also encounter our first instance of a restriction on the scaling dimension, d. Provided $1 \le d < 2$, we can perform the spectral integral to obtain,

$$\langle 0|\,T\big(\sigma(x)\sigma(y)\big)\,|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{A_d}{2}\,\frac{1}{\sin\pi d}\,\frac{i}{(-k^2-i\varepsilon)^{2-d}}. \tag{26}$$

In the case where $d \to 1$, the expression inside the integrand becomes

$$\lim_{d\to 1}\frac{A_d}{2}\,\frac{1}{\sin\pi d}\,\frac{i}{(-k^2-i\varepsilon)^{2-d}} = \frac{i}{k^2+i\varepsilon}. \tag{27}$$

upon substituting the detailed form for A_d, which reproduces the appropriate propagator for a massless scalar particle with the standard kinetic term in its action.

Further, if we integrate over the momentum in the expression for the propagator, we arrive at its position-space representation,[1]

$$\langle 0|\, T\big(\sigma(x)\sigma(y)\big)\, |0\rangle = \frac{1}{(2\pi)^{2d}} \frac{1}{|x-y|^{2d}}. \tag{28}$$

Sometimes the normalization of the field is chosen so that the coefficient is $(2\pi)^{-2}$ rather than $(2\pi)^{-2d}$, which requires only a simple redefinition of A_d. However, we shall leave A_d as defined above.

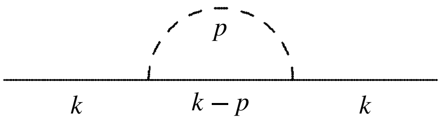
An example of a radiative correction

Although there are many, many possibilities for how these scaleless fields might interact with more conventional particles, we shall look a one particular example. It is chosen to illustrate how the scaling properties of the propagator of these exotic fields can produce radiative corrections that are completely finite when they would have been divergent had we replaced the scaleless field with an ordinary particle field.

Consider a scaleless field σ interacting with a massive field φ, whose mass is m, through the operator,

$$g\Lambda^{2-d}\varphi^2\sigma. \tag{29}$$

Λ is a mass parameter introduced so that the coupling g remains dimensionless. This interaction produces radiative corrections to the φ propagator, the simplest of which is the one-loop graph shown in this picture,



Here the solid line corresponds to the φ field and the dashed line to the σ field.

Evaluating the amplitude according to the usual Feynman rules, using the spectral form for the propagator of the scaleless field, yields,

$$\mathscr{A} = g^2\Lambda^{2(2-d)}\frac{A_d}{2\pi}\int \frac{d^4p}{(2\pi)^4}\int_0^\infty dM^2\, \frac{-i}{(p-k)^2-m^2+i\varepsilon}\frac{(M^2)^{d-2}}{p^2-M^2+i\varepsilon}. \tag{30}$$

As mentioned earlier, it is simpler to introduce a Feynman parameter x first,

$$\begin{aligned}\mathscr{A} &= -ig^2\Lambda^{2(2-d)}\frac{A_d}{2\pi}\int_0^1 dx\int \frac{d^4p}{(2\pi)^4} \\ &\quad \int_0^\infty dM^2\, \frac{(M^2)^{d-2}}{\left[p^2+x(1-x)k^2-xm^2-(1-x)M^2+i\varepsilon\right]^2},\end{aligned} \tag{31}$$

[1] We may have been a little careless in the phase in this equation.

before integrating over the spectral parameter, M^2,

$$\begin{aligned}\mathscr{A} &= g^2\Lambda^{2(2-d)}\frac{A_d}{2\pi}\int_0^1 \frac{dx}{(1-x)^{d-1}} \\ &\quad \int \frac{d^4p_E}{(2\pi)^4}\frac{\Gamma(d-1)\Gamma(3-d)}{\left[p_E^2-x(1-x)k^2+xm^2-i\varepsilon\right]^{3-d}}. \end{aligned} \tag{32}$$

The subscript E indicates that after integrating over M^2 we performed a Wick rotation, $p_0 \to ip_4$, to transform the momentum integral into a Euclidean one. Although the superficial degree of divergence would appear to be p^{d+1}, the fact that the denominator is raised to a nonintegral power produces a finite expression upon integrating over the momentum,

$$\mathscr{A}(k^2) = \frac{g^2}{(16\pi)^d}\Lambda^{2(2-d)}\frac{\Gamma(1-d)}{\Gamma(d)}\int_0^1 dx\left[\frac{xm^2-x(1-x)k^2-i\varepsilon}{1-x}\right]^{d-1}. \tag{33}$$

Unlike the case of a radiative correction from another ordinary particle, there was no need to regularize the integral. On-shell, $k^2 = m^2$, the amplitude simplifies still further, and we obtain

$$\mathscr{A}(m^2) = \frac{g^2m^2}{(16\pi)^d}\left(\frac{\Lambda}{m}\right)^{2(2-d)}\frac{\Gamma(1-d)\Gamma(2-d)\Gamma(2d-1)}{\Gamma(d)\Gamma(d+1)}. \tag{34}$$

Recall that the $d \to 1$ limit reproduces a massless scalar field with the standard kinetic term—the one we encountered in the very beginning of these notes. The loop amplitude here contains a factor of $\Gamma(1-d)$ which diverges in this same limit—in effect, the peculiar scaling of the field has regularized the divergence on its own. This observation resembles the old, rarely used, technique of 'analytic renormalization' developed by Speer in 1967, except that there divergences are regularized by raising a *massive* propagator in its momentum representation to a nonintegral power,[2]

$$\frac{i}{k^2-m^2+i\varepsilon} \to \frac{i}{(k^2-m^2+i\varepsilon)^{2-d}}. \tag{35}$$

Scaleless fields of higher spin

Although we have been examining solely a scalar field throughout, we shall end these notes by briefly mentioning fields with nonzero intrinsic spin. These

[2] Eugene R. Speer, "Analytic Renormalization," Journal of Mathematical Physics **9**, 1404 (1968).

fields can also be constructed so that their actions remain invariant when we rescale the coordinates by $x^\mu \to \lambda x^\mu$ and the field by λ^{-d}. However, just as in the case of a scalar field, the scaling dimension d is not entirely arbitrary, but it subject to bounds if the theory is to remain unitary. The structure of the propagator also contains some nontrivial dependence on d as well.

For example, a *gauge-invariant*, scale-invariant vector field Σ_μ has the following propagator,

$$\begin{aligned}\langle 0|\, T\big(\Sigma_\mu(x)\Sigma_\nu(y)\big)\,|0\rangle \;&=\; \int \frac{d^4}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\frac{A_d^v}{2}\frac{1}{\sin \pi d} \\ &\qquad \frac{-i}{(-k^2-i\varepsilon)^{2-d}}\left[g_{\mu\nu} - \frac{2(d-2)}{d-1}\frac{k_\mu k_\nu}{k^2}\right],\end{aligned}$$

up to a normalization A_d^v which we have not specified, though it can be fixed by an appropriate "renormalization condition" just as for the scalar field. Here, $g_{\mu\nu} = \mathrm{diag}[1,-1,-1,-1]$ is the Minkowski metric.

In the case of a scaleless scalar field, unitarity requires that $d \geq 1$, but for a gauge-invariant vector field the condition is a little more stringent, being $d \geq 3$. Such details and their derivations can be in an article by Grinstein, Intriligator and Rothstein that begins on page 367 of *Physics Letters B* **662** (2008).