Free-body diagrams

Newton's Laws

For this lecture, I shall assume that you have already been introduced to forces and to Newton's laws of motion, so I shall begin by summarizing them here,

Newton's First Law

The momentum of a mass that is not subject to any forces acting on it remains constant.

Newton's Second Law

$$\vec{F} = m\vec{a} \tag{1}$$

Newton's Third Law

An object that exerts a force on another will experience a force from that other object that is equal in magnitude but opposite in its direction from that which it is exerting.

$$\vec{F}_{1\text{ acting on }2} = -\vec{F}_{2\text{ acting on }1} \tag{2}$$

Although these are called 'laws', they are not absolute rules governing the natural world, but are rather only approximate statements about it. They usually work extremely well for describing most of what occurs in one's daily experience, as long as one does not ask for too much precision. For example, in the second law, we have treated the mass as though it is a fixed quantity, but that is really just an idealization. The mass of an ordinary brick is not a perfectly defined quantity—a bit of it is worn off as it slides over a rough surface, or it might scrape off and attach some of that surface to itself. And at a subtler level, at every moment the atoms of the brick a sublimating back and forth into the air around it. But if we do not need to be overly finicky about the mass of the brick, it is usually a useful approximation to treat it as though it has a constant, well defined mass.

It is equally important to realize that these laws are empirical observations about what happens in the natural world, instead of being mathematical axioms or definitions. We do not *define* a force to be something that accelerates masses—a force is supposed to have an independent existence in nature. What these laws tell us is how each of these independent things—forces, masses and accelerations—influence each other. Since physics relies so much on mathematics, it is very tempting to think of these laws as the starting axioms of a mathematical structure. In fact, it is possible to do exactly that and thereby to create a beautiful branch of mathematics. However, that mathematics is not the same as the physical world, even though at times that mathematics might provide a useful way for organizing a description of what happens there. But the real physical world is messier than what follows just from Newton's laws; and in some more extreme circumstances—for example at tiny molecular distances and smaller—it is described not at all by classical mechanics.

So while it is essential to know how to apply Newton's laws, as well as other laws that we shall encounter later, it is always a good idea to remember that physics is only a description of the natural world, and one which works in various degrees of idealizations of it. There is nothing wrong with this approach, and it has over history led to a tremendous amount of genuine and profound understanding about the world; without these idealizations—constant masses, simple force laws—it would have been extremely difficult to understand the messy, complicated natural world all at once. After we have understood how one idealized version of nature works, we are then free to add new details and complications to the picture.

Forces

Essentially all of the forces that occur in mechanics are the result of just two basic forces, *gravity* and *electromagnetism.* Gravity always appears in a completely undisguised form; that is, when we are speaking of gravity, we are not secretly talking about some other more fundamental force that produces something else that we call gravity. Over the centuries the theory of gravity has grown more subtle, but it remains a basic force of nature, felt by all matter, and so far it still appears to be unconnected with any other force.

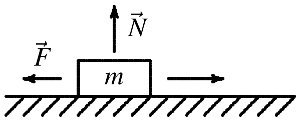
Near the surface of the earth, gravity produces a very familiar force, the *weight* of an object. This force is always directed downward, or more correctly toward the center of the earth, and it produces a uniform acceleration on object that we call g. Empirically, $g = 9.81$ ms^{-2}, and if we call $\vec{g}$ the vector pointing downward with this length, the weight of a mass m is

$$\vec{W} = m\vec{g}. \tag{3}$$

Of course, this too is just an approximate statement since g really depends on the distance of the mass from the earth. A person on the top of Mount Everest weighs slightly less than at the bottom of the mountain. Later, we shall learn why this happens, but for a beginning, we shall keep closer to ordinary experience and not worry about changes in $\vec{g}$.

There is nothing in this respect that is special about the earth and we could equally define a similar weight for any other celestial body. For example, on the moon, $g_{\text{moon}} = 1.63\,\text{ms}^{-2}$, which is about one-sixth that of the earth, so your weight on the moon would be about one-sixth that on the earth.

Aside from gravity, it would seem that there are many, many other forces—the push of the ground beneath our feet, the friction of pulling a heavy box across the floor, the drag of a bit of chalk across a chalkboard, the lift provided by the air beneath a flying bird, the tug of bit of rope. All of these many different mechanical forces are just electromagnetism in disguise. The electromagnetic forces within a bit of rope hold it together and transmit a pull from one end to the other; the electromagnetic repulsions of air molecules impinging on the broad feathers of a bird hold it aloft. But the details of the electromagnetic interactions that make up these effects are so complicated and their net result is usually so simple that it makes sense to speak of them as separate empirical forces. Electromagnetism also appears in a direct form too, just like gravity, as in the force between two charged bodies; but that is a subject for a different course.



Let us consider one of these empirical forces, a very familiar one, *friction*. In its origin, it is a very complicated force. The atoms in the sliding body knock against those of the surface, some attract and cling momentarily together. This process produces a force that opposes the motion. Some of this jostling of the atoms is transferred into the general motion of the atoms in the body and the surface, which is thermal energy or heat. That is why friction heats things up. But to a good approximation, all of these complex interactions can be summarized in a simple empirical force that does not even mention electricity or magnetism. The force of friction $\vec{F}_f$ is proportional in its magnitude to the normal force $\vec{N}$ pushing up from the surface on the sliding body,

$$||\vec{F}_f|| = \mu||\vec{N}||. \tag{4}$$

The constant of proportionality μ depends on exactly what is sliding across what, and it can sometimes be different depending on whether the object is at rest or is already in motion. Note that the vector $\vec{N}$ points orthogonally to the surface and acts on the sliding body; $\vec{F}_f$ points in the direction opposite to the motion of the mass.

Now that we have introduced a few forces and how they affect the motion of bodies, we shall next look at how to apply Newton's laws systematically before turning to a few specific examples.

Free-body diagrams

A *free-body diagram* is a prescription for solving certain classes of problems where forces are acting on a set of interacting masses. It is not itself a set of physical laws, but rather a set of steps for applying Newton's laws in a routine form.

The prescription

or, "How to solve a problem using a free-body diagram"

1. Take the physical system and separate it into its individual massive pieces, m_n, $n = 1, \ldots, N$ (where N is the total number of masses).

2. Identify all the forces acting on each of these masses. Some of these forces might be acting from the outside of the system, such as gravity, while others will be forces where one mass acts on another. In this second case, remember to apply Newton's third law so that the force of one is acting with an equal magnitude, but the opposite direction, on the other. Do not include any fictitious forces that result from using a non-inertial frame.

3. Apply Newton's second law to each of the masses,

$$\sum_j \vec{F}_n^{(j)} = m_n \vec{a}_n. \tag{5}$$

   Here, $\vec{a}_n$ is the acceleration of the nth mass, and $\vec{F}_n^{(j)}$ is one of the forces acting on it.

4. Choose a suitable coordinate system and then write each of the equations in its components,

$$\sum_j \vec{F}_n^{(j)} = m_n \vec{a}_n \qquad \rightarrow \qquad \begin{array}{c} \sum_j F_{nx}^{(j)} = m_n a_{nx} \\ \sum_j F_{ny}^{(j)} = m_n a_{ny} \\ \vdots \quad \vdots \quad \vdots \end{array} \tag{6}$$

   Remember to include any minus signs correctly.

5. Solve all of the equations. Any unbalanced forces in any of the directions will lead to an acceleration in that direction.

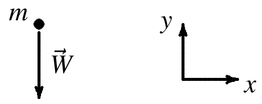
Thus far, this introduction to free-body diagrams has been kept fairly general, so for the rest of the lecture let us illustrate the usefulness of this method through a few examples.

Example. A freely falling mass

Draw the free-body diagram for a mass m that is falling freely (that is, neglect the air resistance).

Solution.

Although this is essentially an example moving in just one dimension, we shall choose the coordinates so that the x coordinate points in the horizontal direction and the y coordinate points in the vertical direction. In this case the only force acting on the mass is gravity, that is, its own weight $\vec{W}$. The free-body diagram is then



In the coordinates that we have chosen, the force is $\vec{W} = -mg\hat{y}$ (remember to include the minus sign). Newton's second law then implies,

$$\vec{F} = m\vec{a} \qquad \Rightarrow \qquad -mg\hat{y} = m\vec{a} \qquad \Rightarrow \qquad \vec{a} = -g\hat{y}. \tag{7}$$

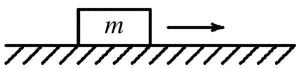
So the acceleration is downward (the minus sign) at g, just as expected.

Example. A brick sliding across the floor

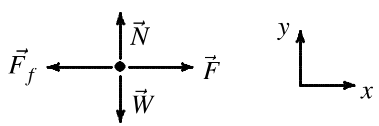
A brick is sliding across a rough floor with a coefficient of friction μ. Calculate the force needed to pull the brick horizontally at a constant speed if its mass is m.

Solution.

step 1. Let us draw a picture of the system to help,



step 2. Next, we identify all of the forces acting on the brick. They are gravity (its weight), $\vec{W}$; the normal force pushing up from the floor, $\vec{N}$; the force of friction, $\vec{F}_f$; and the force with which we pull the brick, $\vec{F}$. Putting each of these into the free-body diagram produces the following picture,



step 3. Since the brick is meant to be sliding at a constant speed, its acceleration vanishes, $\vec{a} = 0$. In this case, all of the forces must be in balance,

$$\vec{W} + \vec{N} + \vec{F}_f + \vec{F} = 0. \tag{8}$$

steps 4 and 5. According to the coordinate system chosen in the free-body diagram, the forces in their components become

$$\vec{W} = -mg\,\hat{y}, \qquad \vec{N} = N\,\hat{y}, \qquad \vec{F}_f = -\mu N\,\hat{x}, \qquad \vec{F} = F\,\hat{x}. \tag{9}$$

Thus the coordinates of Newton's law imply two equations. The y component determines the size of the normal force,

$$N - mg = 0 \qquad \Rightarrow \qquad N = mg, \tag{10}$$

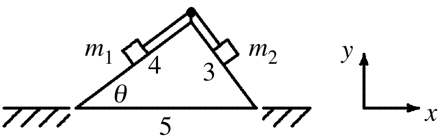
while the x component then determines F, the size of the force we must exert,

$$F - \mu N = 0 \qquad \Rightarrow \qquad F = \mu N = \mu mg. \tag{11}$$

Example. Two blocks on a triangle

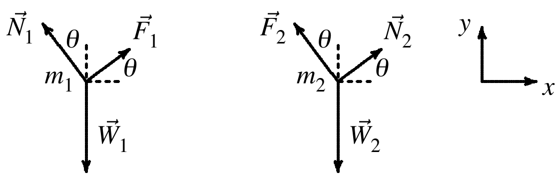
Two blocks, connected by a string, are placed on two upward facing sides of a triangle, which are respectively 4 and 3 units long, while the downward facing side is 5 units long. What should be the ratio of the masses of the blocks so that they stay at rest (neglecting any friction)?

Solution. First, sketch a picture of the system,



step 1. This time there are two masses; call the left one m_1 and the right one m_2.

step 2. Each block experiences three forces: its weight $(\vec{W}_1, \vec{W}_2)$, a normal force pushing up from the triangular block, $(\vec{N}_1, \vec{F}_2)$, the tension or pull from the string, $(\vec{F}_1, \vec{F}_2)$. Having identified all of the forces, we can draw the free-body diagram for each mass,



step 3. The blocks are to be at rest; so for each one, all of the forces on it must add to zero,

$$\vec{W}_1 + \vec{N}_1 + \vec{F}_1 = 0 \qquad \text{and} \qquad \vec{W}_2 + \vec{N}_2 + \vec{F}_2 = 0. \tag{12}$$

step 4. Choose the coordinate system to be aligned with the ground (note that the calculation would be simpler in a frame tilted by θ). The forces then become

$$\text{Block 1:} \qquad \begin{aligned} \vec{W}_1 &= -m_1 g\,\hat{y} \\ \vec{N}_1 &= -N_1 \sin\theta\,\hat{x} + N_1 \cos\theta\,\hat{y} \\ \vec{F}_1 &= F\cos\theta\,\hat{x} + F\sin\theta\,\hat{y} \end{aligned}$$

and

$$\text{Block 2:} \qquad \begin{aligned} \vec{W}_2 &= -m_2 g\,\hat{y} \\ \vec{N}_2 &= N_2 \cos\theta\,\hat{x} + N_2 \sin\theta\,\hat{y} \\ \vec{F}_2 &= -F\sin\theta\,\hat{x} + F\cos\theta\,\hat{y}. \end{aligned}$$

Notice that the magnitude of the force transmitted by the string must be the same at both ends, $||\vec{F}_1|| = ||\vec{F}_2|| \equiv F$.

step 5. Solve the x and y components of Newton's second law.

$$\text{Block 1:} \qquad \begin{aligned} F\cos\theta - N_1 \sin\theta &= 0 \\ F\sin\theta + N_1\cos\theta - m_1 g &= 0 \end{aligned}$$

and

$$\text{Block 2:} \qquad \begin{aligned} -F\sin\theta + N_2 \cos\theta &= 0 \\ F\cos\theta + N_2\sin\theta - m_2 g &= 0. \end{aligned}$$

Solve first for the normal forces,

$$N_1 = F\cot\theta \qquad \text{and} \qquad N_2 = F\tan\theta, \tag{13}$$

then substitute these results into the remaining equations,

$$\begin{aligned} F\sin\theta + F\cot\theta\cos\theta = \frac{F}{\sin\theta}\left[\sin^2\theta + \cos^2\theta\right] = \frac{F}{\sin\theta} &= m_1 g \\ F\cos\theta + F\tan\theta\sin\theta = \frac{F}{\cos\theta}\left[\cos^2\theta + \sin^2\theta\right] = \frac{F}{\cos\theta} &= m_2 g, \end{aligned}$$

so that

$$m_1 g \sin\theta = F = m_2 g \cos\theta, \tag{14}$$

or

$$\frac{m_2}{m_1} = \tan\theta = \frac{3}{4}. \tag{15}$$