Enhancement of inflaton loops in an α-vacuum

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While inflaton loops in the Euclidean vacuum generally have a negligible contribution to the power spectrum, loop effects can be substantially larger when the inflaton is in a nonthermal vacuum state. As an example, we show that in a truncated α-vacuum these loop effects are enhanced by the ratio of the Planck scale to the Hubble scale during inflation. The details of the inflationary models determine whether the coupling constants suppress the loop corrections relative to the tree-level result.

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I. INTRODUCTION

The extremely rapid expansion that occurs during inflation naturally connects quantum fluctuations on the smallest scales to large astrophysical distances today. If inflation lasted for a sufficiently long time, the scales associated with cosmological structure would have been generated when they were at the Planck length or smaller. Observations of the cosmic microwave background (CMB) radiation, such as the recent results of the Wilkinson Microwave Anisotropy Probe [1], are reaching a level of precision which are able not only to confirm the generic predictions of inflation, but also should be capable of distinguishing various specific models for generating inflation. Physics at energies above the Hubble scale during inflation will also be generically imprinted on the CMB and could, if the Hubble parameter is sufficiently large, be observable to future experiments such as Planck [2].

Within a general inflationary model, the variations in the temperature of the CMB radiation we observe today had their origin in the density fluctuations of a scalar inflaton field about its vacuum state. The enormous stretching of the fluctuations during inflation provides a unique possibility to study scales which are not accessible to accelerator experiments by precisely measuring the power spectrum of the variations in the CMB. But to understand what the detailed shape of the power spectrum reveals about short-distance physics, we must first consider which effects might produce deviations from the standard prediction as well as their expected magnitude. One obvious source for these deviations lies in the shape of the inflationary potential itself; variations in the slope of the potential will produce features in the CMB at the scales corresponding to when they occurred during inflation. Another source, and the one which we shall consider in this article, is that some or all of the deviations in the power spectrum arose from features in the vacuum state of the inflaton.

The standard calculation of the power spectrum of density fluctuations assumes that the Universe is in a particular state that matches with the flat space vacuum at short distances. In the case of a de Sitter background, this state is the Euclidean or Bunch-Davies vacuum [3]. Even when in this vacuum state, the existence of new physics with a mass scale $\Lambda_{\rm phys}$ above the Hubble scale H generically leaves an imprint on the CMB, suppressed by a factor $H^2/\Lambda_{\rm phys}^2$ [4]. When the Universe is in another state during inflation, the effects of this short-distance physics on the shape of the CMB power spectrum can be substantially larger, scaling as $H/\Lambda_{\rm phys}$ [5–12]. If future detectors are sufficiently sensitive to observe these larger effects, we should be prepared with an understanding of their possible origins whether they are dynamical [5] or transplanckian [6–12].

Both of these predicted scalings only included the tree-level contributions. In the Euclidean vacuum, loop effects are strongly suppressed by the slow-roll conditions but for nonthermal states, the loop corrections have not been analyzed before and in some instances their effect can be large even compared with the tree-level result.

The motivation for studying a state other than the standard Euclidean vacuum is to determine the sensitivity of the CMB to nontrivial dynamics above H, including possible transplanckian physics. The details of the physics near the Planck scale becomes important for inflation when we recall that a length scale of cosmological relevance today was exponentially smaller during inflation, being further blueshifted the further we look back. If inflation lasted sufficiently long, more than the necessary 60 e-folds needed to solve the flatness and homogeneity problems, scales that we observe today were Planck size at some point during inflation. Although it has been argued that physics near the Planck scale should naturally select the Euclidean vacuum [13], other authors have showed that it is possible to introduce nonthermal features into the vacuum [14,15], for example, by applying a stringy uncertainty relation [7,8] or by some nontrivial dynamics of a heavy field below the Planck scale [5]. Given the unknown properties of the transplanckian regime, we can use a nonthermal vacuum to estimate

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whether nonstandard features near the Planck scale could appear in the CMB.

In this article we shall use a truncated α-vacuum, as an example of a nonthermal state, to evaluate the loop corrections to the CMB power spectrum. This state is based upon the one parameter family of SO(1,4) invariant states that exist in a de Sitter background [16–18]. The Euclidean vacuum corresponds to a special case among these states. Since they are invariant under the de Sitter symmetries, nonthermal features are not redshifted away during inflation. From the perspective of the Euclidean vacuum these other α-vacua are populated up to arbitrarily high wave numbers. This peculiar short-distance behavior leads to pathologies[1] in an interacting theory based on a naïve extension of the Feynman propagator which appear alternately as nonrenormalizable ultraviolet divergences [22] or pinched singularities [23]. However, Danielsson [6] has emphasized that to be physically relevant for the transplanckian problem in inflation, the modes in the α-vacuum should not be defined to arbitrarily high energies, but only up to a cutoff of the order of the Planck scale, $\Lambda_{\rm phys} \sim M_{\rm pl}$. Imposing a cutoff on the theory removes the divergences since the loop momenta, which led to the nonrenormalizable terms, no longer can be arbitrarily large. To maintain a steady truncated α-state requires that above the cutoff, the theory onto which we are matching steadily replenishes the modes as they redshifted. Whether this behavior is natural requires knowing the high energy completion of the theory. Our goal here is to determine under what conditions such potentially nonstandard features would be visible in the CMB, using this truncated version of the α-vacuum as a test case.

The short-distance behavior that produced divergences in the true α-vacuum still produces a general enhancement in the truncated α-states which is proportional to the cutoff scale, $\Lambda_{\rm phys}$, where we match onto the high energy completion to the theory. In particular, we shall show that the loop corrections for a theory with a cubic interaction with a coupling of λ are only suppressed by the dimensionless factor,

$$e^{\alpha+\alpha^*}\frac{\lambda^2}{H^2}\frac{\Lambda_{\rm phys}}{H}, \tag{1.1}$$

relative to the tree-level result. The $\Lambda_{\rm phys}/H$ factor depends only on the structure of the loop corrections in the α-vacuum while the initial $(\lambda/H)^2$ factor depends on the details of the inflationary model. For example, in chaotic inflation, the small size of the coupling is sufficient to overcome the loop enhancement; however, in hybrid models the couplings can be significantly larger so that the total effect of the loop corrections can be comparable to the tree-level result.

In the next section we review the invariant vacua of de Sitter space and introduce the truncated α-vacuum we shall use. Our calculation of the one-loop corrections to the power spectrum in the truncated α-vacuum appears in Sec. III. The perturbation theory is developed using the Schwinger-Keldysh method for studying the finite time-evolution of a quantum field theory from an initial state. Section IV describes constraints on the size of the coupling of the inflaton and we present our conclusions in Sec. V.

II. A TRUNCATED α-VACUUM

The theory of a free scalar field in a de Sitter spacetime has a one parameter family of vacua labeled by the complex parameter α. Each of these vacua is invariant under the symmetries of de Sitter space and this property is most easily demonstrated by showing that the Wightman two-point function in an α-vacuum depends only on the de Sitter invariant distance between the points [16–18,24]. With respect to the space-time symmetries then, any of these α-vacua provides an acceptable choice for the vacuum state. In this section we shall first review the properties of the true α-vacuum before introducing the phenomenologically more realistic truncated α-vacuum. Our notation follows [22] where perturbation theory in an α-vacuum is developed more fully.

At distances shorter than the inherent curvature length associated with de Sitter space, the space-time appears approximately flat. In this limit, it should be possible to apply the same prescription for defining positive and negative frequency modes as in Minkowski space. The unique vacuum which matches with the Minkowski vacuum in this limit is the Euclidean, or Bunch-Davies vacuum [3]. In addition to reducing to the flat space vacuum at high energies, this vacuum also is thermal—an Unruh detector placed in this vacuum satisfies the principle of detailed balance as though the background is at the de Sitter temperature, $T_{\rm dS} = H/2\pi$ [25]. For these reasons, the Euclidean vacuum is frequently assumed to be the correct choice when calculating the power spectrum of the primordial fluctuations which seed the temperature fluctuations that appear in the CMB radiation.

A convenient choice of coordinates for studying de Sitter space is provided by conformally flat coordinates,

$$ds^2 = \frac{d\eta^2 - d\vec{x}^2}{H^2\eta^2}, \qquad \eta \in [-\infty, 0]. \tag{2.1}$$

These coordinates are simply related to the standard coordinates used in inflation,

[1]These pathologies might also be signaling that another prescription for the propagator ought to be used. For example, a time-ordering prescription based on a real time-ordering or by regarding the α-vacua as squeezed states eliminates these nonrenormalizable divergences [19–21].

$$ds^2 = dt^2 - e^{2Ht} d\vec{x}^2, \qquad (2.2)$$

through $\eta = -H^{-1}e^{-Ht}$. The Hubble constant is related to the cosmological constant which is equal to $6H^2$.

In a model in which the density perturbations are seeded by the fluctuations of a scalar field, the inflaton is divided into terms describing, respectively, the spatially homogeneous zero mode, $\phi(\eta)$, and a term for the fluctuations, $\Phi(\eta, \vec{x})$,

$$\hat{\Phi}(\eta, \vec{x}) = \phi(\eta) + \Phi(\eta, \vec{x}). \qquad (2.3)$$

$\phi(\eta)$ is the mode which drives inflation. The expansion of a free scalar field with respect to the Euclidean vacuum is then given by

$$\Phi(\eta, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} [U_k^E(\eta) e^{i\vec{k}\cdot\vec{x}} a_{\vec{k}}^E + U_k^{E*}(\eta) e^{-i\vec{k}\cdot\vec{x}} a_{\vec{k}}^{\dagger E}] \qquad (2.4)$$

where the operator $a_{\vec{k}}^E$ annihilates the Euclidean vacuum, $|E\rangle$. The Euclidean mode functions $U_k^E(\eta)$ are solutions to the Klein-Gordon equation which in conformally flat coordinates is

$$\left[\eta^2 \frac{\partial^2}{\partial \eta^2} - 2\eta \frac{\partial}{\partial \eta} + \eta^2 k^2 + \frac{m^2}{H^2}\right] U_k^E(\eta) = 0. \qquad (2.5)$$

Since the Euclidean modes should become those of the Minkowski vacuum at short distances, or as $H \to 0$, they are fixed to be

$$U_k^E(\eta) = \frac{\sqrt{\pi}}{2} H \eta^{3/2} H_\nu^{(2)}(k\eta) \qquad (2.6)$$

where $H_\nu^{(2)}(k\eta)$ is the Hankel function and where

$$\nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}. \qquad (2.7)$$

To obtain the correct tree-level prediction for the CMB power spectrum, we shall need the effective mass of the inflaton, m, to be small compared with the Hubble scale so we shall frequently consider the limit of a massless, minimally coupled field although it is important to remember that a small but finite mass is present to avoid infrared divergences in the theory. In the limit $m \to 0$, the Euclidean mode function simplifies to

$$U_k^E(\eta) = \frac{iH}{k\sqrt{2k}} (1 + ik\eta) e^{-ik\eta}. \qquad (2.8)$$

The mode functions for the α-vacua can be obtained from those for the Euclidean vacuum through

$$U_k^\alpha(\eta) = N_\alpha [U_k^E(\eta) + e^\alpha U_k^{E*}(\eta)] \qquad (2.9)$$

where

$$N_\alpha = (1 - e^{\alpha + \alpha^*})^{-1/2} \qquad (2.10)$$

since the operator that annihilates the α-vacuum, $|\alpha\rangle$, is a Bogolubov transformation of the Euclidean vacuum creation and annihilation operators. Here $\mathrm{Re}\,\alpha < 0$ and note that we recover the Euclidean vacuum in the limit, $\alpha \to -\infty$.

The mode functions that appear in Eq. (2.6) and (2.9) are valid for a free scalar field in a classical de Sitter background. At sufficiently high energies, $\Lambda_{\rm phys} \sim M_{\rm pl}$, we expect that gravity becomes strongly interacting and there is no reason to assume that the mode functions for the full theory will continue to satisfy the Klein-Gordon equation (2.5). In order to study the size of loop effects at energies where gravity is weakly interacting, we shall consider a simple scenario in which the strongly interacting theory rapidly damps the mode functions at high energies while at lower scales the mode functions are simply those of the free theory,

$$U_k(\eta) = \begin{cases} U_k^\alpha(\eta) & \text{for } k \le \Lambda \\ 0 & \text{for } k > \Lambda \end{cases}. \qquad (2.11)$$

Note that k is the comoving and not the physical momentum, so the cutoff Λ depends on the conformal time, $\Lambda = \Lambda_{\rm phys}/(-H\eta)$. Equation (2.11) defines a "truncated α-vacuum."

Although we shall use this truncated α-vacuum to estimate the size of loop corrections to the power spectrum, the most general case could include some dependence on the momentum as well when we match onto the short-distance physics,

$$U_k^{\rm gen}(\eta) = \frac{A_k}{k\sqrt{2k}} (1 + ik\eta) e^{-ik\eta} + \frac{B_k}{k\sqrt{2k}} (1 - ik\eta) e^{ik\eta} \qquad (2.12)$$

for $k < \Lambda$. A_k and B_k are not independent since they are related by the normalization of the state. The final degree of freedom for the mode is then fixed by some assumption about the matching of the mode to the high energy theory, at $k = \Lambda$ [6–12]. Depending upon the matching condition, the coefficients A_k and B_k may introduce some additional k-dependence which will appear in the power spectrum. Here we shall study the case when the high energy theory matches onto an α-vacuum at $k = \Lambda$. Although this case will not introduce any new k-dependence into the power spectrum, it will allow us to estimate the size of perturbative corrections when the Universe is not in the Euclidean vacuum during inflation. Our results then will indicate how large similar loop effects from other k-dependent vacua could be.

III. THE POWER SPECTRUM

In the standard inflationary picture, the seeds of the large scale structure are provided by the quantum fluctuations of a scalar field during inflation. This scalar field is given by a linear combination of the fluctuations of the inflaton, $\Phi(\eta, \vec{x})$, and the scalar component of the metric fluctuations [26,27]. In the limit approaching purely exponential inflation, the inflaton fluctuations become the dominant component of this linear combination so in this section we shall neglect the contribution of the metric fluctuations. The power spectrum of density fluctuations, which produce the CMB temperature fluctuations and which eventually become the large scale structure, in this limit is proportional to the power spectrum of the scalar field, $\mathcal{P}^\alpha(\eta, k)$, defined by

$$\langle\alpha|\Phi(\eta,\vec{x})\Phi(\eta,\vec{y})|\alpha\rangle \equiv \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left[\frac{2\pi^2}{k^3}\mathcal{P}^\alpha(\eta,k)\right] \tag{3.1}$$

up to corrections suppressed by the slow-roll parameters which are small as we approach a purely de Sitter limit.

At tree level, substituting in the mode expansion of Eq. (2.4) and using massless Euclidean mode functions in Eq. (2.8), the power spectrum for the Euclidean vacuum is

$$\mathcal{P}^E(\eta,k) = \frac{H^2}{4\pi^2}(1 + k^2\eta^2). \tag{3.2}$$

When a mode has been redshifted well outside the horizon during inflation, $k\eta \ll 1$, the power spectrum becomes flat. In a general α-vacuum, the power spectrum to leading order in $k\eta$ is also flat,

$$\mathcal{P}^\alpha(\eta,k) = \frac{H^2}{4\pi^2}N_\alpha^2|1 - e^\alpha|^2[1 + \mathcal{O}(k^2\eta^2)]. \tag{3.3}$$

As the α-dependent prefactor is not large, it is observationally difficult to distinguish from other cosmological parameters unless α were to have some k-dependence [6–12].

To study the perturbative corrections to the power spectrum in a truncated α-vacuum, we shall consider a theory with a cubic interaction. This example provides a simple setting since the first nontrivial corrections to the two-point function (3.1) appear already at one-loop order.

One of the difficulties in formulating perturbation theory in a de Sitter background is the lack of a well-defined S-matrix. Therefore, we should apply a quantization procedure that evolves a matrix element over a finite conformal time interval rather than one that evaluates the matrix element between asymptotic "in" and "out" states. This approach also allows us to avoid the transplanckian problem since we can choose the initial state as that given at the matching scale Λ. If we had attempted to follow an in state back earlier, then we would eventually need to evaluate the state when the physical scales relevant today would have been blueshifted above the Planck scale.

The closed time contour formalism developed by Schwinger, Keldysh and others [28–30] describes a perturbative approach for solving the evolution of a matrix element over a finite time interval. Unlike the usual S-matrix calculation which essentially requires only a single insertion of the time-evolution operator, in the Schwinger-Keldysh approach both the $\langle\alpha|$ and the $|\alpha\rangle$ states are evolved from a given initial state at η_0 to a finite time later, η, when we wish to evaluate the expectation value of the operator. Both of these time-evolution operators can be grouped into a single time-ordered operator by formally doubling the field content of the theory to include "+" fields associated with the time-evolution of the $|\alpha\rangle$ state and "−" fields associated with the evolution of the $\langle\alpha|$ state. In the interaction picture, since the interacting part of the Hamiltonian H_I is used to evolve the states in the theory, we effectively double the interactions present—for every interaction of the "+" fields, there exists a "−" field interaction with a coupling of the opposite sign. Thus the evolution of the expectation value of an operator $\mathcal{O}$ is given by

$$\langle\alpha|\mathcal{O}|\alpha\rangle(\eta) = \frac{\langle\alpha|T\{\mathcal{O}_I e^{-i\int_{\eta_0}^{0} d\eta'[H_I(\Phi^+)-H_I(\Phi^-)]}\}|\alpha\rangle}{\langle\alpha|T\{e^{-i\int_{\eta_0}^{0} d\eta'[H_I(\Phi^+)-H_I(\Phi^-)]}\}|\alpha\rangle}. \tag{3.4}$$

Here T is the time-ordering operator which orders events along the time contour so that the arguments of the Φ^- fields always occur after, and in the opposite order than, those of the Φ^+ fields. The subscript in $\mathcal{O}_I$ indicates that the operator is evaluated in the interaction picture. The field doubling automatically removes the acausal portion of the matrix element so that although the time-evolution operators integrate to the infinite future, $\eta' = 0$ in conformal coordinates, terms involving propagators depending upon $\eta' > \eta$ are cancelled in Eq. (3.4). A more complete description of the Schwinger-Keldysh method for a de Sitter background is provided in [22].

To evaluate the perturbative corrections to the power spectrum in an interacting theory, we take the expectation value of the two-point operator $\mathcal{O}_I = \Phi(\eta,\vec{x})\Phi(\eta,\vec{y})$ in a truncated α-vacuum. In the interaction picture, the time-evolution of operators is produced by the free Hamiltonian so that the fields evolve correctly when the mode functions satisfy the free Klein-Gordon equation (2.5). In a theory with a cubic interaction,

$$H_I = \int \frac{d^3\vec{x}}{H^4\eta^4}\left[J\Phi + \frac{1}{2}\delta m^2\Phi^2 + \frac{1}{3!}\lambda\Phi^3\right], \tag{3.5}$$

the first nontrivial corrections to the two-point operator appear at one-loop order. The linear and quadratic terms correspond to allowed counterterms; the former is used to cancel tadpole subdiagrams and the latter cancels a logarithmic ultraviolet (UV) divergence in the matrix elements of a true α-vacuum [22].

When J is chosen to cancel the tadpole subgraphs, the power spectrum is given to second order in the coupling constant λ by

$$\mathcal{P}^{\alpha}(\eta,k) = \frac{k^3}{2\pi^2}|U_k^{\alpha}(\eta)|^2 - \frac{\delta m^2}{\pi^2}\frac{k^3}{H^4}\int_{\eta_0}^{\eta}\frac{d\eta_1}{\eta_1^4}\mathrm{Im}\{[U_k^{\alpha}(\eta)U_k^{\alpha*}(\eta_1)]^2\} + \frac{2\lambda^2}{\pi^2}\frac{k^3}{H^8}\int_{\eta_0}^{\eta}\frac{d\eta_1}{\eta_1^4}\mathrm{Im}\{U_k^{\alpha}(\eta)U_k^{\alpha*}(\eta_1)\} \times \int_{\eta_0}^{\eta_1}\frac{d\eta_2}{\eta_2^4}\mathrm{Im}\{U_k^{\alpha}(\eta)U_k^{\alpha*}(\eta_2)L_k^{\alpha}(\eta_1,\eta_2)\} + \cdots \tag{3.6}$$

where the loop integral is defined by

$$L_k^{\alpha}(\eta_1,\eta_2) \equiv \int_{\mu}^{\Lambda} d^3\vec{p}\, U_p^{\alpha}(\eta_1)U_p^{\alpha*}(\eta_2)U_{|\vec{p}-\vec{k}|}^{\alpha}(\eta_1)U_{|\vec{p}-\vec{k}|}^{\alpha*}(\eta_2). \tag{3.7}$$

Diagrammatically, the order λ^2 correction to the power spectrum, $\mathcal{P}^{(2)}(\eta,k)$, is generated by the self-energy graph shown in Fig. 1.

The spatial momenta in the loop in Eq. (3.7) are bounded in the UV by the structure of the truncated α-vacuum since we have assumed that the short-distance behavior is highly suppressed as in Eq. (2.11). As we shall show, once we have extracted an overall factor of the physical cutoff Λ_{phys}, the leading contribution to $\mathcal{P}^{(2)}(\eta,k)$ does not depend sensitively on how we truncate the modes since it arises among terms from the loop whose $p \equiv |\vec{p}|$ dependent phases cancel. The integral of these terms add coherently and receive their largest contributions well away from the UV cutoff.

A massless, minimally coupled scalar theory in a de Sitter background contains infrared (IR) divergences [3,31] so we have imposed an IR cutoff, μ, on our loop integrals. However true inflation is not in a pure de Sitter background and the scalar which provides the source for the perturbations is light but not massless. We could incorporate a small mass, $m \ll H$, in Eq. (2.6), and expand in m/H so that no IR cutoff would be necessary. Yet since we find that the dominant contribution to the loop integral is not strongly sensitive to μ we focus on the UV physics.

With these bounds on the large and short-distance behavior, any of the comoving momenta appearing in the integrand of Eq. (3.7) should always be between the bounds,

$$\mu \le p \le \Lambda \qquad \mu \le |\vec{p}-\vec{k}| \le \Lambda. \tag{3.8}$$

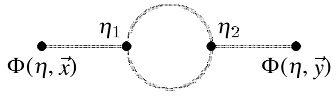

FIG. 1. The one-loop correction to the two-point function in a theory with a cubic interaction used to generate the power spectrum.

Since μ and Λ are the bounds on the comoving momenta, to relate them to the fixed IR and UV cutoffs we must specify how they depend on the conformal times at which they are evaluated. The limits of the η_1 and η_2 integrations in Eq. (3.6) impose that $\eta_2 \le \eta_1 (<0)$. Therefore, we should choose the UV bound in the loop integration to scale as

$$\Lambda = \frac{\Lambda_{\mathrm{phys}}}{-H\eta_2} \equiv -\frac{\tilde{\Lambda}}{\eta_2} \tag{3.9}$$

since by the time we have arrived at the η_1 vertex of the loop, all comoving momenta will have been redshifted and will therefore still be below Λ. Conversely, we would not like any momenta to be redshifted below the IR cutoff by the later time, η_1, so we choose the IR bound to scale as

$$\mu = \frac{\mu_{\mathrm{phys}}}{-H\eta_1} \equiv -\frac{\tilde{\mu}}{\eta_1}. \tag{3.10}$$

For example, when the IR cutoff is larger than the horizon size during inflation, we have that $\tilde{\mu} \ll 1$.

In addition to establishing the appropriate bounds on the spatial momenta appearing in the one-loop correction to the power spectrum, we must determine the limits on the conformal time integrals. The natural choice for the initial time, η_0, from which we are evolving the matrix element is at the UV scale since in our model we assume that when the comoving momentum equals this scale that the corresponding mode is that for the alpha vacuum. Thus we take

$$k\eta_0 = -\tilde{\Lambda}. \tag{3.11}$$

The lower scale depends upon when the state is assumed to become essentially classical, so that we can neglect the quantum corrections. Here we shall take the low energy cutoff to be given by the conformal time at which the momentum of the mode k has been red-shifted to the size of the IR cutoff,

$$k\eta_{\mu} = -\tilde{\mu}. \tag{3.12}$$

After integrating over the spatial loop momenta, the only remaining spatial momentum is $k \equiv |\vec{k}|$ and it becomes convenient to define dimensionless variables by rescaling the conformal times,

$$x = k\eta, \qquad x_1 = k\eta_1, \qquad x_2 = k\eta_2. \tag{3.13}$$

The leading contribution to the $\mathcal{O}(\lambda^2)$ correction to the power spectrum comes from the region of the loop integral,

$$L_k^\alpha(x_1, x_2) = -\frac{2\pi}{k^3} H^4 N_\alpha^4 e^{\alpha+\alpha^*} x_1^2 x_2 \tilde{\Lambda} \times \left[\frac{\sin(x_1 - x_2)}{x_1 - x_2} + \frac{\sin(x_1 + x_2)}{x_1 + x_2}\right] + \cdots, \tag{3.14}$$

and is given by

$$\begin{aligned}\mathcal{P}^{(2)}(\eta, k) = -\frac{\lambda^2}{\pi} N_\alpha^4 e^{\alpha+\alpha^*} \tilde{\Lambda} \bigg\{ &\int_{\tilde{\Lambda}-\tilde{\mu}}^{x} \frac{dx_1}{x_1^2} [(1 + xx_1)\sin(x - x_1) - (x - x_1)\cos(x - x_1)] \\ &\times \int_{\tilde{\Lambda}-\tilde{\mu}}^{x_1} \frac{dx_2}{x_2^3} [(1 + xx_2)\sin(x - x_2) - (x - x_2)\cos(x - x_2)] \times \left[\frac{\sin(x_1 - x_2)}{x_1 - x_2} + \frac{\sin(x_1 + x_2)}{x_1 + x_2}\right] \\ &+ \mathcal{O}(\tilde{\Lambda}^{-2}) + \mathcal{O}[\tilde{\Lambda}^{-1}\ln(\tilde{\mu})] \cdots \bigg\}.\end{aligned} \tag{3.15}$$

The momentum dependence of this correction is not of a form that can be canceled by the counterterms in Eq. (3.5) [22]. The dimensionless integral receives its dominant contribution from the region $0.1 \lesssim x_1, x_2 \lesssim 4$ and is therefore not strongly sensitive to the location of the cutoffs. Integrating numerically, we find that

$$\begin{aligned}\mathcal{I} \equiv &\int_{\tilde{\Lambda}-\tilde{\mu}}^{x} \frac{dx_1}{x_1^2} [(1 + xx_1)\sin(x - x_1) - (x - x_1)\cos(x - x_1)] \\ &\times \int_{\tilde{\Lambda}-\tilde{\mu}}^{x_1} \frac{dx_2}{x_2^3} [(1 + xx_2)\sin(x - x_2) - (x - x_2)\cos(x - x_2)] \times \left[\frac{\sin(x_1 - x_2)}{x_1 - x_2} + \frac{\sin(x_1 + x_2)}{x_1 + x_2}\right] \\ = &-0.2618 + \mathcal{O}(\tilde{\Lambda}^{-2}) + \mathcal{O}(\tilde{\mu}^3),\end{aligned} \tag{3.16}$$

so that the leading behavior of the corrections is linear in the UV cutoff,

$$\mathcal{P}^{(2)}(\eta, k) = 0.2618 \times \frac{\lambda^2}{\pi} N_\alpha^4 e^{\alpha+\alpha^*} \frac{\Lambda_{\rm phys}}{H} + \cdots. \tag{3.17}$$

If we define a dimensionless coupling $\tilde{\lambda} \equiv \lambda/H$, then the power spectrum for $k\eta \ll 1$ to leading order is

$$\begin{aligned}\mathcal{P}^\alpha(\eta, k) = &\frac{H^2}{4\pi^2} N_\alpha^2 |1 - e^\alpha|^2 \\ &\times \left\{1 + 1.047\pi \frac{N_\alpha^2 e^{\alpha+\alpha^*}}{|1 - e^\alpha|^2} \tilde{\lambda}^2 \frac{\Lambda_{\rm phys}}{H} + \cdots\right\}.\end{aligned} \tag{3.18}$$

Note that the loop correction scales inversely with the Hubble scale H so that in low-scale inflation this portion of the correction becomes larger. In the next section we estimate the size of the small dimensionless coupling $\tilde{\lambda}$ for some specific models.

A fully complete scenario would require a mechanism for α to decay since the current measurements from cosmic rays severely constrain the possible size of e^α today [32]. For example, Danielsson [6] has proposed "locally Lorentzian" matching conditions which yield $e^{\alpha(\eta)} \sim H(\eta)/\Lambda_{\rm phys}$ so that e^α is nearly constant during inflation and falls off significantly as the Universe expands. In this case the overall loop correction would be suppressed by $H(\eta)/\Lambda_{\rm phys}$, but since the scaling of higher order terms is not clear at this point we shall continue to allow e^α to be generic. Goldstein and Lowe have also presented a model [10] with an initially nonthermal vacuum which relaxes to the Euclidean vacuum after inflation. The α-vacua are only invariant states in a de Sitter background and nothing precludes α from changing since over the current age of the Universe this de Sitter symmetry is broken.

Note that while the leading loop correction to the power spectrum does not contain any k-dependence, this feature arose from our vacuum choice which, from the perspective of the Euclidean vacuum, is an excited state with all modes populated equally up to the matching scale, $\Lambda_{\rm phys}$. A more general scenario would introduce some k-dependence either through the background, as in a generic Robertson-Walker universe, or through a matching condition which varies with time. In such models we could incorporate this dependence on the wave number by writing $e^\alpha \to e^{\alpha(k)}$; then Eq. (3.18) should provide a reasonable estimate of the amplitude of these nonthermal state effects relative to the tree-level terms.

IV. THE INFLATON POTENTIAL

While the loop contribution to the power spectrum is enhanced by a factor of $\Lambda_{\rm phys}/H$, the dimensionless coupling constant $\tilde{\lambda}$ suppresses the total loop term relative to the tree contribution. The α-vacuum enhancement is generic for any cubic self-interaction, but the coupling constant suppression is model dependent. We briefly consider two common inflationary models to illustrate the size of the coupling and hence the overall size of the loop term.

To determine the size of the coupling we need the normalization constraint from the Cosmic Background Explorer (COBE) [33],

$$\frac{V^{3/2}(\phi)}{M_{\rm pl}^3 V'(\phi)} \approx 5 \times 10^{-4}. \tag{4.1}$$

Here a prime denotes differentiation with respect to the field ϕ, $V' = \partial V/\partial\phi$. We also require that the modes observed in the CMB left the horizon about 50 e-folds before the end of inflation, which determines the value of the inflaton zero mode, ϕ, when these fluctuations left the horizon

$$\frac{1}{M_{\rm pl}^2}\int_{\phi_{\rm end}}^{\phi} d\phi' \frac{V(\phi')}{V'(\phi')} \approx 50. \tag{4.2}$$

For chaotic inflation, where the potential is dominated by the inflaton terms such as, $V(\phi) = \frac{1}{3!}H\tilde{\lambda}\phi^3$, inflation occurs when the inflaton field is much larger than the Planck scale and ends when $\phi_{\rm end} \approx M_{\rm pl}$. From Eq. (4.2) we have $\phi \approx 20M_{\rm pl}$ and the COBE normalization gives $\tilde{\lambda} \approx 10^{-11}M_{\rm pl}/H$. Use of the Friedmann equation gives the Hubble scale, $H \approx 10^{-4}M_{\rm pl}$ and if we set $\Lambda_{\rm phys} \sim M_{\rm pl}$, we finally arrive at the result we seek,

$$\tilde{\lambda}^2 \frac{\Lambda_{\rm phys}}{H} \approx 10^{-10}. \tag{4.3}$$

Clearly this suppression is quite significant.

For a hybrid inflation model, the potential is dominated by a constant term, V_0, as long as ϕ is above a critical value, ϕ_c. V_0 must be roughly constant while the modes of interest are leaving the horizon, but contains operators which end inflation at $\phi = \phi_c$. We use the potential $V(\phi) = V_0 + \frac{1}{3!}H\tilde{\lambda}\phi^3$. If ϕ is not fine tuned to be close to ϕ_c, then Eq. (4.2) gives

$$\phi_c \approx \frac{H}{10\tilde{\lambda}} < \phi. \tag{4.4}$$

Dropping numerical factors close to 1, the COBE normalization now gives, $10^{-4} \approx H^2/(\tilde{\lambda}\phi^2)$. It is consistent to choose $\phi \approx \mathcal{O}(\phi_c)$, in which case $\tilde{\lambda} \approx 10^{-6}$. The size of the enhancement is difficult to predict without more details of the theory, specifically, the Hubble scale. However, by requiring that the constant term dominate the potential, $V_0 \gg \frac{1}{3!}H\tilde{\lambda}\phi^3$, as it must for this to be a model of hybrid inflation, we arrive at

$$\tilde{\lambda}^2 \frac{\Lambda_{\rm phys}}{H} \gg 10^{-8}. \tag{4.5}$$

For small enough inflationary energy density,

$$V_0 \lesssim (10^{12}\ {\rm GeV})^4, \tag{4.6}$$

the Hubble parameter is small enough that the loop effects are as large as the tree-level effect, or larger. Of course, if we are to remain in the perturbative regime, we require a suppression from the α-dependent factors; the state at the matching scale may only deviate from the Euclidean state by a small amount.

Models for inflation with a cubic self-coupling may not be common, but the enhancement demonstrated in this work is not limited to cubic theories. The naïve power counting argument discussed in [22] indicates that the loop corrections to the two-point function in a quartic theory should scale as

$$\mathcal{O}\left(\lambda_4^2 \frac{\Lambda_{\rm phys}^3}{H^3}\right). \tag{4.7}$$

Although there is again a suppression due to the model dependent coupling constant, the loop enhancement effect is general.

In models in which the vacuum state of the inflaton is nonthermal, loop effects can generally be larger than what would be suggested by the counting of coupling constants alone. In some cases, the model may not even be perturbative for $|\alpha| \sim \mathcal{O}(1)$ and thereby will constrain the degree to which the vacuum can deviate from the Euclidean, or thermal, vacuum.

Thus far we have only considered a single scalar field, corresponding to the inflaton. However, any other fields present will be sensitive to the rapid expansion of the Universe and will require a matching condition at the cutoff scale as well. These fields could also be in a nonthermal vacuum below this scale and their coupling to the inflaton would then provide an additional source for large loop corrections—even if the inflaton itself was in the Euclidean vacuum. For example, when the inflaton interacts with an additional scalar field Ψ through a trilinear coupling, $g\Phi\Psi^2$, then the power spectrum receives potentially large loop corrections from the diagram shown in Fig. 2. When Ψ is in a truncated α-vacuum, the product of α-vacuum propagators in the loop integral will again produce a $\Lambda_{\rm phys}$ enhancement.

Furthermore, any simple model of inflation, such as those mentioned in this section, must ultimately be embedded in a more fundamental theory with many massive degrees of freedom, such as Ψ. To arrive at an inflationary model, those massive fields must be integrated out leaving an effective Lagrangian for the inflaton. Given the loop

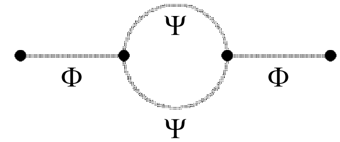

FIG. 2. Even if the inflaton Φ is in the Euclidean vacuum, couplings to other scalar fields in a nonthermal vacuum, Ψ, will produce large corrections to the power spectrum.

enhancements expected for a generic α-vacuum, or in any nonthermal vacuum, this process of integrating out the high energy physics may contain previously unexpected subtleties.

V. CONCLUSIONS

In this article we have investigated the perturbative corrections to the power spectrum due to inflaton interactions when the field is in a truncated α-vacuum. In a theory with a cubic interaction with a coupling λ, the size of the perturbative corrections are of the order $(\lambda^2/H^2) \times (\Lambda_{\rm phys}/H)$. The large second factor depends only on the form of the free-field propagators in the loop, and is therefore quite general. The initial suppressing factor from the coupling constant is constrained more or less restrictively depending upon the specific inflationary model studied.

For a single self-interacting inflaton in chaotic inflation, slow-roll or observational constraints require a sufficiently small coupling λ, compared with the Hubble scale H, that the overall loop effects in these models will be negligible. However, in a hybrid model the self-coupling is not nearly as constrained so that loop effects could be potentially observed in the cosmic microwave background radiation, or could even be larger than tree-level effects so that such a theory could not be treated perturbatively. Moreover, as a result of inverse scaling of the Hubble constant in the $\Lambda_{\rm phys}/H$ factor, such loop corrections are actually more significant in low-scale inflationary theories where tree-level corrections in a nonthermal background would be negligible.

Precisely measuring the shape of the CMB power spectrum provides a unique opportunity to study physics at scales well above those accessible to accelerator experiments. Details in the inflationary potential or in the state of the inflaton during inflation translate directly into features in the power spectrum. Studying the loop corrections to the power spectrum in a nonthermal vacuum provides a further resource for constraining inflationary models or the vacuum state during inflation. If deviations from the Euclidean vacuum expectation are observed, it will be important to distinguish their possible origins—whether from nonthermal tree or loop effects or from the dynamics of other fields near the Planck scale or from the inflationary potential—to understand what they are telling us about the very early Universe.

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