Randall-Sundrum scenario with an extra warped dimension

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We investigate a scenario with two four branes embedded in six dimensions. When the metric is periodic and compact in one of the dimensions parallel to the branes, the value of the effective cosmological constant for the remaining five dimensions can assume a variety of values, determined by the dependence of the metric on the sixth dimension. The picture that emerges resembles the Randall-Sundrum model but with an extra warped dimension that allows the usual brane-bulk fine tuning to be satisfied *without* finely tuning any of the parameters in the underlying six-dimensional theory. Although the action contains terms with four derivatives of the metric, we show that when the branes have a finite, natural thickness, such terms have only a small effect on the Randall-Sundrum structure. The presence of these four derivative terms also allows a configuration that resembles that produced by a domain wall but which results from gravity alone.

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I. INTRODUCTION

Two of the most enigmatic features of the universe are the weakness of gravity compared to electroweak interactions—the hierarchy problem—and the small size of the cosmological constant. A recent approach [1] to the hierarchy problem has observed that the presence of a small warped extra dimension could naturally produce an exponential hierarchy between the scales of gravitational and standard model interactions. The new element in this and related [2] scenarios is the introduction of solitonic three-dimensional hypersurfaces, or three-branes, to which the standard model fields are confined while gravity propagates in all the dimensions. These models still require a fine tuning to produce a low-energy effective $3+1$-dimensional theory with no cosmological constant.

Extra dimensions might also provide a framework for addressing the cosmological constant problem. Instead of setting the cosmological constant to an unnaturally small value, we can demand only that the theory should admit a nearly flat effectively $3+1$-dimensional theory below some high-energy scale—regardless of the value of the cosmological constant. This picture was introduced by Rubakov and Shaposhnikov with a six-dimensional model [3]. The idea is that the cosmological constant distorts, or warps, some or all of the extra dimensions while leaving the theory with a $3+1$-dimensional Poincaré symmetry. If this idea is extended so that this warping is accomplished with a metric that is both smooth and periodic in the extra dimensions, then there is no need to cut off the space or to encounter singularities in the extra dimensions.

An explicit realization occurs in $4+1$ dimensions [4] when the metric is smooth, nonsingular, and periodic in the extra dimension. We can then choose the extra dimension to be compact with its size given by the period. At large distances compared to this period, the universe appears four dimensional (4D). The 4D cosmological constant is determined by both the 5D cosmological constant and the geometry of the extra dimension. Therefore, we can achieve a $3+1$-dimensional Poincaré invariance even when the 5D cosmological constant is not zero, by choosing the solution to the field equations with the appropriate behavior in the extra dimension. However, some further mechanism is still required to explain why this particular solution should be preferred.

This paper intends to combine these ideas into a scenario that incorporates the Randall-Sundrum picture [1] but without finely tuning any of the parameters in the action. The scenario starts with an effective action for gravity in six dimensions including terms with up to four derivatives of the metric. It also includes two parallel four-branes, which are compact in one dimension and extend infinitely in the other three. With some mild bounds on the parameters in the action, we find that the equations of motion allow the geometry of the resulting universe to contain a five-dimensional anti–de Sitter (AdS_5) subspace with a warped metric that is periodic in the sixth dimension.

A generic set of four derivative terms in the action apparently implies an infinite tension on the branes, however, this singularity appears as an artifact of the vanishing thickness of the branes. When the brane has a finite thickness, it is possible to show explicitly that the higher derivative terms can be neglected. An action that contains *only* gravity—including these four derivative terms but without any scalar fields—also admits solutions in which gravity is localized about a hypersurface of codimension one. Far from this hypersurface, the metric approaches an AdS metric as in the second Randall-Sundrum model [5], however, it is the four derivative terms and not a brane or a scalar field that effects this localization of gravity.

II. PRELIMINARIES AND EFFECTIVE ACTION DESCRIPTIONS

Randall and Sundrum [1] proposed that if the universe were to consist of two three branes bounding a bulk region of five-dimensional anti–de Sitter space-time, then the redshift induced by the bulk metric at one of the branes could gen-

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erate an exponential hierarchy between the Planck scale and the scale of electroweak symmetry breaking. The action for their scenario contains an Einstein-Hilbert term for the bulk

$$S_{\rm bulk}^{\rm RS}=M_5^3\int d^4x dr\,\sqrt{-\hat{g}}[2\Lambda_{\rm RS}+R], \tag{1}$$

while the branes located at $r=0$ and $r=r_c$ only contribute through their surface tensions,

$$S_{\rm branes}^{\rm RS}=M_5^3\int_{r=0}d^4x\,\sqrt{-\hat{h}}[-2\sigma_{\rm RS}] +M_5^3\int_{r=r_c}d^4x\,\sqrt{-\hat{h}}[2\sigma_{\rm RS}]. \tag{2}$$

Here $\hat{g}_{MN}$ is the metric for AdS_5,

$$\hat{g}_{MN}\,dx^M dx^N=e^{-2|r|/l}\eta_{\mu\nu}dx^\mu dx^\nu+dr^2, \tag{3}$$

and $\hat{h}_{\mu\nu}$ is the induced metric on the branes. We shall denote the usual space-time directions by x^μ, with $\mu,\nu,\ldots=0,1,2,3$, and r describes the direction orthogonal to the branes with $M,N,\ldots=0,1,2,3,r$. M_5 denotes the bulk Planck mass. The bulk Einstein equations determine $\Lambda_{\rm RS}=6/l^2$ and the specific choice of $\sigma_{\rm RS}=6/l$ for the brane tensions is necessary for the low-energy four-dimensional (4D) theory to be free of a cosmological constant.

As the cosmological constant and the surface tension appear in the action, they represent fundamental parameters of the theory and we have no reason *a priori* that the fine-tuning condition is satisfied. If instead the quantities that appear in the action arise from some more fundamental theory, then it might be possible for a dynamical mechanism to exist that favors solutions in which the low-energy, four-dimensional theory is nearly flat.

We can adapt the picture developed in Ref. [4] without branes to one which resembles the Randall-Sundrum construction but where the AdS_5 length is not uniquely determined by the higher-dimensional cosmological constant. The structure for such a model would include *two* extra dimensions—one small periodic dimension to avoid fine-tuning the cosmological constant and a second to generate the electroweak Planck hierarchy (Fig. 1).

As in Ref. [4], we consider gravity as an effective theory, expanded in powers of derivatives, with a scalar field ϕ,

$$S_{\rm bulk}=M_6^4\int d^4x\,dr\,dy\sqrt{-g}(2\Lambda+R+aR^2+bR_{ab}R^{ab} +cR_{abcd}R^{abcd}+\cdots) +M_6^4\int d^4x\,dr\,dy\sqrt{-g}(-\tfrac{1}{2}\nabla_a\phi\nabla^a\phi+\Delta\mathcal{L}). \tag{4}$$

Here y represents the coordinate of the extra periodic dimension, with $a,b,\ldots=0,1,2,3,r,y$. Λ, and M_6 denote the cosmological constant and the six dimensional Planck mass, respectively.

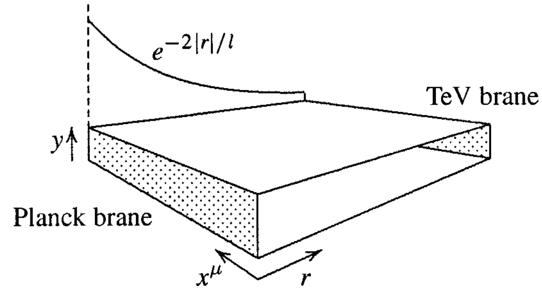

FIG. 1. The geometry of a six-dimensional model with two four-branes. The small periodic coordinate is y. The direction orthogonal to the four-branes, r, becomes the extra coordinate of the Randall-Sundrum model when we integrate out the y dimension. We can recover the second Randall-Sundrum model [5] by letting $r_c\to\infty$. The model assumes an orbifold geometry about $r=0$.

The existence of metric solutions that are periodic in y appears to be a fairly generic feature of actions that include terms beyond the standard Einstein-Hilbert terms [4]. For example, when the theory is truncated so that the gravitational action contains only terms with up to four derivatives of the metric and including a scalar field ϕ with $\Delta\mathcal{L}=k(\nabla_a\phi\nabla^a\phi)^2$, the equations of motion for Eq. (4) admit periodic solutions over a range of the parameters $\{\Lambda,a,b,c,k\}$.

We instead shall focus upon the simpler case where $\Delta\mathcal{L}$ describes a Casimir effect. Since the finite y direction explicitly breaks $5+1$-dimensional Poincaré symmetry, any quantum vacuum contribution can differ in the y and (x^λ,r) directions. We can include this effect by adding the following energy-momentum tensor to the field equations:

$$T^{{\rm vac}\,b}_{\ \ a}=\mathrm{diag}(C,C,C,C,C,-C). \tag{5}$$

Any contribution proportional to the identity can be absorbed into the definition of the cosmological constant.[1] Satisfactory, periodic solutions then exist when C is above a mild bound.

To produce a Randall-Sundrum scenario in five of the dimensions, we shall examine a metric of the form

$$ds^2=g_{ab}(x^\lambda,r,y)\,dx^a\,dx^b =e^{A(y)}\hat{g}_{MN}(x^\lambda,r)\,dx^M\,dx^N+dy^2 \tag{6}$$

with an AdS_5 metric for the (x^λ,r) subspace,

$$d\hat{s}^2=\hat{g}_{MN}\,dx^M\,dx^N=e^{-2|r|/l}\eta_{\mu\nu}\,dx^\mu\,dx^\nu+dr^2. \tag{7}$$

When $A(y)$ is a periodic function of y, we can obtain a compact extra dimension with a very nontrivial y dependence without any singularities. The shape of $A(y)$ determines the effective cosmological constant of the $\hat{g}_{MN}$ metric. Since this metric is conformally flat, the linear combinations

[1] Since we do not require that Λ vanish, quantum contributions no longer need to cancel the classical contribution to Λ; rather, they are only regarded as some component of the total Λ.

of the R^2 terms that represents the squared Weyl tensor will not contribute to the equations of motion. It is convenient to parametrize the two remaining linear combinations by

$$\tilde{\mu} \equiv 20a + 6b + 4c, \quad \tilde{\lambda} \equiv 15a + \tfrac{5}{2}b + c. \tag{8}$$

Before examining the detailed form of the equations that determine the warp function $A(y)$ with a Randall-Sundrum 5D subspace, we derive a 5D effective action by integrating out the sixth dimension in the warped background given by the metric (6). There, the scalar curvature R is related to the scalar curvature $\hat{R}$ of the 5D metric $\hat{g}_{MN}$ and derivatives of the warp function

$$R = e^{-A(y)}\hat{R} - 5A'' - \tfrac{15}{2}(A')^2. \tag{9}$$

Here the prime denotes a y derivative. Similarly, the components of the Ricci and Riemann tensors in the full theory, R_{ab} and $R^a{}_{bcd}$, can be expanded in terms of the analogous tensors for the Randall-Sundrum subspace, $\hat{R}_{MN}$ and $\hat{R}_{MNPQ}$.

Integrating out the small y_c dimension in a background such as Eq. (7) where $\hat{R}$ is constant produces a five dimensional effective action,

$$S^{\rm eff}_{\rm bulk} = M_5^3 \int d^4x\, dr \sqrt{-\hat{g}}\,(2\Lambda_{\rm eff} + \hat{R} + a_{\rm eff}\hat{R}^2 + b_{\rm eff}\hat{R}_{MN}\hat{R}^{MN} + c_{\rm eff}\hat{R}_{MNPQ}\hat{R}^{MNPQ} + \cdots). \tag{10}$$

This action should reproduce the leading behavior for small x^μ-dependent perturbations about AdS_5 (7), $\mathbf{R}^{4,1}$, or de Sitter space backgrounds. The new parameters that appear in this effective action depend partially upon the "fundamental" parameters of the original action but also upon the behavior of the warp function. Thus, in the low-energy theory, the 5D cosmological constant is

$$M_5^3\Lambda_{\rm eff} = M_6^4 \int_0^{y_c} dy\, e^{(5/2)A(y)}\Big[\Lambda - \tfrac{1}{4}(\phi')^2 + \tfrac{5}{2}(A')^2 + \tfrac{5}{8}\tilde{\mu}(A'')^2 - \tfrac{5}{24}\tilde{\lambda}(A')^4\Big] \tag{11}$$

while the 5D Planck mass is

$$M_5^3 = M_6^4 \int_0^{y_c} dy\, e^{(3/2)A(y)}\Big[1 - \tfrac{1}{8}(3\tilde{\mu} - 4\tilde{\lambda})(A')^2\Big]. \tag{12}$$

The coefficients of the $\hat{R}^2$ terms are

$$M_5^3 a_{\rm eff} = M_6^4 a \int_0^{y_c} dy\, e^{(1/2)A(y)}, \tag{13}$$

with analogous expressions for $b_{\rm eff}$ and $c_{\rm eff}$. In these expressions we have freely integrated by parts.

For the theory to resemble the standard Randall-Sundrum picture, the five dimensional theory of gravity should be weak, $M_5 l \gg 1$. Since $\Lambda_{\rm eff} \sim l^{-2}$, we require the effective 5D cosmological constant to be small, which can easily occur when the contribution from the bulk cosmological constant is partially canceled by effects from the warp function in Eq. (11).

In the weak 5D gravity limit, $M_5 l \gg 1$, the self-coupling terms become negligible and the leading behavior is governed by the Einstein-Hilbert terms in Eq. (10). If we include two four-branes at $r=0$ and $r=r_c$ with respective tensions $\sigma^{(0)}$ and $\sigma^{(r_c)}$, then we recover the action considered by Randall and Sundrum [1],

$$S^{\rm eff} = M_5^3 \int d^4x\, dr \sqrt{-\hat{g}}(2\Lambda_{\rm eff} + \hat{R}) + M_5^3 \int_{r=0} d^4x \sqrt{-\hat{h}}[-2\sigma^{(0)}_{\rm eff}] + M_5^3 \int_{r=r_c} d^4x \sqrt{-\hat{h}}[-2\sigma^{(r_c)}_{\rm eff}] + \cdots. \tag{14}$$

Here the effective tension on the $r=0$ brane is

$$M_5^3 \sigma^{(0)}_{\rm eff} = M_6^4 \sigma^{(0)} \int_0^{y_c} dy\, e^{2A(y)}, \tag{15}$$

with an analogous expression for the $r=r_c$ brane. $\hat{h}_{MN}$ represents the metric induced on the branes at $r=0$ or $r=r_c$ by the metric $\hat{g}_{MN}$.

The fine tunings of the tensions on the Planck brane ($\sigma^{(0)}_{\rm eff}$) and the TeV brane ($\sigma^{(r_c)}_{\rm eff}$) in Ref. [1] are

$$\sqrt{6\Lambda_{\rm eff}} = \sigma^{(0)}_{\rm eff} = -\sigma^{(r_c)}_{\rm eff}. \tag{16}$$

Numerically, we find solutions periodic in the y direction provided that the effective cosmological constant is of the same order or smaller than the full cosmological constant, $|\Lambda_{\rm eff}| \lesssim \mathcal{O}(\Lambda)$. Using the desired value of $\Lambda_{\rm eff}$ from Eq. (16) and applying Eqs. (11), (12), and (15), we can find solutions that are periodic in y and satisfy Eq. (16) without finely tuning any of the parameters when

$$(\sigma^{(0)})^2 \lesssim \mathcal{O}(\Lambda). \tag{17}$$

In making this transition from the effective parameters back to the 6D parameters, any exponential factors are either small or tend to cancel. In the Appendix we show two representative examples, with $\sigma^{(0)} = 0.586$ and $\sigma^{(0)} = 0.396$ in units where $\Lambda = -1$.

Although Eq. (16) actually contains two fine tunings, Goldberger and Wise [6] showed that including a massive bulk scalar field with quartic couplings to the brane, thereby generating a nontrivial effective potential for r_c, eliminates one of these fine tunings. Since we expect that some such mechanism can be adapted to our picture, we are left with one condition on $\Lambda_{\rm eff}$ in terms of $\sigma^{(0)}_{\rm eff}$. When Eq. (17) is satisfied, solutions of the equations of motion from Eq. (4) can satisfy this condition without finely tuning any of the parameters in the action. The vanishing of the 4D effective cosmological constant in the Randall-Sundrum scenario in this picture thus reduces to a dynamical question as to why the flat solutions are favored.

III. EXACT ANALYSIS

When we include one or two codimension one branes orthogonal to the periodic y direction, in order to find a solution in which the low-energy effective theory for an observer on one of these branes is flat, the tension of the branes requires a nonvanishing value of the effective 5D cosmological constant, $\Lambda_{\rm eff}$, Eq. (16). The method for numerically demonstrating the existence of such solutions has been detailed in Ref. [4] for a 5D theory with a flat 4D subspace. The qualitative results from Ref. [4] do not change when we start from a theory in $5+1$ dimensions and alter only slightly when a small cosmological constant appears in the 5D subspace. The flat solutions, for which $\hat{g}_{MN}=\eta_{MN}$ in Eq. (6), exist in the region of parameter space, where $\Lambda,\tilde{\mu}<0$, and

$$\tilde{\lambda}\Lambda>\begin{cases} -\frac{25}{4}\tilde{\mu}\Lambda+5\sqrt{10\tilde{\mu}\Lambda}-\frac{15}{2} & \text{for}\ \ 0<\tilde{\mu}\Lambda\leqslant\frac{9}{10},\\ \frac{25}{12}\tilde{\mu}\Lambda & \text{for}\ \ \frac{9}{10}\leqslant\tilde{\mu}\Lambda, \end{cases} \tag{18}$$

although the exact location of the boundary corresponding to the second case of Eq. (18) has not been precisely determined.

When a 5D cosmological constant $\Lambda_{\rm eff}$ is included, the form of $A(y)$ alters slightly from its $\Lambda_{\rm eff}=0$ value; however, numerical integrations show that the warp function remains periodic as long as $|\Lambda_{\rm eff}|\lesssim\mathcal{O}(|\Lambda|)$, so that periodic solutions exist without any additional fine tunings as long as Eq. (17) is satisfied. This bound includes the case wherein the 5D effective theory of gravity is weak. Thus it is possible to satisfy $M_5^2\Lambda_{\rm eff}^{-1}\gg 1$ even while $M_6^2\Lambda^{-1}\sim 1$.

The bulk equations of motion for a universe with an AdS_5 subspace (7), are obtained by varying the full action (4),

$$\begin{aligned}&\tilde{\mu}[\tfrac{1}{2}A''''+\tfrac{5}{2}A'A'''+\tfrac{15}{8}(A'')^2+\tfrac{25}{8}A''(A')^2]\\ &\quad+\tilde{\lambda}[A''(A')^2+\tfrac{5}{8}(A')^4]-2A''-\tfrac{5}{2}(A')^2\\ &\quad-l^{-2}e^{-A(y)}\{6+(3\tilde{\mu}-4\tilde{\lambda})[A''+\tfrac{3}{4}(A')^2]\}\\ &\quad+2l^{-4}e^{-2A(y)}[\tfrac{1}{4}\tilde{\mu}+\tilde{\lambda}]\\ &\quad=-\Lambda-C+\frac{1}{4}(\phi')^2\end{aligned} \tag{19}$$

$$\begin{aligned}&\tfrac{5}{4}\tilde{\mu}[A'A'''-\tfrac{1}{2}(A'')^2+\tfrac{5}{2}A''(A')^2]+\tfrac{5}{8}\tilde{\lambda}(A')^4\\ &\quad-\tfrac{5}{2}(A')^2\\ &\quad-l^{-2}e^{-A(y)}[10+\tfrac{5}{4}(3\tilde{\mu}-4\tilde{\lambda})(A')^2]\\ &\quad+10l^{-4}e^{-2A(y)}[\tfrac{1}{4}\tilde{\mu}+\tilde{\lambda}]\\ &\quad=-\Lambda+C-\tfrac{1}{4}(\phi')^2.\end{aligned}$$

The equation of motion for the scalar field is

$$\phi''+\tfrac{5}{2}A'\phi'=0 \tag{20}$$

when the scalar field depends only on the y direction, $\phi=\phi(y)$. The sum of the equations in Eq. (19) provides a single differential equation for $A(y)$ which we can integrate numerically. Once a periodic solution is found, the difference of these equations determines the behavior of the scalar field. Note that since the scalar field appears in Eqs. (19) and (20) only through its derivatives, only ϕ' is guaranteed to be periodic; $\phi(y)$ monotonically increases and must thus assume values only over a compact range.

We have found periodic solutions throughout the parameter space (18) for small 5D cosmological constants of either sign. Two such examples are sketched in the Appendix. Since the theory is invariant under y translations and rescalings in the other five coordinates, we are free to choose coordinates in which $A(0)=A'(0)=0$ without any loss of generality. We also set $A'''(0)=0$. We then numerically integrated Eq. (19) for an initial choice for $A''(0)$. What we find is that for a particular value of $A''(0)$, the warp function returns to its initial conditions after some finite y_c: $A(y)=A(y+y_c)$.

Note that when we substitute the equations of motion (19) into the expression for the effective cosmological constant (11)

$$\begin{aligned}M_5^3\Lambda_{\rm eff}=M_6^4\int_0^{y_c}dy\, e^{3/2A(y)}\bigg[&\frac{6}{l^2}-\frac{6}{l^2}(b+c)(A')^2\\ &-\frac{4}{l^4}e^{-A(y)}(10a+2b+c)\bigg],\end{aligned} \tag{21}$$

and rewrite this expression in terms of the parameters of the effective theory (11)–(13), we find

$$\Lambda_{\rm eff}=\frac{6}{l^2}-\frac{4}{l^4}(10a_{\rm eff}+2b_{\rm eff}+c_{\rm eff})=\frac{6}{l^2}-\frac{8}{l^4}\lambda_{\rm eff}. \tag{22}$$

We would have obtained the same result by inserting the metric in Eq. (7) into the equations of motion for the effective action, Eq. (10).

IV. THICK BRANES

In passing to the weak gravity limit in order to ignore the $\hat{R}^2$ terms in the 5D theory (14), we might worry that while their effect on the bulk dynamics is small, they nevertheless might significantly alter the brane tension. Indeed, the particular linear combination of terms for which

$$\mu_{\rm eff}\equiv 16a_{\rm eff}+5b_{\rm eff}+4c_{\rm eff}\neq 0 \tag{23}$$

generates terms of the form $\partial_r^2\delta(r)$ or $[\delta(r)]^2$ for Eq. (7), which implies that the brane tension receives an infinite correction. Note that this feature is present in the original Randall-Sundrum scenario. Therefore, in its original form, even in the weak-field limit and when the coefficients of the $\hat{R}^2$ terms are small but generic, small effects in the bulk can have a large effect on the brane tension.

The origin for these δ-function divergencies is the zero thickness of the brane. Yet while we can remove such severe divergencies by giving the brane a finite thickness, in order to recover the Randall-Sundrum picture, we must additionally show that the $\hat{R}^2$ terms can be neglected in this case. Therefore, we shall examine a solution of the form

$$\hat{g}_{MN}\,dx^M\,dx^N = e^{\sigma(r)}\eta_{\mu\nu}\,dx^\mu dx^\nu + dr^2 \tag{24}$$

for the exact $a_{\rm eff}=b_{\rm eff}=c_{\rm eff}=0$ case, and then study the size of $\hat{R}$ compared to M_5^2, both near the brane and deeply within the bulk.

An elegant formalism for obtaining a thick domain wall generated by a scalar field $\Phi(r)$,

$$S = M_5^3\int d^4x\,dr\sqrt{-\hat{g}}[\hat{R} - \tfrac{1}{2}\hat{\boldsymbol{\nabla}}_M\Phi\hat{\boldsymbol{\nabla}}^M\Phi - V(\Phi)], \tag{25}$$

has been used by several authors [7–9] who noted that when the scalar potential has the following "superpotential" form:

$$V(\Phi) = \frac{1}{2}\left(\frac{\partial W}{\partial\Phi}\right)^2 - \frac{1}{3}W^2(\Phi), \tag{26}$$

the 5D warp function and domain-wall profile are, respectively,

$$\partial_r\sigma = -\frac{1}{3}W(\Phi), \quad \partial_r\Phi = \frac{\partial W}{\partial\Phi}. \tag{27}$$

We shall use the solution of Gremm [9] for a single domain wall at $r=0$ ($r_c\to\infty$) with

$$W(\Phi) = \frac{6}{l}\sin\left(\sqrt{\frac{6}{\kappa l}}\Phi\right)$$

and

$$\sigma(r) = -\frac{2}{\kappa l}\ln[2\cosh(\kappa r)],$$

$$\Phi(r) = 2\sqrt{\frac{6}{\kappa l}}\arctan\left[\tanh\left(\frac{1}{2}\kappa r\right)\right]. \tag{28}$$

Here l corresponds to the asymptotic AdS_5 length as in Eq. (7) and κ^{-1} is the thickness of the brane. The scalar curvature is now free of singularities everywhere,

$$\hat{R} = \frac{8\kappa}{l} - \left(\frac{8\kappa}{l} + \frac{20}{l^2}\right)\tanh^2(\kappa r). \tag{29}$$

From this result we notice that in the bulk ($r\gg\kappa^{-1}$), since $\hat{R}\to -20l^{-2}$, the weak gravity condition is $M_5 l\gg 1$. At the brane ($r\approx 0$), in order to be able to neglect the $\hat{R}^2$ terms relative to $\hat{R}$, we require

$$M_5\sqrt{\frac{l}{\kappa}}\gg 1. \tag{30}$$

Thus, assuming that gravity is weakly coupled in the bulk so that $M_5 l\gg 1$, both $\kappa\sim l^{-1}$ and $\kappa\sim M_5$ automatically satisfy Eq. (30).

The above analysis applies to the positive tension brane, or "Planck brane" [1], although the same infinite corrections to the negative tension brane arise when $\mu_{\rm eff}\neq 0$. Negative tension branes do not admit a thick wall description, at least in the regime in which the $\hat{R}^2$ terms become negligible. However, we can also add a warped, compact extra dimension to a scenario that does not contain any thin negative tension branes, such as that of Lykken and Randall [10].

The purely gravitational action of Eq. (10), which includes $\hat{R}^2$ terms, can also generate a warp function in the 5D subspace (24) that resembles that in Eq. (28) but without the need for a scalar field to generate a domain wall. In this case, we still have[2]

$$\sigma(r) = -\frac{2}{\kappa l}\ln[2\cosh(\kappa r)], \tag{31}$$

but where the width κ^{-1} and the asymptotic AdS_5 length l are, respectively,

$$\kappa = \left(\frac{3-4\sqrt{2\Lambda_{\rm eff}\mu_{\rm eff}}}{-2\mu_{\rm eff}}\right)^{1/2},$$

$$l = \left(\frac{3-4\sqrt{2\Lambda_{\rm eff}\mu_{\rm eff}}}{-\Lambda_{\rm eff}}\right)^{1/2}, \tag{32}$$

with $0\leqslant\Lambda_{\rm eff}\mu_{\rm eff}\leqslant\frac{9}{32}$, $\Lambda_{\rm eff}<0$ and $\mu_{\rm eff}\leqslant 0$. This configuration requires one fine tuning among $\Lambda_{\rm eff}$, $\lambda_{\rm eff}$, and $\mu_{\rm eff}$, given in Ref. [4] by

$$\Lambda_{\rm eff}\lambda_{\rm eff} = -(3-4\sqrt{2\Lambda_{\rm eff}\mu_{\rm eff}})(\tfrac{9}{8}-\tfrac{1}{2}\sqrt{2\Lambda_{\rm eff}\mu_{\rm eff}}), \tag{33}$$

which can presumably be effected by the appropriately warped compactification in the sixth dimension.

The $\hat{R}^2$ terms in this case are in no sense negligible—they play the same role as $\Phi(r)$ above and balance against the $\hat{R}$ term to produce the solution (31). In particular, $|\hat{R}|\gg|\lambda_{\rm eff}\hat{R}^2|$ for $r\gg\kappa^{-1}$ translates into

$$\tfrac{45}{2} - 10\sqrt{2\Lambda_{\rm eff}\mu_{\rm eff}}\ll 1, \tag{34}$$

which is nowhere satisfied in the allowed range for $\Lambda_{\rm eff}\mu_{\rm eff}$.

[2]This configuration corresponds to Eq. (4.6) of Ref. [4].

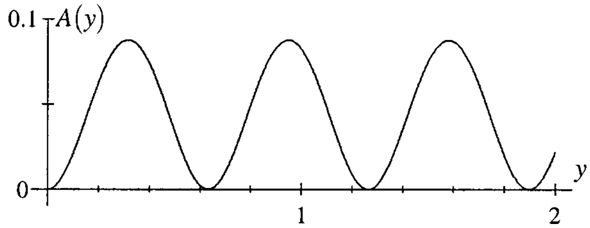

FIG. 2. A periodic warp function $A(y)$ for $\Lambda=-1$ and $\lambda=\mu=-0.1$. The initial condition is $A''(0)=4.639\,239\,91$. The value of the AdS length, which appears in Eq. (7), is $l=10$, which corresponds to $\sigma^{(0)}=0.586$.

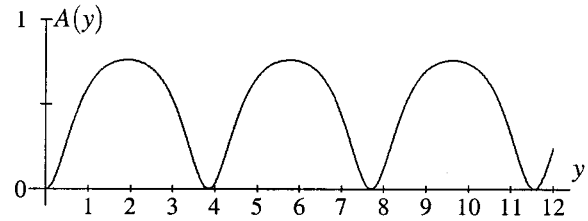

FIG. 3. A periodic warp function $A(y)$ for $\Lambda=-1$, $\lambda=-2$, and $\mu=-0.9$. The initial condition is $A''(0)=3.199\,870\,015$. The value of the AdS length is $l=10$, which corresponds to $\sigma^{(0)}=0.396$.

V. CONCLUDING REMARKS

An intriguing feature of this model is that for each choice of the parameters in the action that admits a periodic warp function, a family of solutions exists. Elements of this family are specified by the value of the 5D effective cosmological constant $\Lambda_{\rm eff}$. Alternatively, since each $\Lambda_{\rm eff}$ is associated with a unique period—or at worst a discrete set of periods—of the warp function, we can also specify an element by the size of the sixth dimension y_c. This behavior differs from a factorizable geometry such as the original Kaluza-Klein picture which has a continuous set of solutions—labeled by the compactification radius—which is not related to $\Lambda_{\rm eff}$.

While this scenario does not require any unnatural choices of the parameters in Eq. (4) to satisfy Eq. (16), this condition is not the unique solution to the equations of motion. Some further dynamical mechanism is still required that favors a bulk $\Lambda_{\rm eff}$ that obeys Eq. (16).

The picture that we have described allows solutions with a stable exponential electroweak Planck hierarchy without an unnatural choice of the parameters in the action. Since this picture crucially relies on the presence of R^2 terms in the action, one might worry whether it persists upon including further higher order R^n terms. However, in Ref. [4] we argued that such periodic solutions should exist generically for an action composed of general powers of the curvature tensors. In particular, the small cosmological constant case, $M_6^2\Lambda^{-1}\gg 1$, with $\Lambda_{\rm eff}=0$, admitted a semianalytic description of these periodic solutions. Including a small 5D cosmological constant does not greatly perturb these solutions, so that the physically interesting case in which the effective 5D theory of gravity is weak, should continue to exist even with higher-order terms in the underlying 6D theory.

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APPENDIX: EXAMPLES OF PERIODIC WARP FUNCTIONS

Since the existence of periodic metrics resulting from Eq. (19) can only be demonstrated numerically, it is helpful to present a couple of examples. For a small $\Lambda_{\rm eff}$, that is, $M_6 l\gg 1$, the shape of the warp function $A(y)$ and the parameter space in which periodic ones exist [Eq. (18)] are nearly the same as in the $\Lambda_{\rm eff}=0$ case. In Figs. 2 and 3, the AdS_5 length l that appears in the $\hat{g}_{MN}$ components of the metric, is the same in both cases, although the forms of $A(y)$ differ markedly in the two examples. Figure 2 shows a typical profile of a warp function for a small cosmological constant, $M_6^2\Lambda^{-1}\gg 1$, while Fig. 3 shows an example in which $M_6^2\Lambda^{-1}\sim 1$. In these numerical solutions it is more convenient to choose l rather than the brane tension, $\sigma^{(0)}$, however, we can accommodate an arbitrary $\sigma^{(0)}$ ($\lesssim\sqrt{|\Lambda|}$) by adjusting the 5D AdS length l which is a property of the solution and does not itself appear in the original 6D action.

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