Brane cosmologies without orbifolds

Hael Collins* and Bob Holdom†
Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7
(Received 4 April 2000; published 17 October 2000)

We study the dynamics of branes in configurations where (1) the brane is the edge of a single anti–de Sitter (AdS) space and (2) the brane is the surface of a vacuum bubble expanding into a Schwarzschild or AdS-Schwarzschild bulk. In both cases we find solutions that resemble the standard Robertson-Walker cosmologies, although, in the latter, the evolution can be controlled by a mass parameter in the bulk metric. We also include a term in the brane action for the scalar curvature. This term adds a contribution to the low-energy theory of gravity which does not need to affect the cosmology, but which is necessary for the surface of the vacuum bubble to recover four-dimensional gravity.

PACS number(s): 11.10.Kk, 04.50.+h, 98.80.Cq

I. INTRODUCTION

A remarkable feature in certain theories with more than the observed $3+1$ dimensions is that while these extra dimensions can extend infinitely, the geometry of the bulk space-time nevertheless is able to confine gravity to a three-dimensional surface within the larger space. Randall and Sundrum (RS) [1] first showed that by attaching two semi-infinite slices of $(4+1)$-dimensional anti–de Sitter space (AdS_5) along a three-dimensional hypersurface, or ''three-brane,'' with orbifold conditions about this three-brane, gravity behaves as though it is confined to its vicinity. This three-brane is identified with our universe. In addition to reproducing ordinary Newtonian gravity, any successful model should also be able to produce a realistic cosmological evolution for the three-brane. The dynamical evolution of the brane is determined by Einstein's equations for the combined bulk and brane system, but these equations might not produce the familiar Robertson-Walker cosmology along the brane. Viewed locally, near the brane the surrounding bulk introduces a new element into the field equations for gravity on the brane through a term for the change in the extrinsic curvature across the brane, as originally derived by Israel [2]. While generalizations of the original RS orbifold [1] have been shown to admit the usual open, flat, and closed Robertson-Walker cosmologies [3], we shall examine more asymmetric geometries for which the AdS curvature lengths on opposite sides of the brane are not necessarily equal. We shall treat in detail the case of a finite and spherical region of AdS space, including the case of a vacuum bubble that expands in an asymptotically flat $(4+1)$-dimensional space.

Most previous studies [3–5] of the dynamics of a brane have only included a surface tension term and a Lagrangian for the matter fields, which generally includes all the standard model fields, in the brane action. Yet without a more fundamental description of the physics that produces the brane, these terms should represent only the leading pieces of an effective action [6] that could include higher-order terms in a derivative expansion, such as a term for the scalar curvature on the brane, $\mathcal{R}$, and higher powers of curvature tensors on the brane, such as $\mathcal{R}^2$ and $\mathcal{R}_{ab}\mathcal{R}^{ab}$. Such terms generically are suppressed by extra powers of the AdS curvature length scale l, so at distances much larger than l we expect that these higher-order terms in the brane action can be neglected. However, at least one fine-tuning is typically made to obtain a vanishing cosmological constant on the brane by canceling the brane tension against a contribution from the bulk. After this fine-tuning is made, a scalar curvature term on the brane can be naturally of the same order as the terms that remain in the field equations for gravity on the brane. The importance of such a term increases when we consider universes very different from the original Randall-Sundrum scenario. For a vacuum bubble in a asymptotically flat bulk, this term is the sole source for four-dimensional gravity.

A brane action that contains powers of the brane curvature tensors has also been used in the context of the AdS/conformal field theory (CFT) correspondence [7] to regularize the action of a bulk AdS_{n+1} space which diverges when the radius of the AdS_{n+1} space becomes infinite [8–10]. Unlike the effective field theory description of the brane action, the requirement that the total action of the theory—the sum of the brane action and the bulk action—be finite in this limit precisely fixes the coefficients of the terms in the brane action. The coefficient of the brane tension gives the usual cancellation of the cosmological constant on the brane; however, we find that for the specific coefficient of the scalar curvature term on the brane, the brane curvature term cancels the leading-order effects coming from the bulk gravity. In light of the AdS/CFT correspondence, this result might be anticipated since the bulk gravitational theory is conjectured to be equivalent to a conformal quantum field theory—without gravity—on the surface.

In the next section, we derive the equations of motion for a bulk AdS_{n+1} space in which a hypersurface is embedded. In Sec. III these equations are used to study a dynamic brane on which the induced metric takes the standard Robertson-Walker form. The general equations for a nonorbifolded geometry including the effects of a scalar curvature term in the brane action are found. Since these equations are difficult to solve exactly, in Sec. IV four we neglect the scalar curvature term and focus on the expansion of a vacuum bubble in an asymptotically flat $(4+1)$-dimensional space-time. In Sec. V we include the scalar curvature in the brane action and

*Email address: hael@physics.utoronto.ca
†Email address: bob.holdom@utoronto.ca

study its effects when the brane is the edge of a single AdS space. Finally, we consider its effect on the vacuum bubble, before concluding in Sec. VI.

II. ACTION FOR AdS$_{n+1}$ WITH A BOUNDARY

We would like to derive the form of Einstein's equations on an n-dimensional hypersurface embedded in an $(n+1)$-dimensional bulk space-time. Later, we restrict ourselves to the interesting case where $n=4$. To be general, we shall treat the bulk space-time as two regions $\mathcal{M}_1$ and $\mathcal{M}_2$, separated by the hypersurface $\mathcal{B}$. Note that these bulk regions do not need to have the same metric on either side of the brane, but only need to satisfy the Israel conditions derived below. Since the boundary corresponds to the observed universe, we include an action on the brane containing, in addition to a surface tension term, a term for the scalar curvature on the brane plus the contributions from matter and gauge fields confined to the brane. At each point on the brane, we define a spacelike unit normal, $N_a=N_a(x)$, to the surface that satisfies $g^{ab}N_aN_b=1$. Here g_{ab} is the bulk metric and the indices a, b run over all the bulk coordinates. The bulk metric induces a metric on the brane,

$$h_{ab}=g_{ab}-N_aN_b; \tag{2.1}$$

while the bulk metric can be discontinuous across the brane, the induced metric on the brane should be the same whether calculated with the bulk metric for either region.

Combining all of these ingredients, the total action is the sum of the actions for the two bulk regions,[1]

$$\begin{aligned}S_1&=\frac{1}{16\pi G}\int_{\mathcal{M}_1}d^{n+1}x\sqrt{-g}\left[R+\frac{n(n-1)}{l_1^2}\right]\\&\quad-\frac{1}{8\pi G}\int_{\mathcal{B}}d^nx\sqrt{-h}K^{(1)},\\S_2&=\frac{1}{16\pi G}\int_{\mathcal{M}_2}d^{n+1}x\sqrt{-g}\left[R+\frac{n(n-1)}{l_2^2}\right]\\&\quad-\frac{1}{8\pi G}\int_{\mathcal{B}}d^nx\sqrt{-h}K^{(2)},\end{aligned} \tag{2.2}$$

and that of the boundary,

$$\begin{aligned}S_{\text{surf}}=\frac{1}{16\pi G}\int_{\mathcal{B}}d^nx\sqrt{-h}\bigg[&-\frac{2(n-1)}{l}\frac{\sigma}{\sigma_c}+b\frac{l}{n-2}\mathcal{R}\\&+16\pi G\mathcal{L}_{\text{fields}}+\cdots\bigg].\end{aligned} \tag{2.3}$$

Here G is the bulk Newton's constant and K is the trace of the extrinsic curvature K_{ab}, defined by

$$K_{ab}=h_a{}^c\nabla_cN_b. \tag{2.4}$$

σ, $\mathcal{R}$, and $\mathcal{L}_{\text{fields}}$ represent the brane tension, the scalar curvature of the *induced* metric, and the Lagrangian of fields confined to the brane. We normalize the brane tension with respect to a critical tension,[2] $\sigma_c=3/8\pi Gl$, as will be useful later, and we allow the two bulk regions to have potentially different curvature lengths l_1 or l_2. This action is a generalization of that which appears in [9] and [10]. From the vantage of writing an effective theory on the brane [6], we simply include the $\sqrt{-h}\mathcal{R}$ term as the next-to-leading term in the brane action in powers of derivatives. The coefficient of this term, $bl/(n-2)$, is determined by some underlying theory, so we leave it unspecified.

Varying the total action yields the usual Einstein equations in the bulk,

$$R_{ab}-\frac{1}{2}R_{ab}=\frac{n(n-1)}{l_{1,2}^2}g_{ab}, \tag{2.5}$$

where the appropriate AdS length is chosen for each region, plus the following equation for the surface:

$$\begin{aligned}\Delta K_{ab}-h_{ab}\Delta K=-\frac{n-1}{l}\frac{\sigma}{\sigma_c}h_{ab}-b\frac{l}{n-2}\left[\mathcal{R}_{ab}-\frac{1}{2}\mathcal{R}h_{ab}\right]\\+8\pi GT_{ab}+\cdots,\end{aligned} \tag{2.6}$$

where $\Delta K_{ab}\equiv K_{ab}^{(2)}-K_{ab}^{(1)}$, T_{ab} is the energy-momentum tensor for the fields confined to the brane,

$$T_{ab}\equiv h_{ab}\mathcal{L}_{\text{fields}}+2\frac{\delta\mathcal{L}_{\text{fields}}}{\delta h^{ab}}, \tag{2.7}$$

and $\mathcal{R}_{ab}$ is the Ricci tensor for the induced metric. Contracting both sides of Eq. (2.6) with h^{ab} and solving for $\Delta K=h^{ab}\Delta K_{ab}$ gives the Israel condition

$$\begin{aligned}\Delta K_{ab}=\frac{1}{l}\frac{\sigma}{\sigma_c}h_{ab}-b\frac{l}{n-2}\left[\mathcal{R}_{ab}-\frac{1}{2(n-1)}\mathcal{R}h_{ab}\right]\\+8\pi G\left[T_{ab}-\frac{1}{n-1}T_c^ch_{ab}\right]+\cdots,\end{aligned} \tag{2.8}$$

which describes the effect of the presence of the bulk space-time on the brane Einstein equations through the appearance of the extrinsic curvature term. A similar equation, although without the term arising from varying the scalar curvature in the brane action, has appeared in earlier studies of domain walls in four [11,19], five [3,4], and an arbitrary number of dimensions [20]. This term might seem unimportant at distances much larger than l, since it contains two more powers of l compared to the term with the brane tension. However, the contributions from $\sqrt{-h}\mathcal{R}$ can be of the same order as the difference between the brane tension and ΔK_{ab}, once the brane tension has been finely tuned.

---

[1] Our convention for the sign of the Riemann tensor is $-R^a{}_{bcd}\equiv\partial_d\Gamma^a_{bc}-\partial_c\Gamma^a_{bd}+\Gamma^a_{ed}\Gamma^e_{bc}-\Gamma^a_{ec}\Gamma^e_{bd}$.

[2] Note that since we are considering more general geometries than the orbifolds of [3], it is convenient to define the critical tension to be half that of the RS universe.

For comparison, in the original RS orbifold we only include the first term on the right-hand side of Eq. (2.8) and the extrinsic curvatures from the two sides are equal and opposite, $K_{ab} \equiv K_{ab}^{(1)} = -K_{ab}^{(2)}$. The bulk AdS_5 space gives $K_{ab} = -h_{ab}/l$, which yields the usual fine-tuning condition [1] for the brane tension, $\sigma = 2\sigma_c$.

III. COSMOLOGY ON THE BOUNDARY

We shall now set $n=4$ and examine some specific solutions of the field equations for gravity on a three-brane between two $(4+1)$-dimensional regions with negative cosmological constants. The metrics for the interior $r<R(\tau)$ and exterior $r>R(\tau)$ regions with respect to the brane can be written in the AdS_5-Schwarzschild form [12]:

$$\begin{aligned} ds^2|_{\rm int} &= -u(r)dt^2 + [u(r)]^{-1}dr^2 + r^2 d\Omega_3^2,\\ u(r) &= \frac{r^2}{l_1^2} + k - \frac{m_1}{r^2},\\ ds^2|_{\rm ext} &= -v(r)dt^2 + [v(r)]^{-1}dr^2 + r^2 d\Omega_3^2,\\ v(r) &= \frac{r^2}{l_2^2} + k - \frac{m_2}{r^2}. \end{aligned} \tag{3.1}$$

An AdS_5 bulk corresponds to setting $m_1 = m_2 = 0$. We have included the $-m_{1,2}/r^2$ terms in the metric since they can have an important effect on the brane cosmology. Their presence leads to black-hole horizons at some distance into the bulk whose masses are determined by m_1 and m_2 [12,13].

We shall frequently refer to the $k=1$ case, for which the brane is a three-sphere and a closed Robertson-Walker cosmology results, but we shall leave k in the expressions with the understanding that flat or open cosmologies can be obtained by setting $k=0$ or -1, respectively:

$$d\Omega_3^2 \equiv \begin{cases} d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2), & k=1,\\ l^{-2}(dx^2+dy^2+dz^2), & k=0,\\ d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2), & k=-1 \end{cases}. \tag{3.2}$$

We are looking for dynamical solutions, so we let the position of the brane be given by

$$(t,r,\chi,\theta,\phi) = (T(\tau),R(\tau),\chi,\theta,\phi), \tag{3.3}$$

where τ is the proper time for an observer at rest with respect to the brane. The normal to the brane is then

$$N_a = (-\dot{R},\dot{T},0,0,0), \tag{3.4}$$

with an overdot denoting differentiation with respect to τ. Since the normal has unit length, $g^{ab}N_aN_b = 1$, we can express $\dot{T}$ in terms of $\dot{R}$,

$$\dot{T} = \frac{[\dot{R}^2 + u(r)]^{1/2}}{u(r)}, \tag{3.5}$$

in the interior and with $u(r)$ replaced by $v(r)$ in the exterior bulk. With the normal in this form we find that the induced metric on the brane is already in the standard Robertson-Walker form; the metric induced from the interior bulk metric is

$$\begin{aligned} ds^2 &= -u(r)dt^2 + [u(r)]^{-1}dr^2 + R^2 d\Omega_3^2\\ &= -u(r)\{\dot{T}^2 - [u(r)]^{-2}\dot{R}^2\}d\tau^2 + R^2 d\Omega_3^2\\ &= -d\tau^2 + R^2(\tau)d\Omega_3^2 \equiv h_{\mu\nu}dx^\mu dx^\nu, \end{aligned} \tag{3.6}$$

where μ, ν run over the coordinates on the brane. The exterior region produces exactly the same induced metric.

In terms of the coordinate system defined by Eq. (3.6), the interior contribution to the extrinsic curvature is

$$K_{\mu\nu}^{(1)}dx^\mu dx^\nu = -\frac{1}{u(R(\tau))\dot{T}}\left[\ddot{R} + \frac{1}{2}\frac{\partial u}{\partial R}\right]d\tau^2 + u(R(\tau))\dot{T}R\, d\Omega_3^2, \tag{3.7}$$

with the exterior region contributing an analogous expression with $u(R(\tau))$ replaced with $v(R(\tau))$.

Let us consider the matter on the brane to be distributed as an isotropic perfect fluid, of density ρ and pressure p, for which the energy-momentum tensor is

$$T_\mu^\nu = \mathrm{diag}(-\rho,p,p,p). \tag{3.8}$$

In this case, the spatial components of Eq. (2.8) together with Eq. (3.5) yield

$$\sqrt{\dot{R}^2 + u(R)} \pm \sqrt{\dot{R}^2 + v(R)} = \frac{R}{l}\frac{\sigma}{\sigma_c} - \frac{bl}{2R}[\dot{R}^2 + k] + \frac{8\pi G}{3}R\rho + \cdots. \tag{3.9}$$

The temporal component of Eq. (2.8) does not give an independent equation once we have imposed the conservation of energy on the brane [3,4], which demands that

$$\frac{d}{d\tau}(\rho R^3) = -p\frac{d}{d\tau}R^3. \tag{3.10}$$

The choice of the relative sign between the extrinsic curvature terms in Eq. (3.9) depends on the geometry of the bulk AdS_5 space that surrounds the brane. In the original RS universe, the orbifold is made of two slices of AdS_5 space attached so that the warp factor—the r^2/l^2 in the AdS metric (3.1)—decreases as we move further from the brane in either direction. Thus, for the orbifold geometry, the plus sign is chosen. When the warp factor behaves differently on opposite sides of the brane, as for a brane simply embedded in a single bulk AdS_5 space, the minus sign is used.

For no scalar curvature term, $b=0$, the Israel condition (3.9) can be rewritten so that the evolution of $R(\tau)$ is determined by a potential,

$$\frac{1}{2}\dot{R}^2+V(R)=-\frac{1}{2}k, \tag{3.11}$$

where

$$V(R)=-\frac{1}{8}\frac{R^2}{l^2}\left\{\frac{(\sigma+\rho)^2}{\sigma_c^2}-2\left(\frac{l^2}{l_1^2}+\frac{l^2}{l_2^2}\right)+\frac{\sigma_c^2}{(\sigma+\rho)^2}\times\left(\frac{l^2}{l_1^2}-\frac{l^2}{l_2^2}\right)^2\right\}-\frac{1}{4}\frac{1}{R^2}\left\{m_1+m_2-\frac{\sigma_c^2}{(\sigma+\rho)^2}\times(m_1-m_2)\left(\frac{l^2}{l_1^2}-\frac{l^2}{l_2^2}\right)\right\}-\frac{1}{8}\frac{\sigma_c^2}{(\sigma+\rho)^2}\frac{l^2(m_1-m_2)^2}{R^6}. \tag{3.12}$$

A similar potential is implicit in [3]. For $R \gg l$ and for a generic tension, this potential does not produce a standard Robertson-Walker cosmology. However, when the brane tension is tuned to

$$\sigma=\pm\sigma_c\left|\frac{l}{l_1}\pm\frac{l}{l_2}\right|, \tag{3.13}$$

the leading R^2/l^2, ρ-independent term drops out of the potential. The appropriate signs in Eq. (3.13) depend on the behavior of the AdS space on either side of the brane.

The simplest example of a system that produces a realistic cosmology is an AdS_5 space that terminates on an ''edge of the universe'' three-brane, in the spirit of [14], with only a tension term in the brane action. This closely resembles the usual orbifold geometry [3] except that here the critical tension, $\sigma=\sigma_c$, is half of that needed for the orbifold. For $l_1=l$, $l_2\to\infty$ and $m_1=m$, $m_2=0$, the potential (3.12) becomes

$$V(R)=-\frac{1}{2}\frac{R^2}{l^2}\frac{1}{\sigma_c^2}[(\sigma+\rho)^2-\sigma_c^2]-\frac{m}{2R^2}. \tag{3.14}$$

Making the fine-tuning of the brane tension to its critical value,

$$\dot{R}^2+k=\frac{R^2}{l^2}\frac{2\rho}{\sigma_c}+\frac{R^2}{l^2}\frac{\rho^2}{\sigma_c^2}+\frac{m}{R^2}=\frac{16\pi G}{3l}\rho R^2+\frac{m}{R^2}+\cdots. \tag{3.15}$$

Here we have inserted the definition of σ_c and assumed $\rho/\sigma_c \ll 1$. For the standard Robertson-Walker universe, the dynamical equation that determines $R(\tau)$ is

$$\dot{R}^2+k=\frac{8\pi G_4}{3}\rho R^2, \tag{3.16}$$

where G_4 is the $(3+1)$-dimensional Newton's constant. Thus identifying $G_4=2G/l$, we recover the familiar cosmologies on the brane driven by the energy density on the brane, as long as m is not too large. A similar result was found in this edge of the universe picture in [15].

IV. VACUUM BUBBLE

When a bubble nucleates in a region having a vacuum energy higher than that in the bubble's interior, the bubble will expand or contract depending upon the surface tension of the bubble and the difference in the bulk vacuum energies. A simple example of this behavior occurs when a bubble of AdS_5 space is surrounded by an asymptotically flat region. The three-brane here is the surface of this bubble. The purpose of this section is to introduce this bubble as an example of an acceptable brane cosmology that is driven by one of the mass parameters in the bulk metric in a relatively simple setting. One obvious difficulty—that the model does not produce a 4D Newton's law—can be removed by adding a scalar curvature term to the brane action. Yet we shall first study the cosmology without this term since in this limit we can solve the behavior exactly and shall find that it is maintained when the brane curvature term is included.

For a bubble in a flat vacuum, the metrics for the interior and exterior regions are then respectively given by

$$u(R)=\frac{r^2}{l^2}+k-\frac{m_1}{r^2}, \quad v(R)=k-\frac{m_2}{r^2}. \tag{4.1}$$

Since $l_2\to\infty$, we have set $l_1=l$ without loss of generality. Then the cosmological evolution is determined by the function

$$V(R)=\frac{1}{8}\frac{R^2}{l^2}\left\{2-\frac{(\sigma+\rho)^2}{\sigma_c^2}-\frac{\sigma_c^2}{(\sigma+\rho)^2}\right\}-\frac{1}{4}\frac{1}{R^2}\times\left\{m_1+m_2-\frac{\sigma_c^2(m_1-m_2)}{(\sigma+\rho)^2}\right\}-\frac{1}{8}\frac{\sigma_c^2 l^2(m_1-m_2)^2}{(\sigma+\rho)^2R^6}. \tag{4.2}$$

Again, this potential does not to lead to a standard Robertson-Walker cosmology on the brane unless we set $\sigma=\sigma_c$. Expanding in the limit where the matter density is small compared to this critical tension, we have

$$V(R)=-\frac{1}{2}\frac{m_2}{R^2}-\frac{1}{8}\frac{l^2(m_2-m_1)^2}{R^6}+\cdots+\frac{1}{2}\frac{\rho}{\sigma_c}\times\left(\frac{m_2-m_1}{R^2}+\frac{1}{2}\frac{l^2(m_2-m_1)^2}{R^6}+\cdots\right)-\frac{1}{2}\frac{\rho^2}{\sigma_c^2}\times\left(\frac{R^2}{l^2}+\frac{3}{2}\frac{m_2-m_1}{R^2}+\frac{3}{4}\frac{l(m_2-m_1)^2}{R^6}+\cdots\right)+\cdots. \tag{4.3}$$

The potential for the vacuum bubble (4.3) does not contain a ρR^2 term, since the same fine-tuning that removes the cosmological constant from the brane also eliminates such a term. However, in the limit in which $R(\tau)\gg l$ and $\rho \ll \sigma_c$, the leading term that determines the cosmology on the surface of the expanding bubble is

$$\dot{R}^2+k=\frac{m_2}{R^2}+\cdots. \tag{4.4}$$

Although this equation seems quite different from Eq. (3.16), the time dependence of its solution is exactly the same as for a radiation-dominated universe in which $\rho \propto R^{-4}$. Notice that if we do not want the $\rho^2 R^2$ term to dominate, we should only consider sufficiently late times in the evolution when

$$\frac{\rho}{\sigma_c} \ll \frac{l}{R}. \tag{4.5}$$

V. EFFECTS OF THE SCALAR CURVATURE IN THE BRANE ACTION

A. Edge of the universe

We now study the effects of including a scalar curvature term for the induced metric in the brane action, $b \neq 0$. We first consider a bulk AdS_5 space that terminates on a three-brane. The Israel equation (3.9) in this case contains only one extrinsic curvature term

$$-[\dot{R}^2 + u(R)]^{1/2} = \frac{R}{l}\frac{\sigma+\rho}{\sigma_c} - b\frac{l}{2R}[\dot{R}^2 + k]. \tag{5.1}$$

Solving for $\dot{R}^2 + k$, we obtain the potential

$$V(R) = -\frac{R^2}{b^2 l^2}\left[1 + b\frac{\sigma+\rho}{\sigma_c} \pm \sqrt{1 + b^2 + 2b\frac{\sigma+\rho}{\sigma_c} - b^2\frac{ml^2}{R^4}}\right]. \tag{5.2}$$

At the critical brane tension, for the lower sign and assuming that we can expand the square root, we find that

$$V(R) = -\frac{1}{1+b}\frac{R^2}{l^2}\frac{\rho}{\sigma_c} - \frac{1}{2(1+b)}\frac{m}{R^2} - \frac{1}{2(1+b)^3}\frac{R^2}{l^2}\frac{\rho^2}{\sigma_c^2} + \cdots. \tag{5.3}$$

This time instead of Eq. (3.15) we have

$$\dot{R}^2 + k = \frac{1}{(1+b)}\left(\frac{16\pi G}{3l}\rho R^2 + \frac{m}{R^2}\right) + \cdots. \tag{5.4}$$

We obtain the standard Robertson-Walker evolution on the brane if we identify the four-dimensional Newton's constant with

$$G_4 = \frac{1}{1+b}\frac{2}{l}G. \tag{5.5}$$

We obtain the same result if we calculate the G_4 by considering variations about the background metric and then integrating over the extra dimension, as described in [16] and [1]. Since a nonzero m is not needed to produce the standard cosmology (5.3), we set it to zero while determining G_4. It is also convenient, rather than working with the coordinates of Eq. (3.1), to define a new radial coordinate through

$$e^{2\rho/l} = \frac{r^2}{l^2} + 1. \tag{5.6}$$

The AdS_5 metric then becomes

$$ds^2 = -e^{2\rho/l}dt^2 + \frac{e^{2\rho/l}}{e^{2\rho/l}-1}d\rho^2 + (e^{2\rho/l}-1)l^2\, d\Omega_3^2. \tag{5.7}$$

In the limit $\rho \gg l$, this metric reduces to the simpler form

$$ds^2 \approx e^{2\rho/l}(-dt^2 + l^2\, d\Omega_3^2) + d\rho^2. \tag{5.8}$$

If we replace the metric on the brane $-dt^2 + l^2\, d\Omega_3^2$ with a metric $\overline{g}_{\mu\nu}(x^\lambda)dx^\mu dx^\nu$ that only depends on the coordinates on the brane, then we find that the 5D scalar curvature is related to the 4D scalar curvature by

$$R_5 = -\frac{20}{l^2} + e^{-2\rho/l}\overline{R}_4 + \cdots. \tag{5.9}$$

Integrating over the AdS_5 region gives then the following term in the effective action:

$$\frac{1}{16\pi G_4}\int_{\mathcal{B}} d^4x\sqrt{-h}\overline{R}_4 \equiv \frac{1}{16\pi G}\int_{\mathcal{B}} d^4x\sqrt{-h}\frac{l}{2}\overline{R}_4. \tag{5.10}$$

Combining this effective brane curvature induced by the bulk zero mode with that included in the brane action gives

$$S_{\mathrm{eff}} = \frac{1}{16\pi G}\int_{\mathcal{B}} d^4x\sqrt{-h}\frac{l}{2}(\overline{R}_4 + b\mathcal{R}) + \cdots, \tag{5.11}$$

so that the effective four-dimensional Newton constant gets renormalized by the factor $1/(1+b)$. From the vantage of an effective field theory on the brane, for which b is determined by some unknown higher-energy theory, this result shows that we recover Newtonian gravity on the brane and at the same time the ability to generate a standard cosmological behavior on the brane, as long as $b > -1$. For comparison, in the standard orbifold picture, with $\sigma = 2\sigma_c$, the effective Newton constant on the brane is

$$G_4 = \frac{1}{2+b}\frac{2}{l}G. \tag{5.12}$$

For the special choice $b = -1$ when $\sigma = \sigma_c$, our effective action on the brane corresponds to the first two terms in the brane counterterm action of [9] and [10] which regularizes the bulk AdS action. The AdS/CFT conjecture [7] suggests that for this action, the theory of gravity in the AdS bulk is equivalent to a conformal field theory on the boundary, without gravity. Indeed, we find that for physical values for the matter density ($\rho \geq 0$) and for a positive mass parameter in the AdS-Schwarzschild metric ($m \geq 0$), we do not recover a realistic cosmological evolution on the brane. For $b = -1$ and $\sigma = \sigma_c$, the Israel equation (3.9) yields a complex potential for $\dot{R}^2 + k$,

$$V(R)=\frac{R^2}{l^2}\left[\frac{\rho}{\sigma_c}\pm\sqrt{-2\frac{\rho}{\sigma_c}-\frac{ml^2}{R^4}}\right],\tag{5.13}$$

so we no longer obtain the ordinary cosmological solutions.

B. Vacuum bubble

When a brane is embedded between arbitrary bulk AdS-Schwarzschild spaces and a scalar curvature term is included in the brane action, the Israel equation (3.9) is a quartic polynomial in $(\dot{R}^2+k)$ which becomes tractable only for special space-time geometries or in the $R\gg l$ limit. Returning to the case of a vacuum bubble expanding into an asymptotically flat region, with $u(r)$ and $v(r)$ as in Eq. (4.1), we find the following leading behavior in the $l/R\ll 1$ limit:

$$\begin{aligned}V(R)\approx&-\frac{1}{2}\frac{m}{R^2}-\frac{1}{8}\frac{l^2}{R^2}\frac{m^2}{R^4}(b+1)^2+\frac{1}{2}\frac{m}{R^2}\frac{\rho}{\sigma_c}(b+1)+\cdots\\&-\frac{1}{2}\frac{\rho^2}{\sigma_c^2}\left(\frac{R^2}{l^2}+\frac{3}{2}\frac{m}{R^2}(b+1)^2+\cdots\right)+\cdots.\end{aligned}\tag{5.14}$$

Here, for simplicity, we have set $m_2=m$ and $m_1=0$. Notice that the presence of a brane curvature term has not generated a ρR^2 term. Therefore we still require that the cosmology be driven by the m/R^2 term in order to obtain the same time evolution as in a radiation-dominated universe. For the new b-dependent terms not to overwhelm the m/R^2 term, we must impose $b\rho/\sigma_c\ll 1$. Comparing with the condition already imposed by Eq. (4.5)—that the m/R^2 term and not the ρ^2R^2 term should drive the cosmology—we see that we can accommodate a *bl* up to cosmological scales without imposing any new constraint.

A curvature term in the brane action plays a crucial role in the vacuum bubble scenario since it produces a 4D Newton's law for distances along the brane smaller than *bl* [17]. A similar result is also found in [18] for a brane embedded with a flat bulk on both sides. As we just have seen, *bl* can be large without affecting the cosmology. One unpleasant feature of this example is that while the correct Newton's law is obtained, the effective 4D Einstein equation contains a term for a scalar graviton [17].

As a more realistic variation, consider a vacuum bubble that expands into another AdS_5-Schwarzschild region, rather than a flat bulk. Unlike the standard Randall-Sundrum picture, we shall let the second AdS length l_2 have a large macroscopic size, but which is yet much smaller than the length associated with the brane curvature: $l_1,l\ll l_2\ll bl$. The leading behavior of the cosmology (3.9) for this universe is then governed by

$$\begin{aligned}V(R)=&-\frac{1}{2}\frac{1}{b}\frac{l_2}{l_1}\frac{m_2}{R^2}-\frac{1}{2}\frac{1}{b}\frac{m_1}{R^2}+\cdots-\frac{\rho}{\sigma_c}\\&\times\left(\frac{1}{b}\frac{R^2}{l^2}-\frac{1}{b^2}\frac{l_2}{l}\frac{R^2}{l^2}+\cdots\right)+\frac{\rho^2}{\sigma_c^2}\left(\frac{3}{2}\frac{1}{b}\frac{l_1}{l}\frac{R^2}{l^2}+\cdots\right)\\&+\cdots.\end{aligned}\tag{5.15}$$

Unlike Eq. (5.14), the ρR^2 term is again present:

$$\dot{R}^2+k=\frac{8\pi G}{3}\frac{2}{bl}\rho R^2+\frac{1}{b}\frac{l_2}{l_1}\frac{m_2}{R^2}+\cdots.\tag{5.16}$$

Provided m_2 is not too large, we recover a standard Robertson-Walker cosmology with an effective 4D Newton's constant, $G_4=2G/bl$. What has happened for this bubble is that above the AdS lengths we expect that the bulk space produces an effectively 4D theory of gravity [16]. Since we have assumed that $l_1,l_2\ll bl$, when we probe distances below l_1,l_2 we do not observe the extra dimensions of the bulk space since we are in the regime in which the effect of the brane curvature term dominates. This argument is borne out in [17] where it is shown that the effective theory of gravity on the surface is governed by a 4D Einstein equation at all scales when $l_1,l_2\ll bl$.

VI. CONCLUSIONS

We have found that, in general, the inclusion of a scalar curvature term in the brane action still allows us to find the standard Robertson-Walker cosmologies for the evolution of the brane. This standard behavior emerges once the size of the universe has grown large in comparison to the AdS length of the bulk space and provided that the usual fine-tuning of the effective cosmological constant on the brane to zero has been made. When the AdS lengths are small, the presence of this brane scalar curvature term simply acts to renormalize the effective Newton's constant on the brane. In the case of an ''end of the universe'' brane, the brane curvature does not affect the cosmology, except when $b=-1$.

We have explored physically intuitive brane universes in which the bulk does not have an orbifold symmetry. In the case of a vacuum bubble expanding into an asymptotically flat space, we encountered an intriguing example of a system in which the existence of the bulk is crucial for the correct cosmological evolution since the ρR^2 term that usually produces a Robertson-Walker cosmology is absent. Instead, the cosmology, which has the same time dependence as a radiation-dominated universe, is driven by a mass term in the bulk Schwarzschild metric. A scalar curvature in the brane action plays a more important role here since it provides the only possible source for 4D gravity up to a scalar graviton. We also examined a variation in which the bubble lies between two regions with potentially very different cosmological constants. For such a bubble, it is possible to recover a completely standard Robertson-Walker cosmology without constraining the bulk AdS lengths to be below a millimeter scale, provided that the brane curvature term is sufficiently strong. These examples should encourage the search for novel extra-dimensional models in which the bulk effects are not small corrections to the standard cosmology, but rather drive its evolution.

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[2] W. Israel, Nuovo Cimento Soc. Ital. Fis., B **44**, 1 (1966).
[3] P. Kraus, J. High Energy Phys. **12**, 011 (1999).
[4] J. Garriga and M. Sasaki, Phys. Rev. D **62**, 043523 (2000).
[5] C. Csaki, M. Graesser, C. Kolda, and J. Terning, Phys. Lett. B **462**, 34 (1999); C. Csaki, M. Graesser, L. Randall, and J. Terning, Phys. Rev. D **62**, 045015 (2000); J. M. Cline, C. Grojean, and G. Servant, Phys. Rev. Lett. **83**, 4245 (1999); E. E. Flanagan, S. H. Tye, and I. Wasserman, Phys. Rev. D **62**, 044039 (2000); P. Binetruy, C. Deffayet, and D. Langlols, Nucl. Phys. **B565**, 269 (2000); T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).
[6] R. Sundrum, Phys. Rev. D **59**, 085009 (1999).
[7] For a review, see O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
[8] M. Henningson and K. Skenderis, J. High Energy Phys. **07**, 023 (1998).
[9] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999).
[10] R. Emparan, C. V. Johnson, and R. C. Myers, Phys. Rev. D **60**, 104001 (1999).
[11] J. Ipser and P. Sikivie, Phys. Rev. D **30**, 712 (1984).
[12] D. Birmingham, Class. Quantum Grav. **16**, 1197 (1999).
[13] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).
[14] P. Horava and E. Witten, Nucl. Phys. **B475**, 94 (1996).
[15] S. S. Gubser, ''AdS/CFT and gravity,'' hep-th/9912001.
[16] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
[17] H. Collins and B. Holdom, Phys. Rev. D (to be published), hep-th/0006158.
[18] G. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000).
[19] M. Cvetic, S. Griffies, and H. H. Soleng, Phys. Rev. D **48**, 2613 (1993).
[20] M. Cvetic and J. Wang, Phys. Rev. D **61**, 124020 (2000); M. Cvetic and H. H. Soleng, Phys. Rep. **282**, 159 (1997).