# A little large $N$ group theory[*]

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## Abstract

We discuss the group theory relevant to the ground-state baryons in large $N_c$ QCD. For very large representation, the group generators become classical variables. We find the form of the classical generators for the completely symmetric $N$ index representation of $SU(m)$ as $N \rightarrow \infty$ and derive an integral formula for the matrix elements of an arbitrary polynomial in the group generators between low-spin baryon states in the large $N$ limit.

The idea of replacing the $SU(3)$ gauge symmetry of QCD with an $SU(N)$ symmetry and studying the $N \rightarrow \infty$ limit, as articulated by 't Hooft in [1], has led to an important qualitative understanding of some of the properties of QCD, such as Zweig's rule and the narrowness of resonances. Witten [2] later provided a conceptual framework to include baryons in the theory. In Witten's picture, a baryon is described by a Hartree-Fock equation with each quark moving in the mean potential generated by all the other quarks in the baryon. Using this description for the baryons, [3–5] (see also [6]) one can show that to leading order in $N$, the low-spin baryons have a spin-flavor symmetry which we will denote as $SU(m)$, e.g. for an $N_f$-flavor theory $m = 2N_f$. In this letter, we study the group theory associated with the large representations of this $SU(m)$. In doing so, we develop an elegant integral formula for matrix elements of $SU(m)$ generators between low-spin baryon states. This formula provides an insight into the nature of the large $N$ enhancement of these matrix elements.

An $N$ quark baryon lives in the completely antisymmetric representation of the color $SU(N)$. An $s$-wave ground state, to satisfy fermi statistics, must then be completely symmetric under the $SU(m)$ symmetry. Therefore, the representation of $SU(m)$ relevant to large $N$ baryons is the completely symmetric combination of $N$ defining representations. We will examine the matrix elements of the group generators in this representation, which we will denote by $T^\alpha_\beta$, to leading order in $N$, and will discover that many properties of the generators can be obtained rather simply in this limit.

Our primary tool will be the fact that for large $N$, the commutator of two group generators is lower order in $N$ than the product. The product is order $N^2$, while the commutator is order $N$ (times the structure constants, which depend on $m$ but not on $N$). Thus to leading order in $N$, the generators can be simultaneously diagonalized – they become essentially classical variables [6,3,7].

Explicitly calculating the traces in the representation space of products of generators and taking the $N \to \infty$ limit leads to the following form for the generators: [1]

$$T_\beta^\alpha |u\rangle = T_\beta^\alpha(u) |u\rangle, \quad T_\beta^\alpha(u) = N \left( u^\alpha \bar{u}_\beta - \frac{1}{m} \delta_\beta^\alpha \right) \tag{1}$$

where $u$ is an $m$ component complex unit vector and $\bar{u}_\beta \equiv u^{\beta *}$. Notice that this form for the generators has a single non-degenerate eigenvalue $((m-1)/m)$ and $(m-1)$ degenerate eigenalues ($-1$ for the eigenvectors orthogonal to $u$) and that moreover the overall phase of $u$ does not matter. The generators in the large $N$ limit belong to the classical coset space,

$$SU(m)/[SU(m-1) \times U(1)] \cong \mathbf{CP}^{m-1} \tag{2}$$

but (1) is stronger than (2), because the normalization is also fixed. For an $SU(2)$ symmetry, this limit reproduces the familiar statement that angular momentum becomes a classical vector when the spin grows very large, since then $SU(2)/U(1)$ is a two-sphere that labels the direction of the classical angular momentum.

We show in the following section how the form for the generators, (1), naturally arises in the $m$ dimensional harmonic oscillator. The harmonic oscillator energy eigenstate live in precisely the same representation of $SU(m)$ as do the ground state baryons in large $N$ QCD. In Section 2, we find the eigenstates of the group generators, and in Section 3, we use these eigenstates to find an integral formula for the leading contribution to matrix elements of an arbitrary polynomial in the generators between low-spin states.

## 1. $m$ dimensional harmonic oscillator

We begin by analyzing the large $N$ limit of the $m$-dimensional harmonic oscillator. In addition to providing a familiar system which has a well-known classical limit, this example will prove useful later when we attempt to construct the low-spin baryons. In the large $N$ limit of QCD, the important behavior of the quarks comprising a baryon can be represented by *colorless, bosonic* creation and annihilation operators, as has been applied in the "quark representation" of [3] and in the analysis of [5]. We will use the energy eigenstates of the harmonic oscillator as an intermediate step bridging between the baryon states and the $SU(m)$ generator eigenstates.

The $m$-dimensional harmonic oscillator corresponds to the Hamiltonian (setting the spring constant and the mass to 1)

$$H = \tfrac{1}{2} \left( x^2 + p^2 \right) = \sum_\alpha a^{\alpha \dagger} a_\alpha + \frac{m}{2} \tag{3}$$

where $a^{\alpha \dagger}$ and $a_\beta$ represent the raising and lowering operators

$$a_\alpha = \frac{x^\alpha - i p^\alpha}{\sqrt{2}}, \quad a^{\alpha \dagger} = \frac{x^\alpha + i p^\alpha}{\sqrt{2}}. \tag{4}$$

Besides the energy, $H$, the classical harmonic oscillator has two additional conserved quanities: the angular momentum,

$$M^{\alpha\beta} = x^\alpha p^\beta - x^\beta p^\alpha \tag{5}$$

and the traceless symmetric tensor

$$S^{\alpha\beta} = x^\alpha x^\beta + p^\alpha p^\beta - \frac{2}{m} \delta^{\alpha\beta} H. \tag{6}$$

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[1] We will present a detailed proof for this result in a future article. The same result also emerges in the geometric quantization of $SU(m)$. [8]

These conserved tensors correspond, respectively to the antisymmetric and the symmetric parts of the $SU(m)$ generators,

$$T^\alpha_\beta = a^{\alpha\,\dagger} a_\beta - \frac{1}{m}\delta^{\alpha\beta} a^{\gamma\,\dagger} a_\gamma = \tfrac{1}{2} S^{\alpha\beta} - \frac{i}{2} M^{\alpha\beta}. \tag{7}$$

The raising operators are tensor operators transforming under $SU(m)$ like the defining $m$ dimensional representation. The quantum states are constructed by acting on the ground state with a product of raising operators. Since the raising operators commute, a state with $N$ raising operators transforms as the completely symmetric $SU(m)$ tensor with $N$ indices – the same representation in which arose for the large $N$ baryons. We would expect that for large $N$, the conserved quantities would approach their classical values. Let us show that this happens.

The most general classical motion of the $m$ dimensional harmonic oscillator is

$$x(t) = v_1 \cos(t) + v_2 \sin(t), \quad p(t) = -v_1 \sin(t) + v_2 \cos(t). \tag{8}$$

The real vectors $v_1$ and $v_2$ satisfy

$$v_1{}^2 + v_2{}^2 = x^2 + p^2 = 2E. \tag{9}$$

In the state with principle quantum number $N$, the energy is $N + m/2$, or approximately $N$ for $N$ large. Thus, in the large $N$ limit,

$$v_1{}^2 + v_2{}^2 = 2N. \tag{10}$$

The traceless tensor $S^{\alpha\beta}$ becomes

$$v_1^\alpha v_1^\beta + v_2^\alpha v_2^\beta - \frac{1}{m}\delta^{\alpha\beta}\left(v_1{}^2 + v_2{}^2\right) \tag{11}$$

while the angular momentum tensor becomes

$$v_1^\alpha v_2^\beta - v_2^\alpha v_1^\beta. \tag{12}$$

Now introduce the complex vector

$$\sqrt{2N}\,u = v_1 + i v_2 \tag{13}$$

so that

$$u^\dagger u = 1. \tag{14}$$

In terms of this complex vector $u$, we discover that

$$T^\alpha_\beta = \tfrac{1}{2} S^{\alpha\beta} - \frac{i}{2} M^{\alpha\beta} = N\left(u^\alpha u^{\beta*} - \frac{1}{m}\delta^{\alpha\beta}\right) \tag{15}$$

which is just (1).

## 2. The eigenstates of the generators

Our ultimate goal is to describe the leading $N$ behavior of matrix elements of the $SU(m)$ generators between low spin baryon external states. The simplest states, however, with which to describe the action of the generators

are the eigenstates introduced in Eq. (1) since on these eigenstates the group generators act diagonally. To connect these two pictures, we must learn how to express the baryons in terms of these generators. Since the set of eigenstates is complete, it should be possible to write any state, such as that for a baryon or for a harmonic oscillator energy eigenstate, as some distribution over the space labeling the eigenstates of the generators. Conceptually, it is simplest to proceed in two steps – first to express the harmonic oscillator energy eigenstates in terms of the $|u\rangle$'s and then to write baryons in terms of these energy eigenstates.

To begin, we choose a system of coordinates to describe an eigenstate of the $SU(m)$ generators. Up to an overall phase, to each generator corresponds a unique unit vector, $u$; we shall use this arbitrariness of phase to set the $m^{\text{th}}$ component of $u$ to be purely real. The remaining components of $u$ can be expressed in terms of $(m-1)$ positive numbers $w_\alpha$ and corresponding phases $\theta_\alpha$. Thus the vector $u(w, \boldsymbol{\theta})$ becomes

$$u_\alpha = e^{i\theta_\alpha} \sqrt{w_\alpha} \quad [\alpha = 1, \ldots, m-1], \qquad u_m = \sqrt{1 - \sum_\alpha w_\alpha}. \tag{16}$$

Since these coordinates are continuous, the eigenstates require a continuum normalization,

$$\langle u(w', \boldsymbol{\theta}') | u(w, \boldsymbol{\theta}) \rangle = \prod_{\alpha=1}^{m-1} \delta(w'_\alpha - w_\alpha) \, \delta(\theta'_\alpha - \theta_\alpha) \tag{17}$$

which has the correct $\delta(w' - w)$ dependence to ensure that the measure is $SU(m)$ invariant.

Having chosen this coordinate system, we next establish the connection between the harmonic oscillator energy eigenstates and the $u(w, \boldsymbol{\theta})$ eigenstates. The oscillator has correctly normalized eigenstates of the form

$$|n\rangle \equiv \left( \prod_{\alpha=1}^{m} \frac{[a^\dagger_\alpha]^{n_\alpha}}{\sqrt{n_\alpha!}} \right) |0\rangle \tag{18}$$

where $N = \sum_\alpha n_\alpha$ goes to infinity. Since the eigenstates are complete, we can generally write $|n\rangle$ as some distribution over the space of eigenstates $|u(w, \boldsymbol{\theta})\rangle$, specified by a "spectral function" $C(\boldsymbol{n}, w, \boldsymbol{\theta})$:

$$|n\rangle = \int dw d\theta \, C(\boldsymbol{n}, w, \boldsymbol{\theta}) \, |u(w, \boldsymbol{\theta})\rangle + \mathcal{O}(1/N). \tag{19}$$

Acting upon both sides of this expansion with the group generators the group generators, $T^\alpha_\beta$, determines the behavior of the function $C(\boldsymbol{n}, w, \boldsymbol{\theta})$:

$$\left( a^{\alpha \dagger} a_\beta - \frac{1}{m} \delta^\alpha_\beta a^{\gamma \dagger} a_\gamma \right) |n\rangle = N \int dw d\theta \left( u^\alpha \bar{u}_\beta - \frac{1}{m} \delta^\alpha_\beta \right) C(\boldsymbol{n}, w, \boldsymbol{\theta}) \, |u(w, \boldsymbol{\theta})\rangle + \mathcal{O}(1/N). \tag{20}$$

For example, for a diagonal element $T^\alpha_\alpha$ (with a fixed $\alpha \leqslant m-1$), this equation gives

$$(n_\alpha/N - 1/m) |n\rangle = \int dw d\theta \, (w_\alpha - 1/m) \, C(\boldsymbol{n}, w, \boldsymbol{\theta}) \, |u(w, \boldsymbol{\theta})\rangle + \mathcal{O}(1/N) \tag{21}$$

which in turn implies

$$(w_\alpha - n_\alpha/N) \, C(\boldsymbol{n}, w, \boldsymbol{\theta}) \approx 0 \tag{22}$$

suggesting that the function $C(\boldsymbol{n}, w, \boldsymbol{\theta})$ peaks sharply near $w_\alpha = n_\alpha/N$.

Looking at the action of off-diagonal elements of the generators further constrains $C(\boldsymbol{n}, w, \boldsymbol{\theta})$ to be of the form

$$C(\boldsymbol{n}, w, \boldsymbol{\theta}) \approx e^{i\boldsymbol{\theta} \cdot \boldsymbol{n}} \, h(w - n/N). \tag{23}$$

Here, $h(w - n/N)$ describes a function narrowly peaked at the origin and which satisfies the condition,

$$\int dw \, |h(w - n/N)|^2 = \left(\frac{1}{2\pi}\right)^{m-1} \tag{24}$$

to be consistent with the normalization of the harmonic oscillator states. The detailed form of the function $h$ will not matter. Note that the vectors $\boldsymbol{n}$, $\boldsymbol{w}$, and $\boldsymbol{\theta}$ are all $(m-1)$ dimensional.

Having determined the form of the function $C(\boldsymbol{n}, \boldsymbol{w}, \boldsymbol{\theta})$ in Eq. (23), we can compute the matrix element between two harmonic oscillator energy eigenstates of any finite degree polynomial of the generators, $F(T)$ using

$$\langle n_1, n_2, \ldots | F(T) | n_1 + m_1, n_2 + m_2, \ldots \rangle = \left(\frac{1}{2\pi}\right)^{m-1} \int d\theta \, e^{i\theta \cdot m} F\big(T(w, \theta)\big) \tag{25}$$

where the $n_\alpha \to \infty$ with the $m_\alpha$ and $w_\alpha = n_\alpha/N$ fixed. Roughly speaking, the magnitudes, $w_\alpha$, record the gross pattern of the $n_\alpha$'s, but because the $n_\alpha$'s get divided by $N$ to give $w_\alpha$, this information is not precise enough enough to distinguish $n_\alpha$ from $n_\alpha + 1$ or $n_\alpha + 2$, etc. Those distinctions are produced by the $\theta_\alpha$ dependence, which records the fine details of the states in a kind of holographic fashion.

## 3. Low spin states

The states that interest us for the application to large $N_c$ baryons are the low spin states. For the remainder of this letter, we restrict to an $SU(4)$ spin-flavor symmetry appropriate for baryons built only from $u$ and $d$ quarks. As described in [3] and [5], since the baryons always have exactly the same antisymmetric behavior in color, we can equivalently examine a baryonic state as though built from bosonic, colorless quarks which have the have $SU(m)$ group theoretical properties.

We specify our representation for the quarks as follows:

$a_1^\dagger$ creates a $u$ quark with spin up,      $a_2^\dagger$ creates a $u$ quark with spin down,

$a_3^\dagger$ creates a $d$ quark with spin up,      $a_4^\dagger$ creates a $d$ quark with spin down. $\qquad$ (26)

This assignment has the structure of a tensor product of the two dimensional flavor space and the two dimensional spin space, and we can take the spin and isospin generators to be half the usual Pauli matrices in the appropriate space.

In a low-spin large $N$ baryon, there are a few valence quarks, but most of the quarks are combined in spin-0 pairs. If there are only two quark flavors, these spin-0 pairs also have isospin 0 (this is what makes the two-flavor case so much simpler that three flavors). Thus up to a normalization, a low spin baryon corresponds to the following combination of bosonic creation operators:

$$|N, k_\alpha\rangle \propto z^{\dagger(N-k)/2} \, a_1^{\dagger k_1} \, a_2^{\dagger k_2} \, a_3^{\dagger k_3} \, a_4^{\dagger k_4} \, |0\rangle \tag{27}$$

where the number of valence quarks, $\tilde{k} = \sum_\alpha k_\alpha$, is much smaller than $N$ and finite as $N \to \infty$, and the operator $z^\dagger$

$$z^\dagger \equiv a_1^\dagger a_4^\dagger - a_2^\dagger a_3^\dagger \tag{28}$$

creates a spin and isospin zero pair. We would like to write $|N, k_\alpha\rangle$ as a sum of harmonic oscillator eigenstates, since from the previous section, we learned how to express them in terms of the eigenstates of the $SU(m)$

generators. We therefore expand $z^{\dagger(N-\tilde{k})/2}$, via the binomial theorem, and use Eq. (18) to express the expansion in oscillator eigenstates: $[n \equiv N - \tilde{k}]$

$$
|N, k_\alpha\rangle \propto \sum_{j=0}^{n/2} (-1)^j \frac{(n/2)!}{j!\,(n/2-j)!} \sqrt{(n/2-j+k_1)!\,(j+k_2)!\,(j+k_3)!\,(n/2-j+k_4)!}
$$
$$
\times |n/2 - j + k_1,\ j + k_2,\ j + k_3,\ n/2 - j + k_4\rangle. \tag{29}
$$

In a typical term, both $n/2 - j$ and $j$ are some finite fraction of $N$ – implicitly much larger than any of the $k_\alpha$'s for low spin baryons – so to leading order this expansion simplifies to

$$
|N, k_\alpha\rangle \propto (n/2)! \sum_{j=0}^{n/2} (-1)^j (n/2 - j)^{(k_1+k_4)/2}\, j^{(k_2+k_3)/2}
$$
$$
\times |n/2 - j + k_1,\ j + k_2,\ j + k_3,\ n/2 - j + k_4\rangle. \tag{30}
$$

To obtain the matrix elements of the $SU(m)$ generators between low spin baryons, we can simplify the calculation by directly evaluating such a matrix element and then determining the normalization. The matrix element for a general polynomial $F(T)$ of the group generators between two baryons specified by the valance numbers $k'_\alpha$ and $k_\alpha$ is

$$
\langle N, k'_\alpha | F(T) | N, k_\alpha\rangle \propto \sum_{j,\ell=0}^{n/2} (-1)^{j-\ell} (n/2 - j)^{(k_1+k_4)/2}\, j^{(k_2+k_3)/2} (n/2 - \ell)^{(k'_1+k'_4)/2}\, \ell^{(k'_2+k'_3)/2}
$$
$$
\times \langle \tfrac{n}{2} - \ell + k'_1, \ell + k'_2, \ell + k'_3, \tfrac{n}{2} - \ell + k'_4 | F(T) | \tfrac{n}{2} - j + k_1, j + k_2, j + k_3, \tfrac{n}{2} - j + k_4\rangle \tag{31}
$$

In terms of the coordinates of the generators (using Eqs. (19) and (23) and neglecting the $\mathcal{O}(1/N)$ terms), this matrix element becomes

$$
\langle N, k'_\alpha | F(T) | N, k_\alpha\rangle \approx \int dw\, d\theta \sum_{j,\ell=0}^{n/2} (-1)^{j-\ell}\, e^{i(j-\ell)(\theta_2+\theta_3-\theta_1)}\, e^{i\theta\cdot(k-k')}\, e^{i\theta_1(\tilde{k}'-\tilde{k})/2}
$$
$$
\times (n/2 - j)^{(k_1+k_4)/2}\, j^{(k_2+k_3)/2} (n/2 - \ell)^{(k'_1+k'_4)/2}\, \ell^{(k'_2+k'_3)/2} h(w_1 - 1/2 + \ell/N,\ w_2 - \ell/N,\ w_3 - \ell/N)^*
$$
$$
\times F\big(\mathcal{T}(w,\theta)\big)\, h(w_1 - 1/2 + j/N,\ w_2 - j/N,\ w_3 - j/N). \tag{32}
$$

In the limit $N \to \infty$, the sum will become an integral. Since the function $h$ sharply peaks at the origin, it imposes that $j/N \approx \ell/N$, which we will call $x/2 \equiv j/N$, as well as that

$$
w_1 \approx w_4 \approx (1 - x)/2, \quad w_2 \approx w_3 \approx x/2. \tag{33}
$$

The two sums over $j$ and $\ell$ can then be reorganized into an integral over $x$ (from 0 to 1) and a sum over $j - \ell$ which becomes

$$
\sum_{j-\ell} (-1)^{j-\ell} e^{i(j-\ell)(\theta_2+\theta_3-\theta_1)} = \sum_{j-\ell} e^{i(j-\ell)(\theta_2+\theta_3-\theta_1-\pi)} \approx \delta(\theta_2 + \theta_3 - \theta_1 - \pi). \tag{34}
$$

Therefore, in the large $N$ limit, the matrix element becomes

$$\langle N, k'_\alpha | F(T) | N, k_\alpha \rangle = A_k A_{k'} \int d\theta_1 \, d\theta_2 \, d\theta_3 \int_0^1 dx \, e^{i\theta \cdot (k-k')} \, e^{i\theta_1 (\tilde{k}' - \tilde{k})/2} \, \delta(\theta_2 + \theta_3 - \theta_1 + \pi)$$

$$\times \, (1-x)^{(k_1+k_4+k'_1+k'_4)/2} x^{(k_2+k_3+k'_2+k'_3)/2} F\big(\mathcal{T}(w,\theta)\big) \qquad (35)$$

with

$$w_1 = w_4 = (1-x)/2, \quad w_2 = w_3 = x/2. \qquad (36)$$

The remaining normalization factor, $A_k$, is fixed by evaluating the integral for $F(T) = 1$ to be

$$A_k = \frac{1}{2\pi} \sqrt{\frac{(1 + \tilde{k})!}{(k_1 + k_4)! \, (k_2 + k_3)!}}. \qquad (37)$$

An important feature of the integral for the matrix element in Eq. (35) is how it depends on the external baryons and in particular on the "valance numbers" $k_\alpha$ and $k'_\alpha$. This dependence occurs in both the $\theta$ terms and in the $(1-x)^{(k_1+k_4+k'_1+k'_4)/2} \cdot x^{(k_2+k_3+k'_2+k'_3)/2}$ factor which arose from the normalization of the harmonic oscillator states. Because of Bose-Einstein statistics, the large $N$ states depend strongly on the few indices that are not combined into spin and isospin zero combinations. This is the essential physics of the order $N$ matrix elements of $T^\alpha_\beta$ between low spin states.

Notice that upon setting $\theta_2 + \theta_3 - \theta_1 = \pi$, as in (34), the linear combinations corresponding to the pure spin or the pure isospin generators vanish identically. This observation amounts to the familiar statement that the matrix elements of pure spin or isospin operators between low spin, large $N$ baryons are only of order $\mathcal{O}(1)$, since the integral captures only the leading behavior in $N$.

### 3.1. Some sample calculations

The integral formula in Eq. (35) provides a simple means for extracting the leading $N$ features for the low spin baryon matrix elements. We will show this simplicity with three examples. To begin, let us calculate the matrix element

$$\langle p, 1/2 | \sigma_3 \tau_3 | p, 1/2 \rangle. \qquad (38)$$

The spin up proton corresponds to $k_1 = 1$ with $k_2 = k_3 = k_4 = 0$ while the operator $\sigma_3 \tau_3$ corresponds to the following linear combination

$$T^{11} + T^{44} - T^{22} - T^{33} \qquad (39)$$

which becomes $(1 - 2x)$, with no theta dependence. Thus

$$\langle p, 1/2 | \sigma_3 \tau_3 | p, 1/2 \rangle = \frac{N}{4\pi^2} \frac{2!}{1! \, 0!} \int_0^1 dx \, (1-x) \, (1-2x) \int d\theta_1 d\theta_2 d\theta_3 \, \delta(\theta_2 + \theta_3 - \theta_1 - \pi) = N/3. \qquad (40)$$

This result is correct, for the exact answer is

$$(N+2)/3. \qquad (41)$$

In fact, this factor is general – the exact large $N$ result for any matrix element between spin $1/2$ states is simply the $N = 1$ result multiplied by $(N+2)/3$. For $N = 3$, this gives the famous factor of $5/3$ for the

renormalization of the axial vector current. Eq. (41) (and (46) and (49) below) can be easily obtained by explicitly constucting the low spin states out of colorless commuting "quark" creation operators, using (27) and (28). One can then derive recursion relations relating the matrix elements for for different values of $N$, and solve them to obtain these exact results. But if we only need the leading contributions, the integral formula (35) captures them all in a much simpler way.

The spin $+3/2$ $\Delta^{++}$ state corresponds to $k_1 = 3$, and the matrix $\sigma_1\tau_1$ corresponds to

$$T^{14} + T^{23} + T^{32} + T^{41}. \tag{42}$$

In the matrix element

$$\langle \Delta, 3/2 \, | \, \sigma_1\tau_1 \, | \, p, 1/2 \rangle. \tag{43}$$

The phase from (35) is $e^{-i\theta_1}$, so only $T^{14}$ (which has an $e^{i\theta_1}$ dependence) can contribute – the others are eliminated by the $\theta_j$ integrations. Then the result is

$$\langle \Delta, 3/2 | \sigma_1\tau_1 | p, 1/2 \rangle = \frac{N}{4\pi^2} \sqrt{\frac{2!}{1!\,0!}} \sqrt{\frac{4!}{3!\,0!}} \int_0^1 dx\,(1-x)^2(1-x)/2 \int d\theta_1 d\theta_2 d\theta_3 \; \delta(\theta_2 + \theta_3 - \theta_1 - \pi)$$

$$= N/\sqrt{8}. \tag{44}$$

The exact result is

$$\sqrt{(N-1)(N+5)/8}. \tag{45}$$

As with (41), this result can be easily generalized. The matrix element between any spin $1/2$ state and any spin $3/2$ state is the $N = 3$ value multipled by

$$\sqrt{(N-1)(N+5)/4}. \tag{46}$$

One more, for good measure – let us calculate

$$\langle \Delta, 3/2 \, | \, \sigma_3\tau_3 \, | \, \Delta, 3/2 \rangle = \frac{N}{4\pi^2} \frac{4!}{3!\,0!} \int_0^1 dx\,(1-x)^3(1-2x) \int d\theta_1 d\theta_2 d\theta_3 \; \delta(\theta_2 + \theta_3 - \theta_1 - \pi)$$

$$= 3N/5. \tag{47}$$

The exact result is

$$3 \times (N+2)/5. \tag{48}$$

As before, the matrix element between any two spin $3/2$ states is the $N = 3$ value multipled by

$$(N+2)/5. \tag{49}$$

## 4. Conclusions

We believe that the integral formula, (35), in addition to providing a simple calculational tool, yields some insight into the nature of the large $N$ enhancement of matrix elements. The basic physics is Bose-Einstein statistics. The low spin large $N$ states contain a large number of spin and isospin zero pairs in addition to the

"valence" quarks. In these pairs, because of Bose-Einstein statistics, the creation operators that duplicate those of the valence quarks dominate over those that do not appear in the valence sector. It is this asymmetry that produces the large $N$ matrix element enhancement.

## Acknowledgements

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