

# Capturing Complementarity in Set Functions by Going Beyond Submodularity/Subadditivity

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## Abstract

We introduce two new “degree of complementarity” measures, which we refer to, respectively, as *supermodular width* and *superadditive width*. Both are formulated based on natural witnesses of complementarity. We show that both measures are robust by proving that they, respectively, characterize the gap of monotone set functions from being submodular and subadditive. Thus, they define two new hierarchies over monotone set functions, which we will refer to as Supermodular Width (SMW) hierarchy and Superadditive Width (SAW) hierarchy, with foundations — i.e. level 0 of the hierarchies — resting exactly on submodular and subadditive functions, respectively.

We present a comprehensive comparative analysis of the SMW hierarchy and the Supermodular Degree (SD) hierarchy, defined by Feige and Izsak. We prove that the SMW hierarchy is strictly more expressive than the SD hierarchy. In particular, we show that every monotone set function of supermodular degree  $d$  has supermodular width at most  $d$ , and there exists a supermodular-width-1 function over a ground set of  $m$  elements whose supermodular degree is  $m - 1$ . We show that previous results regarding approximation guarantees for welfare and constrained maximization as well as regarding the Price of Anarchy (PoA) of simple auctions can be extended without any loss from the supermodular degree to the supermodular width. We also establish almost matching information-theoretical lower bounds for these two well-studied fundamental maximization problems over set functions. The combination of these approximation and hardness results illustrate that the SMW hierarchy provides not only a natural notion of complementarity, but also an accurate characterization of “near submodularity” needed for maximization approximation. While SD and SMW hierarchies support nontrivial bounds on the PoA of simple auctions, we show that our SAW hierarchy seems to capture more intrinsic properties needed to realize the efficiency of simple auctions. So far, the SAW hierarchy provides the best dependency for the PoA of Single-bid Auction, and is nearly as competitive as the Maximum over Positive Hypergraphs (MPH) hierarchy for Simultaneous Item First Price Auction (SIA). We also provide almost tight lower bounds for the PoA of both auctions with respect to the SAW hierarchy.

## 1 Introduction

For a ground set  $X = [m] = \{1, 2, \dots, m\}$ , a set function  $f : 2^X \rightarrow \mathbb{R}$  assigns each subset  $S \subseteq X$  a real value.<sup>1</sup> Function  $f$  is *monotone* if  $f(T) \geq f(S), \forall S \subseteq T \subseteq X$ , and *normalized* if  $f(\emptyset) = 0$ . In this paper, we will focus on normalized monotone set functions, which by definition are non-negative.

Like graphs to network analysis, set functions provide the mathematical language for many applications, ranging from combinatorial auctions (economics) to coalition formation (cooperative game theory; political science) [24, 25] to influence maximization (viral marketing) [23, 16]. Because of its exponential dimensionality, set functions — which are as rich as weighted hypergraphs — are far more expressive mathematically and challenging algorithmically than graphs [27]. However, when monotone set functions are *submodular*

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<sup>1</sup>Throughout the paper we use  $m$  to denote the number of elements in the ground set.

[21, 28] or — more generally — *complement-free* [7], algorithms with remarkable performance guarantees have been developed for various optimization, social influence, economic, and learning tasks [2, 16, 19, 3, 22].

In this paper, we study two new *degree-of-complementarity* measures of monotone set functions, and demonstrate their usefulness for several optimization and economic tasks. We prove that our complementarity measures — which are based on natural *witnesses* of complementarity — introduce hierarchies (over monotone set functions) that smoothly move beyond submodularity and subadditivity.

## 1.1 Witnesses to Complementarity: Supermodular Sets and Superadditive Sets

For any sets  $S, T \subseteq X$ , let  $f(S|T) := f(S \cup T) - f(T)$  be the *margin* of  $S$  given  $T$ . Recall that  $f$  is *subadditive* if  $f(S \cup T) \leq f(S) + f(T)$ ,  $\forall S, T \subseteq X$ , and *submodular* if for all  $S, T$  and  $v \in X \setminus T$ ,  $f(v|S \cup T) \leq f(v|S)$ . It is well known that every submodular set function is also subadditive.

If there are sets  $S, T \subseteq V$  such that  $f(S \cup T) > f(S) + f(T)$ , then one may say that  $(S, T)$  is a witness to *complementarity* in monotone set function  $f$ . Motivated by a line of recent work [1, 13, 9, 8, 10, 5], we consider the following fundamental question about set functions:

*Are there other natural, and preferably more general, forms of witnesses to complementarity that have algorithmic consequences?*

The supermodular degree of Feige and Izsak [9] is among the first measures of complementarity that are connected with algorithmic solutions to monotone-set-function maximization and combinatorial auctions. The supermodular degree is defined based on a notion of positive dependency between elements:  $u \in X$  positively depends on  $v \in X \setminus \{u\}$  (denoted by  $u \rightarrow^+ v$ ), if there exists  $S \subseteq X$  such that  $f(u|S) > f(u|S \setminus \{v\})$ .

**Definition 1** (Supermodular Degree). *The supermodular degree of a set function  $f : 2^X \rightarrow \mathbb{R}^+$ ,  $\text{SD}(f)$ , is defined to be  $\text{SD}(f) = \max_{u \in X} |\text{Dep}_f^+(u)|$ , where  $\text{Dep}_f^+(u) = \{v | u \rightarrow^+ v\}$ .*

Although supermodular degree has been shown useful in a number of settings, it is not clear whether it provides the tightest possible characterization of supermodularity. For example, consider a customer who wants any two or more items out of  $m$  items, but not zero or one item. That is, the customer has a valuation function, where any subset of  $[m]$  of size at least 2 provides utility 1, and any subset of size at most 1 provides utility 0. For this function, according to Feige and Izsak’s definition, any two items depend positively on each other. In particular, any item depends positively on all other items, so the supermodular degree of this valuation function is  $m - 1$  — the largest degree possible. This seems to contradict the intuition that there is only very limited complementarity.

Below, we will provide two perspectives, with the first highlighting *supermodularity* and the second highlighting *superadditivity*. In the rest of the paper, we will study how these two complementarity measures can be used to capture the performance of basic computational solutions in optimization and auction settings where the utilities are modeled by monotone set functions. In particular, our measure of supermodularity refines supermodular degree, and avoids the kind of overestimation discussed above.

Our first definition focuses on modularity:

**Definition 2** (Supermodular Set). *Given a normalized monotone set function  $f$  over a ground set  $X$ , a set  $T \subseteq X$  is supermodular w.r.t.  $f$  if*

$$\exists S \subseteq X \text{ and } v \in X \setminus T, \text{ such that: } f(v|S \cup T) > \max_{T' \subsetneq T} f(v|S \cup T').$$

First note that if  $f$  is submodular, then  $f(v|S \cup T) \leq f(v|S \cup T')$ ,  $\forall T' \subsetneq T$ , implying  $f$  has no supermodular set. Thus, if a set function  $f$  has a supermodular set, then it is not submodular.

We say that such a set  $T$  (in Definition 2) *complements* item  $v$  given  $S$ . In other words,  $S$  provides the setting that demonstrates the complementarity between  $v$  and  $T$ . In the customer example given after Definition 1, we can easily check that any singleton is a supermodular set, but any set with size at least two is not a supermodular set, because any single item in the set already provides all the complementarity

for any other single item. A supermodular set behaves similarly to the typical example of complements, namely *complementary bundles*,<sup>2</sup> in the sense that the set as a whole provides more complement to a single item than any of its strict subsets. However, supermodular sets have richer structures while preserving the strong complementarity of such bundles, making them potentially more challenging to deal with mathematically/algorithmically than complementary bundles of a similar size.

Our next definition focuses on additivity:

**Definition 3** (Superadditive Set). *Given a normalized monotone set function  $f$  over a ground set  $X$ , a set  $T \subseteq X$  is superadditive w.r.t.  $f$  if*

$$\exists S \subseteq X \setminus T \text{ such that: } f(S|T) > \max_{T' \subsetneq T} f(S|T').$$

In Definition 3, we say such a set  $T$  *complements* set  $S$ . Note that if  $f$  is subadditive, then for  $T' = \emptyset$ ,  $f(S|T) = f(S \cup T) - f(T) \leq (f(S) + f(T)) - f(T) = f(S) = f(S) - f(T') = f(S|T')$ , implying  $f$  does not have a superadditive set. In other words, if  $f$  has any superadditive set, then it is not subadditive.

Supermodular/superadditive sets correspond to witnesses that exhibit different kinds of complementarity. Supermodular sets are sensitive to the presence of an environment, and superadditive sets model complements to sets instead of items. The cardinality of the largest supermodular sets or superadditive sets provides a measure of the “level of complementarity”, similar to the supermodular degree ([9]), the size of the largest bundle, and the hyperedge size ([8]) (also see Definition 11) in previous work.

**Definition 4** (Supermodular Width). *The supermodular width of a set function  $f$  is defined to be*

$$\text{SMW}(f) := \max\{|T| \mid T \text{ is a supermodular set w.r.t. } f\}.$$

**Definition 5** (Superadditive Width). *The superadditive width of a set function  $f$  is defined to be*

$$\text{SAW}(f) := \max\{|T| \mid T \text{ is a superadditive set w.r.t. } f\}.$$

Each measure classifies monotone set functions into a hierarchy of  $m$  levels:

**Definition 6** (Supermodular Width Hierarchy (SMW- $d$ )). *For any integer  $d \in \{0, \dots, m-1\}$ , a set function  $f : 2^{[m]} \rightarrow \mathbb{R}$  belongs to the first  $d$ -levels of the supermodular width hierarchy, denoted by  $f \in \text{SMW-}d$ , if and only if  $\text{SMW}(f) \leq d$ .*

**Definition 7** (Superadditive Width Hierarchy (SAW- $d$ )). *For any integer  $d \in \{0, \dots, m-1\}$ , a set function  $f : 2^{[m]} \rightarrow \mathbb{R}$  belongs to the first  $d$  levels of the superadditive width hierarchy, denoted by  $f \in \text{SAW-}d$ , if and only if  $\text{SAW}(f) \leq d$ .*

We will show that functions at level 0 of the above two hierarchies, respectively, are precisely the families of submodular and subadditive functions. In both hierarchies,  $\text{SMW-}(m-1)$  and  $\text{SAW-}(m-1)$  contains all monotone set functions over  $m$  elements. Coming back to the customer example again, we see that the utility of the customer has supermodular width of 1. Comparing to its supermodular degree of  $m-1$ , our hierarchy characterizes this utility function at a much lower level, which matches our intuition that the complementarity of this customer’s utility function should be limited. We will further show below that this difference would also have significant algorithmic implications.

## 1.2 Our Results and Related Work

We now summarize the technical results of this paper. Structurally, we provide strong evidence that our definitions of supermodular/superadditive sets are natural and robust. We show that they — respectively — capture a set-theoretical gap of monotone set functions to submodularity and subadditivity. Algorithmically, we prove that our characterization based on supermodular width is *strictly stronger* than that of Feige-Izsak’s based on supermodular degree, by establishing the following:

<sup>2</sup> $S$  is a *complementary bundle* if  $f(S) > 0$  and  $\max_{S' \subsetneq S} f(S') = 0$ .

1. For every set function  $f : 2^{[m]} \rightarrow \mathbb{R}$ ,  $\text{SD}(f) \leq \text{SMW}(f)$ , and there exists a function whose supermodular degree is much larger than its supermodular width.
2. The SMW hierarchy offers the same level of algorithmic guarantees in the maximization and auction settings as the SD hierarchy.

We will also compare both hierarchies with the MPH hierarchy of [8].

### 1.2.1 Robustness: Capturing the Set-Theoretical Gap to Submodularity/Subadditivity

We interpret the level of complementarity in our formulation of supermodular and superadditive sets from a dual perspective: We prove that they characterize the gaps from a monotone set function to submodularity and subadditivity, respectively. Our characterization uses the following definition.

**Definition 8** (*d*-Scopic Submodularity). *For integer  $d \geq 0$ , a normalized monotone set function  $f$  is  $d$ -scopic submodular if and only if,*

$$f(v|T) \leq \max_{T': T' \subseteq T, |T'| \leq d} f(v|S \cup T'), \quad \forall S, T \subseteq X, v \in X \text{ satisfying } S \subseteq T, v \notin T \quad (1)$$

Note that in Condition (1), the family  $\{S \cup T' | T' \subseteq T, |T'| \leq d\}$  defines a set-theoretical *neighborhood* around  $S$ . Our definition of  $d$ -scopic submodularity means that even if the submodular condition  $f(v|T) \leq f(v|S)$  may not hold for some  $S \subseteq T$ , it holds for some set in  $S$ 's  $d$ -neighborhood inside  $T$ . Thus, the parameter  $d$  provides a set-theoretical scope for examining submodularity.

Similarly, we define:

**Definition 9** (*d*-scopic Subadditivity). *For integer  $d \geq 0$ , a set function  $f$  is  $d$ -scopic subadditive if and only if,*

$$f(S|T) \leq \max_{T': T' \subseteq T, |T'| \leq d} f(S|T'), \quad \forall S, T \subseteq X \text{ satisfying } S \cap T = \emptyset. \quad (2)$$

In Section 2, we prove the following two theorems.

**Theorem 1** (Set-Theoretical Characterization of the SMW Hierarchy). *For any integer  $d \geq 0$  and set function  $f : 2^X \rightarrow \mathbb{R}$ ,  $f$  is  $d$ -scopic submodular if and only if  $\text{SMW}(f) \leq d$ .*

**Theorem 2** (Set-Theoretical Characterization of the SAW Hierarchy). *For any integer  $d \geq 0$  and set function  $f : 2^X \rightarrow \mathbb{R}$ ,  $f$  is  $d$ -scopic subadditive if and only if  $\text{SAW}(f) \leq d$ .*

With matching supermodularity/submodularity and superadditivity/subadditivity characterization, Theorems 1 and 2 illustrate that our definitions of supermodular/superadditive sets are both natural and robust. We note that while monotone submodular functions are all subadditive, some  $d$ -scopic submodular functions are not  $d$ -scopic subadditive. As shown in Propositions 2.4 and 2.5, these two hierarchies are not comparable. We will show that they model different aspects of complementarity that can be utilized in different algorithmic and economic settings.

### 1.2.2 Expressiveness: Strengthening Supermodular Degree

We will show that our characterization based on supermodular width strengthens Feige-Izsak's the characterization based on supermodular degree [9]. The statement has two parts. We first prove, that supermodular sets extend positive dependency (as used in supermodular degree), which — in essence — can be viewed as a graphical approximation of supermodular sets.

**Theorem 3.** *Every monotone set function  $f$  with supermodular degree  $d$  has supermodular width at most  $d$  (i.e., it is  $d$ -scopic submodular). Moreover, there exists a monotone set function  $f : 2^{[m]} \rightarrow \mathbb{R}^+$  with  $\text{SMW}(f) = 1$  and  $\text{SD}(f) = m - 1$ .*

In other words, the SMW hierarchy strictly *dominates* the SD hierarchy. <sup>3</sup>

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<sup>3</sup>Formally, when comparing two set-function hierarchies, say with name  $\{Y_d\}_{d \in [0, m-1]}$  and  $\{Z_d\}_{d \in [0, m-1]}$ , we say  $Y$  *dominates*  $Z$ , if for all  $d \in [0, m-1]$  and  $f, g \in Z_d$  implies  $f \in Y_d$ .

### 1.2.3 Usefulness: Algorithmic and Economic Applications

We then show, algorithmically, the SMW hierarchy — while being more expressive than the SD hierarchy — is almost as useful as the latter (Theorems 8, 10 and 19).

We will illustrate the usefulness of our hierarchies in algorithm and auction design with two archetypal classes of problems, *set function maximization* and *combinatorial auctions*, which traditionally involve measures of complementarity. Motivated by previous work [9, 13, 10, 8], we will characterize the *approximation guarantee* of polynomial-time set-function maximization algorithms and *efficiency* of simple auction protocols in terms of the complementarity level in our hierarchies. In these settings, we will compare our hierarchies with two most commonly cited complementarity hierarchies: the supermodular degree (SD) hierarchy and the Maximum over Positive Hypergraphs (MPH) hierarchy.

- *Set-Function Maximization* We will consider both constrained and welfare maximization. The former aims to find a set of a given cardinality with maximum function value. The latter aims to allocate a set of items to  $n$  agents,<sup>4</sup> with potentially different valuations, such that the total value of all agents is maximized. As a set function has exponential dimensions in  $m$ , in both maximization problems, we assume that the input set functions are given by their value oracles.
- *Combinatorial Auctions and Simple Auction Protocols* We will consider two well-studied simple combinatorial auction protocols: Single-bid Auction and Simultaneous Item First Price Auction (SIA). In both settings, there are multiple agents, each of which has a (potentially different) valuation function over subsets of items. The former auction protocol proceeds by asking each bidder to bid a single price, and letting bidders, in descending order of their bids, buy any available set of items paying their bid for each item. The latter simply runs first-price auctions simultaneously for all items.

#### APPROXIMATION GUARANTEES ACCORDING TO SUPERMODULAR WIDTHS

We will prove that the elegant approximability results for constrained maximization by [13] and for welfare maximization by [9] can be extended from supermodular degree to supermodular width. We obtain the same dependency (see Theorems 8 and 10) — that is,  $1 - e^{-1/(d+1)}$  and  $\frac{1}{d+2}$  respectively — on the supermodular width  $d$  as what the supermodular degree previously provides for these problems.

Because our SMW hierarchy is strictly more expressive, our upper bounds for SMW- $d$  cover strictly more monotone set functions than previous results for SD- $d$ . We will also complement our algorithmic results with nearly matching information theoretical lower bounds (see Theorems 9 and 11), for these two well-studied fundamental maximization problems. Our approximation and hardness results illustrate that the SMW hierarchy not only captures a natural notion of complementarity, but also provides an accurate characterization of the “nearly submodular property” needed by approximate maximization problems.

#### EFFICIENCY OF SIMPLE AUCTIONS ACCORDING TO SUPERADDITIVE/SUPERMODULAR WIDTH

Next, we will analyze the efficiency of two well-known simple auction protocols in terms of superadditive width. To state our results and compare them with previous work, we first recall a notation from [8]:

**Definition 10** (Closure under Maximization). *For any family of set functions  $\mathcal{F}$  over  $X$ , the closure of  $\mathcal{F}$  under maximization, denoted by  $\max(\mathcal{F})$ , is the following family of set functions:  $f \in \max(\mathcal{F})$  if and only if*

$$\exists k \in \mathbb{N}, f_1, \dots, f_k \in \mathcal{F}, \text{ s.t. } f(S) = \max_{i \in [k]} f_i(S), \forall S \subseteq X.$$

We will prove the following:

**Theorem 4.** *Single-bid Auction and SIA are approximately efficient — with Price of Anarchy (PoA)  $O(d \log m)$  — for valuation functions in  $\max(\text{SAW}(d))$ .*

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<sup>4</sup>Throughout the paper we use  $n$  to denote the number of agents (whenever applicable) unless otherwise specified.

	SD- $d$	MPH- $(d + 1)$	SMW- $d$	SAW- $d$
constrained maximization	$1 - e^{1/(d+1)}$ [13]	?	$1 - e^{1/(d+1)}$ (Thm 8)	?
welfare maximization	$1/(d + 2)$ [9]	$1/(d + 2)$ [8]	$1/(d + 2)$ (Thm 10)	?
PoA of Single-bid Auction	$O(d^2 \log m)$ [10]	?	$O(d^2 \log m)$ (Thm 18)	$O(d \log m)$ (Thm 14)
PoA of SIA	$O(d)$ [8]	$O(d)$ [8]	$O(d^2 \log m)$ (Thm 19)	$O(d \log m)$ (Thm 15)

Table 1: Comparison of hierarchies of complementarity. Note that the  $O(d)$  bound for PoA of SIA with SD- $d$  valuations follows from the fact that  $\text{SD-}d \subseteq \text{MPH-}(d + 1)$ , which is not clearly comparable with the PoA bound of SIA with SMW- $d$  valuations. See corresponding references and theorems for more accurate statements.

We will also complement our PoA results by almost tight (up to a factor of  $O(\log m)$ ) lower bounds:

**Theorem 5.** *For any  $d > 0$ , there is an instance with SAW- $d$  valuations, where the Price of Stability (PoS) of Single-bid Auction is at least  $d + 1 - \varepsilon$  for any  $\varepsilon > 0$ , and the PoA of SIA is at least  $d$ .*

Although supermodular width strictly strengthens supermodular degree, superadditive width is not comparable with supermodular degree. Nevertheless, our PoA bound of  $O(d \log m)$  is a factor of  $d$  tighter than the  $O(d^2 \log m)$  supermodular-degree based bound of [10] for Single-bid Auction. This improvement of dependency on  $d$ , together with the nearly matching lower bound, suggests that the SAW hierarchy might be more capable in capturing the smooth transition of efficiency of simple auctions. Furthermore, as a byproduct of our efficiency results for the SAW hierarchy, we also obtain similar results, but with a worse dependency on  $d$ , for the SMW hierarchy.

**Theorem 6.** *Single-bid auction and SIA are approximately efficient — with PoA  $O(d^2 \log m)$  — for valuations in  $\max(\text{SMW-}d \cap \text{SUPADD})$ , where SUPADD denotes the class of monotone superadditive set functions.*

For Single-bid Auction, this result strengthens the central efficiency result of [10] by replacing the supermodular degree with the more inclusive supermodular width. For the PoA analysis of SIA, the the Maximum over Positive Hypergraphs (MPH) hierarchy of [8] remains the gold standard, by providing asymptotically matching upper and lower bounds. MPH is defined based on the following hypergraph characterization of set functions: Every normalized monotone set function over ground set  $X$  can be uniquely expressed by another set function  $h$  such that  $f(S) = \sum_{T \subseteq S} h(T)$ ,  $\forall S \subseteq X$ , where  $h(T)$  for each  $T$  is called the weight of hyperedge  $T$ .

**Definition 11** (Maximum over Positive Hypergraphs [8]). *Let PH- $d$  be the class of set functions whose hypergraph representation  $h$  satisfies: (1)  $h(S) \geq 0$  for all  $S$ , and (2)  $h(S) > 0$  only if  $|S| \leq d$ . The  $d$ -th level of the MPH hierarchy is defined as  $\text{MPH-}d = \max(\text{PH-}d)$ .*

MPH provides the best characterization to the efficiency of SIA as well as ties with SD and SMW regarding the approximation ratio of welfare maximization (although it requires access to the much stronger demand oracles). However, it remains open whether it can be used to analyze constrained set function maximization and Single-bid Auction. See Table 1 for a comparison.

We will prove the following theorem which states that, in general, the SAW hierarchy is not comparable with MPH.

**Theorem 7.** *There is a subadditive function that lives in an upper (i.e.  $\geq m/2$ ) MPH level. On the other direction, there is a function on level 2 of MPH, whose superadditive and supermodular widths are both  $m - 1$ .*

It remains open whether  $\text{MPH-}(d + 1)$  — which subsumes SD- $d$  as a subset — contains SMW- $d$ . In particular, the proof that  $\text{SD-}d \subseteq \text{MPH-}(d + 1)$  in [8] does not appear easily applicable to SMW- $d$ .

## 1.2.4 Other Related Work

### SET FUNCTION MAXIMIZATION

There is a rich body of research focusing on set function maximization with complement-free functions, e.g. [21, 28, 7]. Various information/complexity theoretical lower bounds have been established for both problems, e.g. [20, 6, 19, 17].

### EFFICIENCY OF SIMPLE AUCTIONS

Single-bid Auction with subadditive valuations has a PoA of  $O(\log m)$  [4]. SIA with subadditive valuations has a constant PoA [11]. Posted price auctions with XOS valuations give a constant factor approximate welfare guarantee [12].

### OTHER MEASURES OF COMPLEMENTARITY

Some other useful measures include Positive Hypergraph (PH) [1] and Positive Lower Envelop (PLE) [8]. Eden *et al.* recently introduce an extensive measure which ranges from 1 to  $2^m$  to capture the smooth transition of revenue approximation guarantee [5].

## 2 Expressiveness of the New Hierarchies

### 2.1 Characterization of Supermodular/Superadditive Widths

We first prove Theorems 1 and 2, which characterize supermodular/superadditive widths with  $d$ -scopic submodular/subadditive functions.

*Proof of Theorem 1.* We now show  $\text{SMW}(f) \leq d$  iff  $f$  is  $d$ -scopic submodular. First, suppose  $\text{SMW}(f) \leq d$ . Consider any triple  $(T, S, v)$  such that  $S \subseteq T \subseteq X$  and  $v \notin T$ . To show  $f$  is  $d$ -scopic submodular, we prove by induction on the size of  $T$ , that

$$f(v|T) \leq \max_{T': T' \subseteq T, |T'| \leq d} f(v|S \cup T'). \quad (3)$$

As the base case, when  $|T| \leq d$ , the inequality of (3) trivially holds because if  $T' = T \setminus S$ , then  $|T'| \leq d$  and  $f(v|S \cup T') = f(v|T)$ . Inductively, assume that the statement is true for all  $\{V \subseteq X : |V| \leq k\}$  for some  $k \geq d$ . Now consider any set  $T$  with  $|T| = k + 1 > d$ . Because  $T$  is not supermodular, there is  $T'' \subsetneq T$ , such that  $f(v|T) \leq f(v|T'')$ . Applying the inductive hypothesis on  $(T'', S, v)$ , we have:

$$f(v|T'') \leq \max_{T': T' \subseteq T'', |T'| \leq d} f(v|S \cup T') \leq \max_{T': T' \subseteq T, |T'| \leq d} f(v|S \cup T').$$

Thus,  $f(v|T) \leq f(v|T'') \leq \max_{T': T' \subseteq T, |T'| \leq d} f(v|S \cup T')$ , and we have demonstrated that  $f$  is  $d$ -scopic submodular.

For the other direction, we assume  $f$  is  $d$ -scopic submodular. There is no supermodular set of size larger than  $d$ , because for any  $S, T, v \notin T$  where  $|T| > d$ , there is some  $T' \subseteq T$  where  $|T'| \leq d$ , such that  $f(v|S \cup T) \leq f(v|S \cup T')$ , i.e.  $T$  is not supermodular. Therefore  $\text{SMW}(f) \leq d$ .  $\square$

**Corollary 2.1.** *Set function  $f$  is submodular iff  $\text{SMW}(f) = 0$  (i.e.,  $f$  has no supermodular set).*

*Proof of Theorem 2.* We prove  $\text{SAW}(f) \leq d$  iff  $f$  is  $d$ -scopic subadditive. Suppose  $\text{SAW}(f) \leq d$ . Consider  $S$  and  $T$  where  $S \cap T = \emptyset$ . We show  $d$ -scopic subadditivity by induction on the size of  $T$ . When  $|T| \leq d$ , the statement trivially holds. Suppose  $d$ -scopic subadditivity holds for  $|T| \leq k$  where  $k \geq d$ . For  $|T| = k + 1 > d$ , since  $T$  is not superadditive, there is  $T'' \subsetneq T$ , such that  $f(S|T) \leq f(S|T'')$ . Applying inductive hypothesis on  $S, T''$  gives  $f(S|T) \leq f(S|T'') \leq \max_{T': T' \subseteq T, |T'| \leq d} f(S|T')$ , i.e.  $f$  is  $d$ -scopic subadditive.

Now assume  $d$ -scopic subadditivity. There is no superadditive set with size larger than  $d$ , because for any  $S$  and  $T$  where  $|T| > d$  and  $S \cap T = \emptyset$ , there is some  $T' \subseteq T$  where  $|T'| \leq d$ , such that  $f(S|T) \leq f(S|T')$ , i.e.  $T$  is not superadditive.  $\square$

**Corollary 2.2.** *A set function  $f$  is subadditive iff  $\text{SAW}(f) = 0$  (i.e.,  $f$  has no superadditive set).*

## 2.2 Supermodular Width vs Supermodular Degree

The following two propositions establish Theorem 3, showing supermodular width strictly dominates supermodular degree.

**Proposition 2.1.** *For any set function  $f$ ,  $\text{SD}(f) \leq \text{SMW}(f)$ .*

*Proof.* Fix  $f$ . Let  $T$  be a supermodular set of size  $\text{SMW}(f)$ , and  $S, v$  be such that  $f(v|T \cup S) > f(v|T' \cup S)$ ,  $\forall T' \subsetneq T$ . Clearly for any  $t \in T$ ,  $f(v|\{t\} \cup (T \setminus \{t\}) \cup S) > f(v|(T \setminus \{t\}) \cup S)$ . In other words,  $v \rightarrow^+ t$  for all  $t \in T$ , so  $\text{SD}(f) \geq \text{deg}^+(v) \geq |T| = \text{SMW}(f)$ .  $\square$

**Proposition 2.2.** *There exists a monotone set function  $f$  with  $\text{SMW}(f) = 1$  and  $\text{SD}(f) = m - 1$ .*

*Proof.* Consider a symmetric<sup>5</sup>  $f$  over a ground set  $X = [m]$ , where  $f(S) = 0$  if  $|S| \leq 1$ , and  $f(S) = 1$  otherwise. Observe that for any  $u \neq v$ ,  $1 = f(u|\{v\}) > f(u|\emptyset) = f(u) = 0$ , so  $u \rightarrow^+ v$ , and  $\text{SD}(f) = |\text{Dep}_f^+(u)| = m - 1$ . On the other hand, consider any  $T$  where  $|T| \geq 2$ . For any  $v, S$ , we have  $|S \cup T| \geq 2$ , so  $0 = f(v|S \cup T) \leq f(v|S)$ . Thus,  $T$  is not supermodular. Since there is no supermodular set with size larger than 1 and  $f$  is not submodular,  $\text{SMW}(f) = 1$ .  $\square$

While the SAW hierarchy does not subsume the MPH hierarchy (see Proposition 2.6), we show that there is a monotone set function in the lowest layer of the SAW hierarchy (i.e. a subadditive function) and a notably high layer of the MPH hierarchy.

**Proposition 2.3.** *There exists a monotone set function  $f$  with  $\text{SAW}(f) = 0$  and  $\text{MPH}(f) = \frac{m}{2}$ .*

*Proof.* The proposition is a direct corollary of Proposition L.2 in [8]. In fact, consider a symmetric valuation  $f$  over  $[m]$ , where  $f([m]) = 2$ ,  $f(\emptyset) = 0$ , and  $f(S) = 1$  otherwise. Clearly  $f$  is subadditive so  $\text{SAW}(f) = 0$ . According to Corollary F.5 of [8],  $\text{MPH}(f) \geq \frac{m}{2}$ .  $\square$

## 2.3 Further Comparisons between Hierarchies

**Proposition 2.4.** *There exists a monotone set function  $f$  with  $\text{SMW}(f) = 1$  and  $\text{SAW}(f) = m/2$ .*

*Proof.* Let  $h_T(S) = \mathbb{I}[T \subseteq S]$ . Consider function  $f : 2^X \rightarrow \mathbb{R}^+$  where  $X = [2t]$  and

$$f(S) = \sum_{i \in [t]} h_{\{i, i+t\}}(S).$$

$\text{SMW}(f) = 1$  because the only complement set to any item  $i \in [t]$  is  $i + t$ . On the other hand,  $T = \{t + 1, \dots, 2t\}$  is a complement set to  $S = [t]$ , so  $\text{SAW}(f) = t = m/2$ .  $\square$

**Proposition 2.5.** *There exists a monotone set function  $f$  with  $\text{SAW}(f) = 0$  and  $\text{SMW}(f) = m - 1$ .*

*Proof.* Consider a symmetric  $f : 2^X \rightarrow \mathbb{R}^+$ , where  $f(\emptyset) = 0$ ,  $f(X) = 2$  and  $f(S) = 1$  otherwise.  $f$  is subadditive so  $\text{SAW}(f) = 0$ . On the other hand,  $X \setminus \{u\}$  for any  $u$  is a complement set to  $u$ , so  $\text{SMW}(f) = m - 1$ .  $\square$

**Proposition 2.6.** *There exists a monotone set function  $f$  with  $\text{MPH}(f) = 2$  and  $\text{SMW}(f) = \text{SAW}(f) = m - 1$ .*

---

<sup>5</sup> $f$  is symmetric if  $f(S)$  depends only on  $|S|$ .



---

**ALGORITHM 1:** Batched Greedy Selection for Constrained Maximization  $(f, k)$ 

---

```
let  $S_0 \leftarrow \emptyset$ ;  $i = 0$ ;  
while  $|S_i| < k$  do  
    | Let  $i = i + 1$ ;  $T_i \leftarrow \operatorname{argmax}_{T' \subseteq [m], |T'| \leq s} f(T'|S_i)$  where  $s = \min\{\operatorname{SMW}(f) + 1, k - |S_{i-1}|\}$ ;  
    | let  $S_i \leftarrow S_{i-1} \cup T_i$ ;  
end  
return  $S^{\operatorname{BatchedGreedy}} := S_i$ ;
```

---

*Proof.* Let  $h_T(S) = \mathbb{I}[T \subseteq S]$ . Consider function  $f : 2^X \rightarrow \mathbb{R}^+$  where

$$f(S) = \sum_{u \neq v} h_{\{u,v\}}(S).$$

$f$  is in MPH-2 since its hypergraph representation consists of only hyperedges of size 2. Now consider any  $u$  and  $T = X \setminus \{u\}$ . For any  $T' \subsetneq T$ ,

$$f(u|T) = |T| > |T'| = f(u|T').$$

In other words,  $T$  is both supermodular and superadditive, and  $\operatorname{SMW}(f) = \operatorname{SAW}(f) = m - 1$ .  $\square$

### 3 Expanding Approximation Guarantees for Classic Maximization

In this section, we focus on the connection between supermodular width and two classical optimization problems, namely, the constrained and welfare set-function maximization. For submodular functions, greedy algorithms provide tight approximation guarantees for both problems [21, 28]. Here, we will show that simple modifications to these greedy algorithms can effectively utilize the mathematical structure underlying the supermodular degree of  $f$ , namely the  $\operatorname{SMW}(f)$ -scopic submodularity, for any set function  $f$ . We prove that these extensions achieve approximation ratios parametrized by the supermodular width with the same dependency as the supermodular degree provides [13, 9] for both maximization problems. We complement our approximation results by nearly tight information-theoretical lower bounds.

#### 3.1 Constrained Maximization

We first focus on cardinality constrained maximization, a problem at the center of resource allocation and network influence [23, 16, 21, 28]. Formally:

**Definition 12** (Cardinality Constrained Maximization). *Given a monotone set function  $f : 2^X \rightarrow \mathbb{R}^+ \cup \{0\}$  and integer  $k > 0$ , compute a set  $S \subseteq X$  with  $|S| \leq k$  that maximizes  $f(S)$ .*

We will analyze an algorithm which performs *batched greedy selection*, — see Algorithm 1 below — where the batch size is a function of the supermodular width of  $f$ . In particular, for an input set function, the batched greedy algorithm chooses a set of size not exceeding  $\operatorname{SMW}(f) + 1$  which maximizes marginal gain, till all  $k$  elements are chosen.

Below, we show that this simple greedy algorithm provides strong approximation guarantees in terms of the supermodular width of the input function.

**Theorem 8** (Extending [13]). *For any monotone set function  $f$  over  $[m]$ , Algorithm 1 achieves  $(1 - e^{-1/(\operatorname{SMW}(f)+1)})$ -approximation for constrained maximization problem after making  $O(m^{\operatorname{SMW}(f)+1})$  value queries.*

*Proof.* The proof uses similar ideas to those in [13], which are originally from [21]. Let  $d = \text{SMW}(f)$  and (w.l.o.g.) let  $S^* = [k] = \{1, \dots, k\}$  be an optimal solution.

$$f(S^*) - f(S_i) \leq f(S^* \cup S_i) - f(S_i) \quad (4)$$

$$\leq f(S^* | S_i)$$

$$= f([k] | S_i)$$

$$= \sum_{j \in [k]} f(j | [j-1] \cup S_i)$$

$$\leq k \max_j f(j | [j-1] \cup S_i)$$

$$\leq k \max_j \max_{U_j: U_j \subseteq [j-1], |U_j| \leq d} f(j | U_j \cup S_i) \quad (5)$$

$$\leq k \max_j \max_{U_j: U_j \subseteq [j-1], |U_j| \leq d} f(\{j\} \cup U_j | S_i) \quad (6)$$

$$\leq k f(S_{i+1} | S_i) \quad (7)$$

$$= k(f(S_{i+1}) - f(S_i))$$

where (4) is by the monotonicity of  $f$ , (5) is by the equivalent  $d$ -scopic submodularity of  $f$ , (6) is again by the monotonicity of  $f$ , and (7) is by the greedy property:  $f(S_{i+1} | S_i) = \max_{S: |S| \leq d+1} f(S | S_i)$ .

Now we have

$$\begin{aligned} f(S^*) - f(S_i) &\leq \frac{k-1}{k} (f(S^*) - f(S_{i-1})) \\ &\leq \left(\frac{k-1}{k}\right)^i (f(S^*) - f(S_0)) \\ &= \left(\frac{k-1}{k}\right)^i f(S^*) \\ &\leq e^{-i/k} f(S^*). \end{aligned}$$

Because  $f$  is monotone, we have  $|T_i| = d+1$ , for all intermediate steps, i.e.,  $i < \lceil \frac{k}{\text{SMW}(f)+1} \rceil$ . Thus, Algorithm 1 takes exactly  $t := \lceil \frac{k}{\text{SMW}(f)+1} \rceil$  steps to terminate. The function value of its output  $f(S^{\text{BatchedGreedy}}) := f(S_t) \geq (1 - e^{-1/(\text{SMW}(f)+1)}) f(S^*)$ .  $\square$

While in general, Theorem 8 establishes a tighter approximation guarantee for the SMW hierarchy than that for the SD hierarchy, we note that in case of submodular degree, if the positive dependency graph is given, the running times are often of the form  $\text{poly}(n) \cdot 2^{O(\text{SD}(f))}$ , which can be significantly better than  $n^{O(\text{SMW}(f))}$  even if the submodular width  $\text{SMW}(f)$  is much smaller than the submodular degree  $\text{SD}(f)$ .

We now provide a nearly-matching information-theoretical lower bound, suggesting that our approximation guarantee is essentially optimal. In the theorem below, the exponent  $k^{0.99}$  can be replaced by any function of  $k$  in  $o(k)$ .

**Theorem 9.** *For any  $d \in \mathbb{N}$ ,  $\varepsilon > 0$ , and a large enough integer  $k$ , there exists a set function  $f : 2^{[m]} \rightarrow \mathbb{R}^+$ , with  $\text{SMW}(f) = d$ , such that any (possibly randomized) algorithm that produces a  $(1/(d+1) + \varepsilon)$ -approximation (with a constant probability if randomized) for the  $k$ -constrained maximization problem makes at least  $\Omega\left((m/2k)^{k^{0.99}}\right)$  value queries.*

*Proof.* The proof is based on similar high-level ideas to those in [19], but the detailed construction and key properties used are different. Consider a ground set  $X$  of  $m$  elements, which contains a subset  $R$  of  $r$  “special” elements. We will specify  $r$  below. We now construct a “hard-to-distinguish” function  $f_R$  such that for any  $S \subseteq X$ ,  $f_R(S) = g_R(|S|, \mathbb{I}[R \subseteq S])$  for a function  $g_R : \mathbb{N} \times \{0, 1\} \rightarrow \mathbb{R}$ . In other words,  $f_R$  depends on

the cardinality of  $S$  and whether or not  $S$  completely contains  $R$ . For discussion below, let  $D = d + 1$ , and let  $c_1$  and  $c_2$  be two integers to be determined later. We set  $|R| = r = c_1 \cdot D + 1$ . We define  $f_R$  as follows:

$$f_R(S) = \begin{cases} \lfloor |S|/D \rfloor, & |S| \leq c_1 D \\ \lfloor (|S| - c_1 D)/D \rfloor + c_1, & c_1 D < |S| \leq (c_1 + c_2)D, R \not\subseteq S \\ |S| - c_1(D - 1), & c_1 D < |S| \leq (c_1 + c_2)D, R \subseteq S \\ |S| - (c_1 + c_2)(D - 1), & (c_1 + c_2)D < |S| \leq (c_1 + c_2)D + c_2(D - 1), R \not\subseteq S \\ c_1 + c_2 D, & (c_1 + c_2)D < |S| \leq (c_1 + c_2)D + c_2(D - 1), R \subseteq S \\ c_1 + c_2 D, & (c_1 + c_2)D + c_2(D - 1) < |S| \leq m \end{cases}.$$

We will use the following three properties of  $f_R$ :

- Whenever  $|S| \bmod D = D - 1$ , for any  $v \notin S$ ,  $f_R(v|S) = 1$ . Consequently,  $\text{SMW}(f_R) \leq d, \forall R \subseteq X$  with  $|R| = r$ . In fact, this property ensures that  $f_R(v|S \cup T') \geq f_R(v|S \cup T)$ , for any  $v \in X$ ,  $S, T \subseteq X$  with  $|T| \geq D = d + 1$ , and any proper subset  $T'$  of  $T$  with  $|S \cup T'| \bmod D = D - 1$ . Note that such a subset  $T'$  always exists.
- $\max \{f_R(S) \mid |S| = (c_1 + c_2)D\} = c_1 + c_2 D$ . The maximum is achieved whenever  $R \subseteq S$ .
- For any  $S \subseteq X$  satisfying  $|S| = c_1 + c_2 D$  and  $R \not\subseteq S$ ,  $f_R(S) = c_1 + c_2$ .

First, consider  $k = (c_1 + c_2)D$ . We have, for any  $S$  with  $|S| = k$ :

$$f_R(S) = \begin{cases} c_1 + c_2 D & \text{if } R \subseteq S \\ c_1 + c_2 & \text{otherwise.} \end{cases}$$

Suppose  $c_1 = o(c_2)$ . To obtain an approximation ratio better than  $(c_1 + c_2)/(c_1 + c_2 D) \rightarrow 1/D$  for  $k$ -constrained maximization of  $f_R$ , any algorithm must find a set with size  $k$  that contains all special elements in  $R$ .

For our lower bound, we will analyze the following slightly relaxed variation of the problem: Let  $K = (c_1 + c_2)D + c_2(D - 1) - 1 > k$ . Find a set of size  $K$  which contains  $R$  as a subset. Note that  $K$  is the largest number where  $f_R(S) = 1$  for  $|S| = K$  depends on whether or not  $S$  contains  $R$ . In this case, note that the algorithm has no incentive to make queries of  $f_R(S)$  for  $|S| < K$  or  $|S| > K$ , because the former reveals no more information than querying any of its supersets of size  $K$ , and the latter simply does not give any information.

We first focus on the query complexity of any deterministic optimization algorithm. Assume the algorithm makes  $T$  queries regarding  $S_1, \dots, S_T$ , where  $|S_i| = K, \forall i \in [T]$ , which are deterministically chosen when the algorithm is fixed. We now establish a condition on  $T$  such that there is a subset  $R$  such that  $R \not\subseteq S_i, \forall i \in [T]$ . Consider the distribution where the  $r$  elements are selected uniformly at random. Let  $C_i$  be the event that  $S_i$  contains  $R$ . Then,

$$\Pr[C_1 \cup \dots \cup C_T] \leq \sum_i \Pr[C_i] < \sum_i \left(\frac{|S_i|}{m}\right)^r = T \left(\frac{(c_1 + c_2)D + (D - 1)c_2 - 1}{m}\right)^{c_1 D + 1} \leq T \left(\frac{2c_2 D}{m}\right)^{c_1 D}.$$

So, if  $T \leq \lfloor m/(2c_2 D) \rfloor^{c_1 D}$  then  $\Pr[C_1 \cup \dots \cup C_T] < 1$ . In other words, for any selections  $S_1, \dots, S_T \subseteq X$  with  $|S_i| = K$ , there is a subset  $R$ , such that  $R \not\subseteq S_i, \forall i \in [T]$ , implying the deterministic algorithm with querying set  $S_1, \dots, S_T$  will not find a good approximation to  $f_R$ . Let  $c_2 = \frac{1}{2}c_1^{1.01}$ , so  $k^{0.99} = ((c_1 + c_2)D)^{0.99} \leq (c_1^{1.01} D)^{0.99} \leq c_1 D$ . We have  $(m/2c_2 D)^{c_1 D} \geq (m/2k)^{k^{0.99}}$ . Thus, we conclude that any  $(1/(d + 1) + \varepsilon)$ -approximation deterministic algorithm must make at least  $(m/2k)^{k^{0.99}}$  value queries.

Now consider a randomized optimization algorithm. Conditioned on the random bits of the algorithm, the above argument still works. Taking expectation of the probability of success, we see that the overall probability of success is at most  $T(2k/m)^{k^{0.99}}$ . Thus, a constant probability of success requires  $T = \Omega\left((m/2k)^{k^{0.99}}\right)$ .  $\square$

---

**ALGORITHM 2:** Batched Greedy for Welfare Maximization  $(f_1, \dots, f_n)$ 

---

```
for  $j \in [n]$  do
  | let  $X_{j,0} \leftarrow \emptyset$ ;
end
Let  $d = \max_j \{\text{SMW}(f_j)\}$ ; let  $i = 0$ ;
while  $\cup_j X_{j,i} \neq X$  do
  | Let  $i = i + 1$ ; let  $(T_i, j_i^*) = \operatorname{argmax}_{(T', j): |T'| \leq s, j \in [n]} f_j(T' | X_{j,i-1})$  where  $s = \min \left\{ d + 1, n - \sum_j |X_{j,i-1}| \right\}$ ;
  | Let  $X_{j_i^*, i} \leftarrow X_{j_i^*, i-1} \cup T_i$ ;
  | for  $j \in [n] \setminus \{j_i^*\}$  do
  |   | let  $X_{j,i} \leftarrow X_{j,i-1}$ ;
  |   end
  | return  $X_j^{\text{BatchedGreedy}} := X_{j,i}$  for every agent  $j$ ;
end
```

---

### 3.2 Welfare Maximization

We now turn our attention to welfare maximization. Formally:

**Definition 13** (Welfare Maximization). *Given  $n$  monotone set functions  $f_1, \dots, f_n$  over  $2^{[m]}$ , compute  $n$  disjoint sets  $X_1, \dots, X_n$  that maximizes  $\sum_{i \in [n]} f_i(X_i)$ .*

Because  $f_1, \dots, f_n$  are monotone, the optimal solution to welfare maximization is a partition of  $X = [m]$ . Thus, welfare maximization can also be viewed as a generalized clustering or multiway partitioning problem.

We will analyze the following greedy algorithm — see Algorithm 2 below — which repeatedly assigns groups of elements to agents. At each step, the algorithm picks a set of size not exceeding  $\max_i \text{SMW}(f_i) + 1$  — as opposed to one — that provides the largest possible marginal gain to some agent and assigns the set to that agent.

We now prove the following approximation guarantee in terms of supermodular width.

**Theorem 10** (Extending [9]). *For any collection of monotone set functions  $f_1, \dots, f_n$  over  $X = [m]$ , Algorithm 2 achieves  $\frac{1}{2 + \max_i \{\text{SMW}(f_i)\}}$ -approximation for welfare maximization, after making  $O(nm^{\max_i \{\text{SMW}(f_i)\} + 1})$  value queries.*

*Proof.* The proof uses similar ideas to those in [9], which are originally from [15]. Following the notation in Algorithm 2, we use  $i$  to denote the step and  $j$  to denote the agent's index. Recall  $d = \max_j \{\text{SMW}(f_j)\}$ . Suppose  $(X_1^*, \dots, X_n^*)$  is an optimal solution to the welfare maximization of  $(f_1, \dots, f_n)$ . Note that  $\cup_j X_{j,i}$  is the subset of elements that has already been assigned at the end of step  $i$ . Let  $T_{j,i} = X_j^* \setminus \cup_j X_{j,i}$  denote the set of elements of  $X_j^*$  still available at the time. Recall at step  $i$ , the set  $T_i$  is allocated to agent  $j_i^*$ . In other words,  $X_{j_i^*, i} = X_{j_i^*, i-1} \cup T_i$  and  $f_{j_i^*}(X_{j_i^*, i}) - f_{j_i^*}(X_{j_i^*, i-1}) = f_{j_i^*}(T_i | X_{j_i^*, i-1})$ . According to Algorithm 2,  $|T_i| \leq d + 1$ . We now prove the following instrumental inequality to our analysis.

$$(d + 2) \cdot (f_{j_i^*}(X_{j_i^*, i}) - f_{j_i^*}(X_{j_i^*, i-1})) = (d + 2) \cdot f_{j_i^*}(T_i | X_{j_i^*, i-1}) \geq \sum_j (f_j(T_{j,i-1} | X_{j,i-1}) - f_j(T_{j,i} | X_{j,i})). \quad (8)$$

We divide the right hand terms according to two cases:

**Case 1** (terms with  $j \in [n] \setminus \{j_i^*\}$ ): Note that  $(T_{1,i-1} \cap T_i, \dots, T_{n,i-1} \cap T_i)$  is a partition of  $T_i$  because  $(X_1^*, \dots, X_n^*)$  is a partition of  $X$ . Let  $d_j = |T_{j,i-1} \cap T_i|$ . We have,

$$\sum_{j \neq j_i^*} d_j \leq |T_i| \leq d + 1.$$

Thus, for any  $j \neq j^*$ , for analysis below, let's name the  $d_j$  elements in  $T_{j,i-1} \cap T_i$  as  $\{u_1^{(j)}, \dots, u_{d_j}^{(j)}\}$ . Note that for  $j \neq j^*$ ,  $X_{j,i-1} = X_{j,i}$  and  $T_{j,i} = T_{j,i-1} \setminus \{u_1^{(j)}, \dots, u_{d_j}^{(j)}\}$ , which implies the first equality below:

$$\begin{aligned} & f_j(T_{j,i-1}|X_{j,i-1}) - f_j(T_{j,i}|X_{j,i}) \\ &= f_j\left(\{u_1^{(j)}, \dots, u_{d_j}^{(j)}\} | T_{j,i} \cup X_{j,i-1}\right) \\ &= \sum_{k=1}^{d_j} f_j\left(u_k^{(j)} | \{u_1^{(j)}, \dots, u_{k-1}^{(j)}\} \cup T_{j,i} \cup X_{j,i-1}\right) \\ &\leq \sum_{k=1}^{d_j} \left( \max_{V_k \subseteq \{u_1^{(j)}, \dots, u_{k-1}^{(j)}\} \cup T_{j,i}, |V_k| \leq d} f_j\left(u_k^{(j)} | V_k \cup X_{j,i-1}\right) \right) \end{aligned} \quad (9)$$

$$\leq \sum_{k=1}^{d_j} \left( \max_{V_k \subseteq \{u_1^{(j)}, \dots, u_{k-1}^{(j)}\} \cup T_{j,i}, |V_k| \leq d} f_j\left(\{u_k^{(j)}\} \cup V_k | X_{j,i-1}\right) \right) \quad (10)$$

$$\leq d_j f_{j_i^*}(T_i | X_{j_i^*,i-1}), \quad (11)$$

where (9) follows from the  $d$ -scopic submodularity of  $f_j$  (note that  $u_k^{(j)} \notin T_{j,i}$  for  $j \neq j^*$ ), (10) follows from monotonicity of  $f_j$ , and (11) follows from the batched greedy selection of Algorithm 2 that  $f_{j_i^*}(T_i | X_{j_i^*,i-1})$  achieves the maximal possible marginal among sets of size at most  $d+1$ . Summing over  $j \neq j^*$ , we have:

$$\sum_{j \neq j^*} (f_j(T_{j,i-1}|X_{j,i-1}) - f_j(T_{j,i}|X_{j,i})) \leq \sum_{j \neq j^*} d_j \cdot f_{j_i^*}(T_i | X_{j_i^*,i-1}) \leq (d+1) \cdot f_{j_i^*}(T_i | X_{j_i^*,i-1}) \quad (12)$$

**Case 2** (term with  $j_i^*$ ):

$$f_{j_i^*}(T_{j_i^*,i-1}|X_{j_i^*,i-1}) + f_{j_i^*}(X_{j_i^*,i-1}) = f_{j_i^*}(T_{j_i^*,i-1} \cup X_{j_i^*,i-1}) \leq f_{j_i^*}(T_{j_i^*,i} \cup X_{j_i^*,i}) = f_{j_i^*}(T_{j_i^*,i} | X_{j_i^*,i}) + f_{j_i^*}(X_{j_i^*,i}).$$

Therefore,

$$f_{j_i^*}(T_{j_i^*,i-1}|X_{j_i^*,i-1}) - f_{j_i^*}(T_{j_i^*,i}|X_{j_i^*,i}) \leq f_{j_i^*}(X_{j_i^*,i}) - f_{j_i^*}(X_{j_i^*,i-1}) = f_{j_i^*}(T_i | X_{j_i^*,i-1}). \quad (13)$$

Combining (12) and (13), we have established (8). Now, suppose the algorithm terminates after  $t$  steps, during which at step  $i$ , subset  $T_i$  is allocated to agent  $j_i^*$ . We have:

$$\begin{aligned} \sum_{j=1}^n f_j(X_j^*) &= \sum_{j=1}^n f_j(T_{j,0}|X_{j,0}) \\ &= \sum_{0 \leq i < t} \sum_j (f_j(T_{j,i}|X_{j,i}) - f_j(T_{j,i+1}|X_{j,i+1})) \\ &\leq (d+2) \sum_{0 \leq i < t} (f_{j_i^*}(X_{j_i^*,i}) - f_{j_i^*}(X_{j_i^*,i-1})) \\ &= (d+2) \sum_{j=1}^n f_j(X_{j,t}) \\ &= (d+2) \sum_{j=1}^n f_j\left(X_j^{\text{BatchedGreedy}}\right). \end{aligned}$$

□

To show that our algorithm is nearly optimal, we prove the following information-theoretical lower bound: Similar to Theorem 9, the exponent  $(m/n)^{0.99}$  in the theorem below, can be replaced by any function of  $m/n$  in  $o(m/n)$ .

**Theorem 11.** For any  $d \in \mathbb{N}$ ,  $\varepsilon > 0$ , there is a family of function  $f_1, \dots, f_n : 2^{[m]} \rightarrow \mathbb{R}^+$  with  $\text{SMW}(f_i) = d, \forall i \in [n]$ , such that any (possibly randomized) algorithm that produces a  $(1/(d+1) + \varepsilon)$ -approximation (with constant probability if randomized) for the  $n$ -agent welfare maximization problem makes at least  $\Omega\left((n/2D)^{(m/n)^{0.99}}\right)$  value queries.

*Proof.* This proof follows from a similar argument as the proof for Theorem 9. Consider a ground set  $X$  of  $m$  elements, which contains a family of subsets  $R_1, \dots, R_n$  of  $r$  “special” elements. We will specify  $r$  below. We construct a family of (a slightly different version of) “hard-to-distinguish” set functions, which have the same supermodular degree.

To formulate these functions, let us first consider a set  $R \subseteq X$  of  $r$  elements. We construct a function  $f_R$  such that for any  $S \subseteq X$ ,  $f_R(S) = g_R(|S|, \mathbb{I}[R \subseteq S])$ , for a function  $g_R : \mathbb{N} \times \mathbb{B} \rightarrow \mathbb{R}$ . Like in Theorem 9,  $f_R$  depends on the cardinality of  $S$  and whether or not  $S$  completely contains  $R$ . In the discussion below, let  $D = d + 1$ , and let  $c_1$  and  $c_2$  be functions of  $m$  and  $n$  to be determined later. We set  $|R| = r = c_1 D + 1$ . We define  $f_R$  as follows:

$$f_R(S) = \begin{cases} \lfloor |S|/D \rfloor, & R \not\subseteq S, |S| \leq c_1 D + c_2 D^2 - 1 \\ c_1 + c_2 D, & R \not\subseteq S, |S| \geq c_1 D + c_2 D^2 \\ |S| - c_1(D - 1), & R \subseteq S, c_1 D < |S| \leq (c_1 + c_2)D - 1 \\ c_1 + c_2 D, & R \subseteq S, |S| \geq (c_1 + c_2)D \end{cases}.$$

Like in Theorem 9 — for similar reasons —  $f_R$  is in  $\text{SMW}(f_R) \leq d$ .

Each instance of the  $n$ -agent welfare maximization is defined by a family of subsets  $R_1, \dots, R_n \subseteq X$  satisfying for any  $i \neq j$ ,  $R_i \cap R_j = \emptyset$ . The  $i$ -th agent’s valuation function is then  $f_i := f_{R_i}$ .

Consider the case  $m = n \cdot s$  for  $s := (c_1 + c_2)D$ . We will use the following properties:

- A partition  $(X_1, \dots, X_n)$  of  $X$  is an optimal solution to the  $n$ -agent welfare maximization problem with value functions  $f_{R_1}, \dots, f_{R_n}$  if and only if  $X_i \supseteq R_i, \forall i \in [n]$ . The maximum welfare achievable is  $n(c_1 + c_2 D)$ .
- Let  $t = c_1 D + c_2 D^2 - 1$ , which is the largest size of  $S$  such that  $f(S)$  is not a constant. If no agent  $i$  receives a set  $X_i$  with  $|X_i| \leq t$  that is a superset of  $R_i$ , then the maximum possible welfare is  $\lfloor ns/D \rfloor \leq n(c_1 + c_2)$ .

So no algorithm can — when  $c_1 = o(c_2)$  — achieve a better approximation ratio than  $\lfloor n(c_1 + c_2) \rfloor / \lfloor n(c_1 + c_2 D) \rfloor \rightarrow 1/D$  without finding a set  $X_i$  of size at most  $t$  containing  $R_i$ , for some  $i$ . We therefore reduce the analysis to a simple problem, where the goal is to find a set of size  $t = c_1 D + c_2 D^2 - 1$  containing some  $R_i$  as a subset: For each query, the algorithm can specify a set  $S$  and an index  $i \in [k]$ , and is informed — by observing  $f_i(S)$  — whether  $S$  contains  $R_i$ . For similar reasons as in Theorem 9, the algorithm has no incentive to make queries of  $f_i(S)$  for  $|S| < t$  or  $|S| > t$ .

We focus on the query complexity of any deterministic welfare optimization algorithms. Assume the algorithm makes  $T$  queries,  $(S_1, k_1), \dots, (S_T, k_T)$  with  $|S_i| = t, \forall i \in [T]$ , which are deterministically chosen when the algorithm is fixed. We now establish a condition on  $T$  such that there is a family of disjoint subsets  $(R_1, \dots, R_n)$  such that  $R_{k_i} \not\subseteq S_i, \forall i \in [T]$ . Consider the distribution  $R_1, \dots, R_n$  uniformly at random conditioned on  $R_i \cap R_j = \emptyset$ . Let  $C_i$  be the event that  $S_i$  contains  $R_{k_i}$ . Then,

$$\Pr[C_1 \cup \dots \cup C_T] \leq \sum_i \Pr[C_i] < \sum_i \left( \frac{|S_i|}{m} \right)^r = T \left( \frac{c_1 D + c_2 D^2 - 1}{m} \right)^{c_1 D + 1} \leq T \left( \frac{2c_2 D^2}{m} \right)^{c_1 D}.$$

So, if  $T \leq \lfloor m/(2c_2 D^2) \rfloor^{c_1 D}$  then  $\Pr[C_1 \cup \dots \cup C_T] < 1$ . In other words, for any queries  $(S_1, k_1), \dots, (S_T, k_T)$  with  $|S_i| = t, \forall i \in [T]$ , there are disjoint subsets  $R_1, \dots, R_n$  such that  $R_{k_i} \not\subseteq S_i, \forall i \in [T]$ , implying the deterministic algorithm with queries  $(S_1, k_1), \dots, (S_T, k_T)$ , will not find a good approximation to  $(f_{R_1}, \dots, f_{R_n})$ . Let  $c_2 = \frac{1}{2} c_1^{1.01}$ . We have

$$\frac{m}{n} = s = (c_1 + c_2)D \geq c_2 D = \frac{2c_2 D^2}{2D} \Rightarrow \frac{m}{2c_2 D^2} \geq \frac{n}{2D},$$

and

$$\left(\frac{m}{n}\right)^{0.99} = s^{0.99} = (c_1 D + 1/2c_1^{1.01} D)^{0.99} \leq c_1 D.$$

So  $(m/2c_2 D^2)^{c_1 D} \geq (n/2D)^{(m/n)^{0.99}}$ . Thus, we conclude that any  $(1/(d+1) + \varepsilon)$ -approximation algorithm must make at least  $(n/2D)^{(m/n)^{0.99}}$  value queries.

Now consider a randomized welfare optimization algorithm. Conditioned on the random bits of the algorithm, the above argument still works. Taking expectation of the probability of success, we see that the overall probability of success is at most  $T(2D/n)^{(m/n)^{0.99}}$ . Thus, a constant probability of success requires  $T = \Omega\left((n/2D)^{(m/n)^{0.99}}\right)$ .  $\square$

## 4 Efficiency of Simple Auctions

In this section, we study the connection between the SAW hierarchy and efficiency of auctions. We will draw extensively on previous work in this area, particularly on the characterization based on the *CH hierarchy* — see definition below — which is arguably the most simple class of set functions with complementarity.

**Definition 14** (*d-Constraint Homogeneous Functions [10]*). *A set function  $f$  over ground set  $X$  is  $d$ -constraint homogeneous (CH- $d$ ) if there exists a value  $\hat{f}$ , and disjoint sets  $Q_1, \dots, Q_h \subseteq X$  with  $|Q_i| \leq d, \forall i \in [h]$ , such that (1)  $f(Q_i) = \hat{f} \cdot |Q_i|, \forall i \in [h]$ , and (2) the value of every set  $S \subseteq [m]$  is simply the sum of values of contained  $Q_i$ 's, i.e.,  $f(S) = \sum_{Q_i \subseteq S} f(Q_i) = \hat{f} \cdot \sum_{Q_i \subseteq S} |Q_i|$ .*

We will show that previous characterization of auction efficiency [10] can be approximately extended from the CH hierarchy to the SAW hierarchy.

### 4.1 Backgrounds: Related Definitions and Results

We first restate a useful definition and a lemma for analyzing the efficiency of auction mechanisms.

**Definition 15** ([26]). *An auction mechanism  $\mathcal{M}$  is  $(\lambda, \mu)$ -smooth for a class of valuations  $\mathcal{F} = \times_i \mathcal{F}_i$  if for any valuation profile  $f \in \mathcal{F}$ , there exists a (possibly randomized) action profile  $a_i^*(f)$  such that for every action profile  $a$ :*

$$\sum_i \mathbb{E}_{a_i^* \sim a_i^*(f)} [u_i(a_i^*, a_{-i}; f_i)] \geq \lambda \cdot \text{OPT}(f) - \mu \sum_i P_i(a),$$

where  $u_i(a_i^*; f_i)$  is the utility of  $i$  given action profile  $(a_i^*, a_{-i})$ ,  $\text{OPT}(f)$  is the optimum social welfare given valuation profile  $f$ , and  $P_i(a)$  is the payment of  $i$  given action profile  $a$ .

**Lemma 4.1** ([26]). *If a mechanism is  $(\lambda, \mu)$ -smooth then the price of anarchy w.r.t. coarse correlated equilibria is at most  $\max\{1, \mu\}/\lambda$ .*

For Single-bid Auction and Simultaneous Item First Price Auction (SIA), we will derive our results from the following results for CH- $d$  and MPH- $d$ .

**Theorem 12** (Smoothness of Single-bid Auction with CH- $d$  Valuations [10]). *Single-bid Auction is a  $((1 - e^{-d})/d, 1)$ -smooth mechanism when agents have CH- $d$  valuations. Consequently, Single-bid Auction has a PoA of  $(1 - e^{-d})/d$  with CH- $d$  valuations w.r.t. coarse correlated equilibria.*

**Theorem 13** (Smoothness of SIA with MPH- $d$  Valuations [8]). *For SIA, when bidders have MPH- $d$  valuations, both the correlated price of anarchy and the Bayes-Nash price of anarchy are at most  $2d$ . The bound follows from a smoothness argument.*

A key concept to extend these results to other valuation classes is the following notion of pointwise approximation defined in [4].

**Definition 16** (Pointwise Approximation [4]). *A class of set functions  $\mathcal{F}$  over ground set  $X$  is pointwise  $\beta$ -approximated by another class  $\mathcal{F}'$  of set functions over  $X$  if  $\forall f \in \mathcal{F}, S \subseteq X, \exists f'_S \in \mathcal{F}'$  such that (1)  $\beta f'_S(S) \geq f(S)$  and (2)  $\forall T \subseteq X, f'_S(T) \leq f(T)$ .*

For example:

**Proposition 4.1** ([10]). *The class  $\max(\mathcal{F})$  is pointwise 1-approximated by the class  $\mathcal{F}$ .*

We say a function  $f' : 2^X \rightarrow \mathbb{R}$  pointwise  $\beta$ -approximates  $f : 2^X \rightarrow \mathbb{R}$  (at  $X$ ), if (1)  $\beta f'(X) \geq f(X)$ , and (2)  $\forall T \subseteq X, f'(T) \leq f(T)$ .

The following lemma of [4] provides a way to translate PoA bounds between classes via pointwise approximation.

**Lemma 4.2** (Extension Lemma [4]). *If a mechanism for a combinatorial auction setting is  $(\lambda, \mu)$ -smooth for the class of set functions  $\mathcal{F}'$ , and  $\mathcal{F}$  is pointwise  $\beta$ -approximated by  $\mathcal{F}'$ , then it is  $(\frac{\lambda}{\beta}, \mu)$ -smooth for the class  $\mathcal{F}$ . And as a result, if a mechanism for a combinatorial auction setting has a PoA of  $\alpha$  given by a smoothness argument for the class  $\mathcal{F}'$ , and  $\mathcal{F}$  is pointwise  $\beta$ -approximated by  $\mathcal{F}'$ , then it has a PoA of  $\alpha\beta$  for the class  $\mathcal{F}$ .*

## 4.2 Efficiency of Simple Auctions Parametrized by SAW

Applying Lemma 4.2, we are able to translate Theorems 12 and 13 to the SAW hierarchy.

**Theorem 14** (Efficiency of Single-bid Auction with SAW- $d$  Valuations). *When agents have valuations  $f_1, \dots, f_n \in \max(\text{SAW-}d)$ , Single-bid Auction has a price of anarchy of at most  $\frac{2d}{1-e^{-2d}} \cdot H_{\frac{m}{2d}}$  w.r.t. coarse correlated equilibria.*

**Theorem 15** (Efficiency of SIA with SAW- $d$  Valuations). *When agents have valuations  $f_1, \dots, f_n \in \max(\text{SAW-}d)$ , SIA has a price of anarchy of at most  $8d \cdot H_{\frac{m}{2d}}$  w.r.t. coarse correlated equilibria.*

Formally, Theorems 14 and 15 follow from Theorems 12 and 13 respectively, with the help of Lemma 4.2, Proposition 4.1, and the technical lemma (Lemma 4.3) that we will establish below, showing that for any  $d \in \mathbb{N}$ , functions in SAW- $d$  can be approximated by CH- $2d$  functions. In particular, Lemma 4.3 establishes the approximation of SAW hierarchy by CH hierarchy with a loss of factor  $O(\log m)$ .

**Lemma 4.3** (Pointwise Approximation of SAW Hierarchy by CH-Hierarchy). *For any  $d \in \mathbb{N}$ , SAW- $d$  is pointwise  $2H_{\frac{m}{2d}}$ -approximated by CH- $2d$ , where  $H_i = \sum_{k \in [i]} \frac{1}{k}$  is the  $i$ -th harmonic number.*

*Proof.* Our proof is inspired by the constructions of [4] and [10].

For any  $f \in \text{SAW-}d$  over  $X = [m]$ , we first apply the following greedy construction to obtain a partition  $\mathcal{Q} = \{Q_i\}_{i \in [q]}$  of  $[m]$  into sets of size not exceeding  $2d$ : At step  $i$ , we select a new set  $Q_i \subseteq [m] \setminus (Q_1 \cup \dots \cup Q_{i-1})$ , with maximum  $f(Q_i)$ , among all sets of size at most  $2d$ .

We first prove by contradiction that there exists a function  $g$  in CH- $2d$  which  $2H_{\frac{m}{2d}}$ -approximates  $f$  at  $[m]$ . That is, (1)  $2H_{\frac{m}{2d}}g([m]) \geq f([m])$  and (2)  $\forall T \subseteq [m], g(T) \leq f(T)$ .

Suppose this statement is not true. Let

$$h_{\mathcal{Q}}(T) = \frac{f([m])}{\beta \cdot |\cup_i Q_i|} \sum_{Q_i \subseteq T} |Q_i|.$$

Note that  $h_{\mathcal{Q}} \in \text{CH-}2d$  because  $|Q_i| \leq 2d, \forall Q_i \in \mathcal{Q}$ . We now construct a series of functions based on  $h_{\mathcal{Q}}$ , and prove that for any  $\beta > 0$ , if there is no  $g$  among these functions that is a  $\beta$ -approximation of  $f$  at  $[m]$  — that is, there is no  $g$  such that (1)  $\beta g([m]) \geq f([m])$  and (2)  $\forall T \subseteq [m], g(T) \leq f(T)$ , (below we will refer to this condition as Assumption (\*)) — then  $\beta < 2H_{\frac{m}{2d}}$ .



First consider  $h_{\mathcal{Q}}$ . Note that  $\beta h_{\mathcal{Q}}([m]) = \beta \frac{f([m])}{\beta} \geq f([m])$ , because  $\mathcal{Q}$  is a partition of  $[m]$ . Assumption (\*) then implies there is a  $T_1$  such that  $h_{\mathcal{Q}}(T_1) > f(T_1)$ . W.l.o.g. assume  $T_1$  is a union of sets from  $\mathcal{Q}$  (such  $T_1$  exists because  $f$  is monotone).

Let  $S_1 = [m]$ . We now iteratively define  $S_i = S_{i-1} \setminus T_{i-1}$ , and construct its associated  $T_i$ . The construction maintains the following invariant: Both  $S_i$  and  $T_i$  are unions of sets from  $\mathcal{Q}$ . The former follows directly from the iterative property that  $S_{i-1}$  and  $T_{i-1}$  are both unions of sets from  $\mathcal{Q}$ . Our construction below will ensure the latter.

Let  $\mathcal{Q}_{S_i} = \{Q \in \mathcal{Q} \mid Q \subseteq S_i\}$ . Let

$$h_{\mathcal{Q}_{S_i}} = \frac{f([m])}{\beta \cdot |\cup_{j:Q_j \in \mathcal{Q}_{S_i}} Q_j|} \sum_{j:Q_j \in \mathcal{Q}_{S_i}} |Q_j|.$$

Again,  $h_{\mathcal{Q}_{S_i}} \in \text{CH-}2d$ , and  $h_{\mathcal{Q}_{S_i}}([m]) = \frac{f([m])}{\beta}$ . Assumption (\*) then implies there is a  $T_i$  such that  $h_{\mathcal{Q}_{S_i}}(T_i) > f(T_i)$ . Again, w.l.o.g. assume  $T_i$  is a union of sets from  $\mathcal{Q}$  (such  $T_i$  exists because  $f$  is monotone). This iterative process terminates, producing a partition  $\{T_i\}_{i \in [t]}$  of  $[m]$ , which satisfies:

$$\sum_i f(T_i) < \sum_i h_{\mathcal{Q}_{S_i}}(T_i) = \frac{f([m])}{\beta} \sum_i \frac{|T_i|}{|S_i|} \leq \frac{f([m])}{\beta} \sum_{i \in [t]} \frac{1}{i} \leq \frac{f([m])}{\beta} H_{\frac{m}{2d}}.$$

We now show that  $\sum_i f(T_i) \geq \frac{1}{2}f([m])$ . Recall that each member in partition  $\{T_i\}_i$  is a unions of sets from  $\mathcal{Q}$ . We renumber  $\{T_i\}_i$ , in a way that for any  $i < j$ , there is some  $T_i \supseteq Q_k \in \mathcal{Q}$ , such that for any  $T_j \supseteq Q_l \in \mathcal{Q}$ ,  $k < l$ . That is, the smallest index  $k$  where  $Q_k \in T_i$  is smaller than the smallest index  $l$  where  $Q_l \in T_j$ , as long as  $i < j$ .

Since  $(T_1, \dots, T_t)$  is a partition of  $[m]$ , we have:

$$\begin{aligned} f([m]) &= \sum_i f(T_i | T_{i+1} \cup \dots \cup T_t) \\ &\leq \sum_i \max\{f(T_i | U_i) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, |U_i| \leq d\} \end{aligned} \quad (14)$$

$$\leq \sum_i \max\{f(T_i \cup U_i) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, |U_i| \leq d\} \quad (15)$$

$$= \sum_i \max\{(f(U_i | T_i) + f(T_i)) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, |U_i| \leq d\}$$

$$\leq \sum_i \max\{(f(U_i | V_i) + f(T_i)) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, |U_i| \leq d, V_i \subseteq T_i, |V_i| \leq d\} \quad (16)$$

$$\leq \sum_i \max\{(f(U_i \cup V_i) + f(T_i)) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, |U_i| \leq d, V_i \subseteq T_i, |V_i| \leq d\} \quad (17)$$

$$\leq \sum_i (f(Q_{k_i}) + f(T_i)), \text{ where } k_i = \min\{k \mid T_i \supseteq Q_k \in \mathcal{Q}\} \quad (18)$$

$$\leq \sum_i 2f(T_i), \quad (19)$$

where (14) and (16) follow from  $d$ -scopic subadditivity of  $f$ , (15), (17) and (19) follow from monotonicity of  $f$ , and (18) holds because, according to the construction of  $\{Q_l\}_l$ ,  $Q_{k_i}$  maximizes  $f$  among all sets of size  $2d$  contained in  $Q_{k_i} \cup \dots \cup Q_q \supseteq T_i \cup \dots \cup T_t$ , and in particular  $U_i \cup V_i \subseteq T_i \cup \dots \cup T_t$ .

Consequently, it follows from  $\sum_i f(T_i) \geq \frac{1}{2}f([m])$  that:

$$\frac{H_{\frac{m}{2d}} f([m])}{\beta} > \sum_i f(T_i) \geq \frac{1}{2}f([m]) \Rightarrow \beta < 2H_{\frac{m}{2d}}.$$

Thus, Assumption (\*) with  $\beta \geq 2H_{\frac{m}{2d}}$  leads to a contradiction. Therefore, we have established that there exists a CH-2d function  $g$  such that (1)  $g([m]) \geq 2H_{\frac{m}{2d}} f([m])$  and (2)  $\forall T \subseteq [m], g(T) \leq f(T)$ .

As in [10], the above proof can be simply extended to prove for any  $S \subseteq X$ , there exists a CH-2d function  $g$  such that (1)  $g(S) \geq 2H_{\frac{m}{2d}} f([m])$  and (2)  $\forall T \subseteq [m], g(T) \leq f(T)$ . Essentially, we restrict the function  $f$  to  $2^S$ , apply the argument above, and then span the obtained function back to  $2^X$ .

Therefore, SAW- $d$  is pointwise  $2H_{\frac{m}{2d}}$ -approximated by CH-2d.  $\square$

We further analyze previously known hard instances to both auctions, and show that they provide almost matching lower bounds to the above two efficiency upper bounds.

**Theorem 16.** *There is an instance with SAW- $d$  valuations for any  $d$ , where the PoS of Single-bid Auction is at least  $d + 1 - \varepsilon/d$  for any  $\varepsilon > 0$ .*

*Proof.* Consider two players with valuations  $f_1$  and  $f_2$  over ground set  $X = [m]$ . Let  $h_T(S) = \mathbb{I}[T \subseteq S]$ ,  $f_1(S) = \sum_{2 \leq i \leq d+1} h_{\{1, i\}}$ , and  $f_2(S) = \mathbb{I}[1 \in S] \left( \frac{d}{d+1} + \varepsilon \right)$ . Both  $f_1$  and  $f_2$  are in SAW- $d$  because there are at most  $d + 1$  items which matter to the valuations. As shown in Proposition 3.9 of [10], Single-bid Auction has a PoS of  $d + 1 - \varepsilon/d$  on this instance.  $\square$

**Theorem 17.** *There is an instance with SAW- $d$  valuations for any  $d$ , where the PoA of SIA is at least  $d + 1/(d + 1)$ .*

*Proof.* Consider the instance given in Theorem 2.5 of [8]. That is, a projective plane of order  $d + 1$ . There are  $d(d + 1) + 1$  players, each desiring only a bundle of size  $d + 1$ , so the valuations of all players are in SAW- $d$ . As shown by Theorem 2.5 of [8] SIA has a PoA of at least  $d + 1/(d + 1)$  on the above instance.  $\square$

### 4.3 Efficiency of Simple Auctions Parametrized by SMW

As a byproduct of our efficiency results for the SAW hierarchy, we prove similar, but slightly weaker, results for the SMW hierarchy. We note that these bounds extend a central result in [10], which states that when agents have valuations in  $\max(\text{SD-}d \cap \text{SUPADD})$ , Single-bid Auction has a PoA of  $O(d^2 \log m)$ .

**Theorem 18** (Extending [10]). *When agents have valuations  $f_1, \dots, f_n \in \max(\text{SMW-}d \cap \text{SUPADD})$ , Single-bid Auction has a price of anarchy of at most  $\frac{(d+1)^2}{1-e^{-(d+1)}} \cdot H_{\frac{m}{d+1}}$  w.r.t. coarse correlated equilibria.*

**Theorem 19.** *When agents have valuations  $f_1, \dots, f_n \in \max(\text{SMW-}d \cap \text{SUPADD})$ , SIA has a price of anarchy of at most  $2(d + 1)^2 \cdot H_{\frac{m}{d+1}}$  w.r.t. coarse correlated equilibria.*

Like Theorems 14 and 15, the two theorems above follow from Theorems 12 and 13 respectively, with the help of Lemma 4.2, Proposition 4.1, and the technical lemma below.

**Lemma 4.4.** *For any  $d \in \mathbb{N}$ ,  $\text{SMW-}d \cap \text{SUPADD}$  is pointwise  $(d + 1)H_{\frac{m}{d+1}}$ -approximated by CH- $(d + 1)$ .*

*Proof.* The proof essentially follows from the same argument as that for Lemma 4.3. For any superadditive  $f$  over  $X = [m]$  with  $\text{SMW}(f) \leq d$ , we first greedily construct a partition  $\{Q_i\}_i$  of  $X$ : At step  $i$ , we select a set  $Q_i$  of at most  $d + 1$  elements from  $X \setminus (Q_1 \cup \dots \cup Q_{i-1})$  that maximizes  $f(Q_i)$ . For  $x \in X$ , let  $\text{index}(x) = i$  iff  $x \in Q_i$ . W.l.o.g., for analysis below, we assume that elements in  $X$  are already sorted (or are renumbered) according to their indices, i.e., if  $\text{index}(x) < \text{index}(y)$  then  $x < y$ .

Following the proof of Lemma 4.3, we focus on proving by contradiction that there exists a CH- $(d + 1)$  function  $g$  that  $(d + 1)H_{\frac{m}{d+1}}$ -approximates  $f$ . That is (1)  $(d + 1)H_{\frac{m}{d+1}} g([m]) \geq f([m])$  and (2)  $\forall T \subseteq [m], g(T) \leq f(T)$ .

Suppose this statement is not true. Letting

$$h_{\mathcal{Q}_{S_i}} = \frac{f([m])}{\beta \cdot |\cup_{j: Q_j \in \mathcal{Q}_{S_i}} Q_j|} \sum_{j: Q_j \in \mathcal{Q}_{S_i}} |Q_j|,$$

and starting with  $S_1 = [m]$ , we can use the same iterative process to construct a sequence  $((S_1, T_1), \dots, (S_t, T_t))$  for some  $t \in \mathbb{N}$  such that (1) for all  $i \in [t]$ , both  $S_i$  and  $T_i$  are unions of sets from  $\mathcal{Q}$ , (2)  $(T_1, \dots, T_t)$  is a partition of  $X$ , and (3)  $\sum_i f(T_i) < \frac{f([m])}{\beta} H_{\frac{m}{d+1}}$ , under that assumption that  $\beta$  is a parameter such that all induced functions (from CH- $(d+1)$ ) satisfying (1)  $h_{\mathcal{Q}_{S_i}}([m]) = \frac{f([m])}{\beta}$ , and (2)  $h_{\mathcal{Q}_{S_i}}(T_i) > f(T_i)$ .

Now we have:

$$\begin{aligned} f([m]) &= \sum_k f(k|[k-1]) \\ &\leq \sum_k \max_{U_k: U_k \subseteq [k-1], |U_k| \leq d} f(k|U_k) \end{aligned} \tag{20}$$

$$\leq \sum_k \max_{U_k: U_k \subseteq [k-1], |U_k| \leq d} f(\{k\} \cup U_k) \tag{21}$$

$$\leq \sum_k f(Q_{\lceil \frac{m-k+1}{d+1} \rceil}) \tag{22}$$

$$\begin{aligned} &= \sum_j |Q_j| f(Q_j) \\ &\leq (d+1) \sum_j f(Q_j) \end{aligned}$$

where (20) follows from the fact that  $f \in \text{SAW-}d$ , (21) follows from monotonicity, and (22) holds because by the construction of  $\{Q_i\}_i$ ,  $Q_{\lceil \frac{m-k+1}{d+1} \rceil}$  maximizes  $f$  among all sets of size  $d+1$  contained in  $[k]$ .

On the other hand, since every  $T_i$  is a union of some  $Q_j$ 's, according to superadditivity of  $f$ ,

$$\begin{aligned} \beta &< \left( H_{\frac{m}{d+1}} f([m]) \right) \left( \sum_i f(T_i) \right)^{-1} \\ &\leq \left( H_{\frac{m}{d+1}} f([m]) \right) \left( \sum_i f(Q_i) \right)^{-1} \\ &\leq \left( H_{\frac{m}{d+1}} (d+1) \sum_i f(Q_i) \right) \left( \sum_i f(Q_i) \right)^{-1} \\ &= (d+1) H_{\frac{m}{d+1}}. \end{aligned}$$

For all other set  $S \subseteq X$ , we can apply a similar restricting-and-spanning-back argument with the above construction to prove that there exists a CH- $(d+1)$  function  $g$  such that (1)  $(d+1)H_{\frac{m}{d+1}}g(S) \geq f([S])$ , and (2)  $\forall T \subseteq [m], g(T) \leq f(T)$ .  $\square$

## 5 Remarks

### 5.1 Further Comparative Analysis

As observed by Eden *et al.* [5], the right measure of complementarity often varies from application to application. This seems to be true even with the supermodular vs superadditive widths. We note that while the SD and SMW hierarchies give nontrivial bounds on the PoA of simple auctions, SAW hierarchy seems to capture the intrinsic property needed by efficiency guarantees for simple auctions. It provides tighter characterization of PoA with a gap of  $\log m$  (instead of  $d \log m$ ) between upper and lower bounds. On the other hand, while SMW hierarchy captures the intrinsic property needed by the constrained/welfare maximization, it remains open whether a small superadditive width provides any approximation guarantee for the two optimization problems.

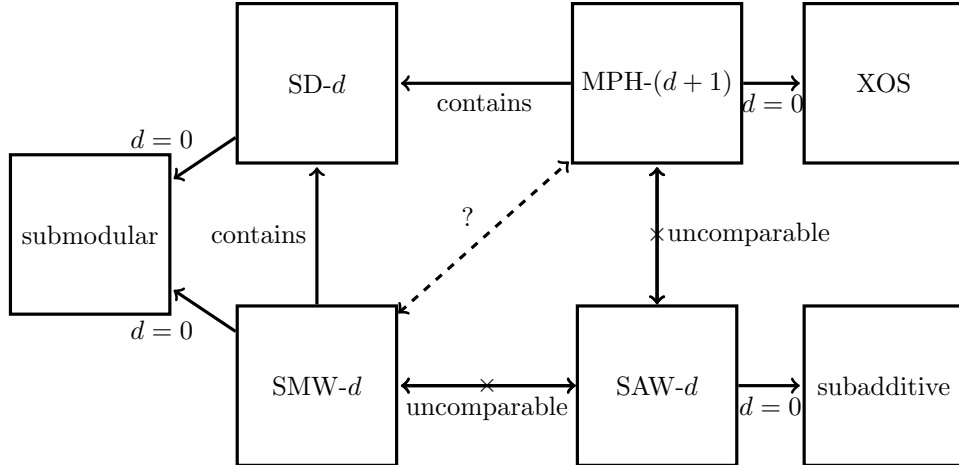


Figure 1: Relationship between hierarchies.

The MPH hierarchy takes a different approach from ours — it relies on a syntactic definition which provides elegant and intuitive structures. In contrast, both SMW and SAW hierarchies — like the SD hierarchy before it — are built on concrete natural concepts of witnesses and semantic intuition of complementarity. In the current definition, the MPH hierarchy is not an extension to submodularity or subadditivity. Rather — as shown in [8] — MPH can be considered as an extension to the fractionally subadditive (or XOS) class proposed in [18]. We therefore consider SMW, MPH and SAW parallel measures of complementarity, just like submodularity, fractional subadditivity and subadditivity in the complement-free case. One key difference is that the three hierarchies seem to diverge at higher levels of complementarity, as opposed to the fact that submodular functions are all fractionally subadditive, and fractionally subadditive functions are all subadditive. This phenomenon provides further evidence that the three hierarchies are likely to capture different aspects of complementarity. See Figure 1 for a comparison.

We also note that all upper bounds supported by our hierarchies are accompanied by almost matching lower bounds, which we consider as a justification of our definitions — they manage to categorize set functions roughly according to their “hardness” in different settings (i.e. optimization for SMW and efficiency for SAW). In contrast, while the less inclusive supermodular degree hierarchy supports a number of upper bounds, to our knowledge, none of those results are proven tight.

## 5.2 Final Remarks and Open Problems

Our SMW and SAW hierarchies may be applied to other problem settings. For example, for the online secretary problem based on supermodular degree [14], we believe that with a slight modification of the algorithms and the analysis, we could replace supermodular degree with supermodular width as well for this problem; also, SMW- $d$  functions are efficiently PAC-learnable under product distributions [29]. It may be possible to look into other venues where SMW and SAW hierarchies are applicable.

There are also a few technical questions to be answered:

- Does MPH- $(d + 1)$  — which subsumes SD- $d$  — include all SMW- $d$  functions?
- Can we improve the SAW-based efficiency characterization of of Single-bid Auction and SIA to  $O(d)$ ?
- Can the MPH hierarchy be used to characterize constrained set function maximization?

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