Internet Appendix for "The Expected Cost of Default"

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Appendix A. Pricing kernel, risk-neutral measure

Given the exogenously specified process for the pricing kernel, the risk-neutral measure can be derived.[1] The pricing kernel, π_t, evolves according to

$$\frac{d\pi_t}{\pi_t} = -r(\nu_t)dt - \varphi^m(\nu_t)dW_t^m\,. \tag{1}$$

Define the density process for the risk-neutral measure by

$$\xi_t = E_t\left[\frac{dQ}{dP}\right]\,. \tag{2}$$

This density process and the pricing kernel are related by

$$\xi_t = B_t\pi_t\,, \tag{3}$$

where

$$B_t = exp\left\{\int_0^t r(\nu_s)ds\right\} \tag{4}$$

is the time t price of a bond paying the riskless rate and B_0 has been normalized to one.[2] Applying Itô's lemma gives

$$d\xi_t = B_t d\pi_t + \pi_t dB_t\,. \tag{5}$$

Plugging in the expression for $d\pi_t$,

$$d\xi_t = B_t[-r(\nu_t)\pi_t dt - \varphi^m(\nu_t)\pi_t dW_t^m] + \pi_t dB_t\,. \tag{6}$$

Replacing π_t with $\frac{\xi_t}{B_t}$ and dividing through by ξ_t gives

$$\frac{d\xi_t}{\xi_t} = -r(\nu_t)dt - \varphi^m(\nu_t)dW_t^m + \frac{1}{B_t}dB_t\,. \tag{7}$$

Itô's lemma implies

$$dB_t = r(\nu_t)dt\,. \tag{8}$$

Thus, the density process, ξ_t, evolves according to

$$\frac{d\xi_t}{\xi_t} = -\varphi^m(\nu_t)dW_t^m\,. \tag{9}$$

[1]Because the horizon is infinite, the risk-neutral measure, $\mathcal{Q}$, that is used for pricing contingent claims is not an equivalent probability measure to the physical measure, $\mathcal{P}$. Still, the risk-neutral measure $\mathcal{Q}$ has the necessary properties for risk-neutral pricing. See **?** for more details.

[2]See **?**.

Applying Girsanov's Theorem, a new Brownian motion under the risk-neutral measure is given by

$$d\widehat{W}_t^m = dW_t^m + \varphi^m(\nu_t)dt\,. \tag{10}$$

The firm-specific Brownian motion, $W_t^{f,n}$, that generates the idiosyncratic shocks to firm n's cash flows is independent of the Brownian motion W_t^m generating systematic shocks to the economy. Thus, $W_t^{f,n}$ is still a Brownian motion under the risk-neutral measure for all firms n. Under the risk-neutral measure, cash flows for firm n evolve according to

$$\frac{dX_t^n}{X_t^n} = \widehat{\mu}^n(\nu_t)dt + \sigma_m^n(\nu_t)d\widehat{W}_t^m + \sigma_f^n dW_t^{f,n}\,, \tag{11}$$

where $\widehat{\mu}^n(\nu_t)$ is the drift under the risk-neutral measure

$$\widehat{\mu}^n(\nu_t) = \mu^n(\nu_t) - \sigma_m^n(\nu_t)\varphi^m(\nu_t)\,. \tag{12}$$

The total volatility of the cash flows of firm n is given by

$$\sigma_X^n(\nu_t) = \sqrt{(\sigma_m^n(\nu_t))^2 + (\sigma_f^n)^2}\,. \tag{13}$$

The two Brownian motions driving the idiosyncratic and systematic shocks to firm n's cash flows under the risk-neutral measure can be aggregated into a single Brownian motion (under the risk-neutral measure) for firm n, which is given by

$$d\widehat{W}_t^n = \frac{\sigma_m^n(\nu_t)}{\sigma_X^n(\nu_t)}d\widehat{W}_t^m + \frac{\sigma_f^n}{\sigma_X^n(\nu_t)}dW_t^{f,n}. \tag{14}$$

So the evolution of firm n's cash flows under the risk-neutral measure can be expressed as

$$\frac{dX_t^n}{X_t^n} = \widehat{\mu}^n(\nu_t)dt + \sigma_X^n(\nu_t)d\widehat{W}_t^n\,. \tag{15}$$

Appendix B. Solving for unlevered firm value

Here I show how to solve for the unlevered firm value.[3] The pair of ordinary differential equations characterizing the unlevered firm value has an associated characteristic function given by

$$g_1(\beta)g_2(\beta) = \lambda_1\lambda_2\,, \tag{16}$$

where

$$g_1(\beta) = \lambda_1 + r - (\mu_1 - \frac{1}{2}\sigma_1^2)\beta - \frac{1}{2}\sigma_1^2\beta^2 \tag{17}$$

[3]The exposition follows **?**. See also **?** and **?**.

and

$$g_2(\beta) = \lambda_2 + r - (\mu_2 - \frac{1}{2}\sigma_2^2)\beta - \frac{1}{2}\sigma_2^2\beta^2\,. \tag{18}$$

This characteristic function has four distinct roots: $\beta_1 < \beta_2 < 0 < \beta_3 < \beta_4$. The general form of the solution is given by

$$A^1(X) = \phi_1(X) + \sum_{i=1}^{4} G_i x^{\beta_i}\,, \tag{19}$$

$$A^2(X) = \phi_2(X) + \sum_{i=1}^{4} H_i x^{\beta_i}\,, \tag{20}$$

and

$$H_i = l(\beta_i)G_i = \frac{g_1(\beta_i)}{\lambda_1}G_i = \frac{\lambda_2}{g_2(\beta_i)}G_i\,. \tag{21}$$

However, boundedness conditions on the unlevered firm value need to be imposed. These are

$$\lim_{x\to\infty} \frac{A^i(x)}{x} < \infty \quad \text{and} \quad \lim_{x\to 0} A^i(x) < \infty\,. \tag{22}$$

These two conditions imply $\beta_i = 0$, $i = 1, ..., 4$. Thus the unlevered firm value has the form

$$A^i(X) = \phi_i(X) \tag{23}$$

Conjecture that the unlevered firm value is affine in X. That is,

$$A^i(X) = c_i X + d_i \tag{24}$$

Furthermore, $d_i = 0,\ i = 1, 2$, because $A^i(0) = 0$.
Thus, the conjecture becomes

$$A^i(X) = c_i X\,. \tag{25}$$

Plugging these expressions into the two ODEs characterizing the unlevered firm value and with some rearranging gives a linear system of two equations in two unknowns:

$$\mu_i c_i X - (\lambda_i + r)c_i X + X + \lambda_i c_j X = 0,\ \ j \neq i\,. \tag{26}$$

Solving these two equations for c_1, c_2 gives the unlevered firm value in state i as

$$A^i(X) = \frac{(\lambda_1 + \lambda_2 + r - \mu_j)X}{\lambda_2(r - \mu_1) + (r - \mu_2)(\lambda_1 + r - \mu_1)}\,. \tag{27}$$

If $\mu_1 = \mu_2$, then the unlevered firm value is the same in both states and is given by

$$A(X) = \frac{X}{r - \mu}\,. \tag{28}$$

Appendix C. Eigenvalue problem

This Appendix describes the eigenvalue problem for the cash flow region in which neither default nor restructuring is an immediate threat. Define the log cash flow process, $x_t = log(X_t)$. By Itô's lemma, under the risk-neutral measure, the log cash flow process evolves according to

$$dx_t = \left[\widehat{\mu}(\nu_t) - \frac{1}{2}\sigma_X(\nu_t)^2\right] dt + \sigma_X(\nu_t) d\widehat{W}_t\,. \tag{29}$$

Under the risk-neutral measure, the price process of any contingent claim on firm cash flows is a martingale with the cash flows discounted by investors at the risk-free short rate, $r(\nu_t)$. Thus, these contingent claims are martingales of the form

$$M_t^f = exp\left(-\int_0^t r(\nu_u)\, du\right) f(\nu_t, x_t) \tag{30}$$

for some function f that depends on the payoffs of the given security.

Applying Itô's lemma gives

$$dM_t^f = exp\left(-\int_0^t r(\nu_u)\, du\right) \left[(\Lambda - R)f + \frac{1}{2}\Sigma f_{xx} + \Theta f_x\right] dt\,. \tag{31}$$

R is the diagonal matrix of r_i 's. Σ is the diagonal matrix of σ_{iX}^2 's. Θ is the diagonal matrix of the risk-neutral drifts of the log cash flow process. Λ is the generator matrix of the Markov chain, ν_t.

Because M_t^f is a martingale, it has zero drift, implying

$$(\Lambda - R)f + \frac{1}{2}\Sigma f_{xx} + \Theta f_x = 0\,. \tag{32}$$

Seeking a separable f of the form

$$f(\nu_t, x_t) = g(\nu_t) exp(-\beta x_t) = g(\nu_t) X_t^{\beta}\,, \tag{33}$$

gives the following equation to be solved in β and g:

$$(\Lambda - R)g + \frac{1}{2}\beta^2 \Sigma g - \beta\Theta g = 0. \tag{34}$$

Pre-multiplying the above equation by $2\Sigma^{-1}$ gives

$$2\Sigma^{-1}(\Lambda - R)g + \beta^2 g - 2\beta\Sigma^{-1}\Theta g = 0. \tag{35}$$

This gives the following system of equations:

$$\begin{aligned} \beta g &= h & (36)\\ \beta h &= 2\Sigma^{-1}\Theta h - 2\Sigma^{-1}(\Lambda - R)g\,. & (37) \end{aligned}$$

This can be written as a standard eigenvalue problem of the form

$$A\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} 0 & I \\ -2\Sigma^{-1}(\Lambda - R) & 2\Sigma^{-1}\Theta \end{pmatrix}\begin{pmatrix} g \\ h \end{pmatrix} = \beta\begin{pmatrix} g \\ h \end{pmatrix}. \tag{38}$$

If (g, β) solve this eigenvalue problem, then

$$M_t^f = exp\left(-\int_0^t r(\nu_u)\,du - \beta x_t\right) g(\nu_t) \tag{39}$$

is a martingale. The matrix A has exactly two eigenvalues with positive real parts and two with negative real parts.

Appendix D. Solving for the w coefficients

For the case in which there are two aggregate states to the Markov chain, there are a total of three relevant cash flow regions and each security has a total of 16 w coefficients (eight for each initial state).

The cash flow regions are
Region 1: $X \in [X_D^1, X_D^2)$
Region 2: $X \in [X_D^2, X_U^{u(1)})$
Region 3: $X \in [X_U^{u(1)}, X_U^{u(2)})$
For $X < X_D^1$ the firm is always in default regardless of the state, and for $X > X_U^{u(2)}$ the firm has already restructured upward for any state.

D.1. Debt

For a given initial state, ν_0, the eight boundary conditions for debt are

$$\begin{aligned}
\lim_{X \uparrow X_D^2} D(X,1,\nu_0) &= \lim_{X \downarrow X_D^2} D(X,1,\nu_0) && (40)\\
\lim_{X \uparrow X_D^2} D_X(X,1,\nu_0) &= \lim_{X \downarrow X_D^2} D_X(X,1,\nu_0) && (41)\\
\lim_{X \uparrow X_U^{u(1)}} D(X,u(2),\nu_0) &= \lim_{X \downarrow X_U^{u(1)}} D(X,u(2),\nu_0) && (42)\\
\lim_{X \uparrow X_U^{u(1)}} D_X(X,u(2),\nu_0) &= \lim_{X \downarrow X_U^{u(1)}} D_X(X,u(2),\nu_0) && (43)\\
D(X_D^1,1,\nu_0) &= (1-\alpha(1))V^U(X_D^1,1) && (44)\\
D(X_D^2,2,\nu_0) &= (1-\alpha(2))V^U(X_D^2,2) && (45)\\
D(X_U^{u(1)},u(1),\nu_0) &= D(X_0,\nu_0) && (46)\\
D(X_U^{u(2)},u(2),\nu_0) &= D(X_0,\nu_0) && (47)
\end{aligned}$$

Eqs. (40) and (42) are the value-matching conditions across cash flow regions, and Eqs. (41) and (43) are the smooth-pasting conditions across regions. Eqs. (46) and (47) are the value-matching boundary conditions for default, and Eqs. (46) and (47) are the value-matching boundary conditions for upward-restructuring.

The initial (par value) of debt at time 0 is given by

$$D(X_0,\nu_0;\nu_0) = w_{2,1}^D(\nu_0)g_{2,1}(\nu_0)exp\{\beta_{2,1}x_0\} + w_{2,2}^D(\nu_0)g_{2,2}(\nu_0)exp\{\beta_{2,2}x_0\} + \\ w_{2,3}^D(\nu_0)g_{2,3}(\nu_0)exp\{\beta_{2,3}x_0\} + w_{2,4}^D(\nu_0)g_{2,4}(\nu_0)exp\{\beta_{2,4}x_0\} + (1-\tau_i)C(\nu_0)\mathbf{b}(\nu_0)$$

$$D(X_0,\nu_0;\nu_0) = \sum_{j=1}^{4} w_{2,j}^D(\nu_0)g_{2,j}(\nu_0)exp\{\beta_{2,j}x_0\} + (1-\tau_i)C(\nu_0)\mathbf{b}(\nu_0) \tag{48}$$

$g_{2,j}(\nu_0)$ is a scalar; that is, it is the ν_0 element of the $g_{2,j}$ eigenvector, where $g_{2,j}$ is the jth eigenvector for the eigenvalue problem for the second cash flow region. Thus, a system of eight equations results to solve for the eight unknown w^D coefficients.

$$\begin{aligned}
G(X)_{LHS}W^D + \xi(X)_{LHS} + \zeta_{LHS} &= G(X)_{RHS}W^D + \xi(X)_{RHS} + \zeta_{RHS} && (49)\\
\left[G(X)_{LHS} - G(X)_{RHS}\right]W^D &= \xi(X)_{RHS} + \zeta_{RHS} - \xi(X)_{LHS} - \zeta_{LHS} && (50)
\end{aligned}$$

Thus,

$$W^D = \left[G(X)_{LHS} - G(X)_{RHS}\right]^{-1}\left(\xi(X)_{RHS} + \zeta_{RHS} - \xi(X)_{LHS} - \zeta_{LHS}\right). \tag{51}$$

D.2. Equity

For a given initial state, ν_0, the 8 boundary conditions for equity are

$$\lim_{X\uparrow X_D^2} E(X,1,\nu_0) = \lim_{X\downarrow X_D^2} E(X,1,\nu_0) \tag{52}$$

$$\lim_{X\uparrow X_D^2} E_X(X,1,\nu_0) = \lim_{X\downarrow X_D^2} E_X(X,1,\nu_0) \tag{53}$$

$$\lim_{X\uparrow X_U^{u(1)}} E(X,u(2),\nu_0) = \lim_{X\downarrow X_U^{u(1)}} E(X,u(2),\nu_0) \tag{54}$$

$$\lim_{X\uparrow X_U^{u(1)}} E_X(X,u(2),\nu_0) = \lim_{X\downarrow X_U^{u(1)}} E_X(X,u(2),\nu_0) \tag{55}$$

$$E(X_D^1,1,\nu_0) = 0 \tag{56}$$

$$E(X_D^2,2,\nu_0) = 0 \tag{57}$$

$$E(X_U^{u(1)},u(1),\nu_0) = \frac{X_U^{u(1)}}{X_0}[(1-q)D(X_0,u(1);u(1)) + E(X_0,u(1);u(1))] - D(X_0,\nu_0;\nu_0) \tag{58}$$

$$E(X_U^{u(2)},u(2),\nu_0) = \frac{X_U^{u(2)}}{X_0}[(1-q)D(X_0,u(2);u(2)) + E(X_0,u(2);u(2))] - D(X_0,\nu_0;\nu_0) \tag{59}$$

These conditions hold for an arbitrary coupon rate, $C(\nu_0)$. For a given initial state, ν_0, the optimal default thresholds (for an arbitrary coupon) satisfy the smooth-pasting conditions for equity such that

$$\left.\frac{\partial}{\partial X}E(X,1;\nu_0)\right|_{X\downarrow X_D^1(\nu_0)} = 0 \tag{60}$$

$$\left.\frac{\partial}{\partial X}E(X,2;\nu_0)\right|_{X\downarrow X_D^2(\nu_0)} = 0\,. \tag{61}$$