Graphs without Odd Holes, Parachutes or Proper Wheels: A Generalization of Meyniel Graphs and of Line Graphs of Bipartite Graphs

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Abstract

We prove that the strong perfect graph conjecture holds for graphs that do not contain parachutes or proper wheels. This is done by showing the following theorem:

If a graph G contains no odd hole, no parachute and no proper wheel, then G is bipartite or the line graph of a bipartite graph or G contains a star cutset or an extended strong 2-join or $\bar{G}$ is disconnected.

To prove this theorem, we prove two decomposition theorems which are interesting in their own rights. The first is a generalization of the Burlet-Fonlupt decomposition of Meyniel graphs by clique cutsets and amalgams. The second is a precursor of the recent decomposition theorem of Chudnovsky, Robertson, Seymour and Thomas for Berge graphs that contain a line graph of a bipartite subdivision of a 3-connected graph.

Key words: perfect graph, odd hole, strong perfect graph conjecture, decomposition, star cutset, 2-join, Meyniel graph, line graph of bipartite graph

Running head: WP-FREE GRAPHS

1 Introduction

A graph is *perfect* if, in all its induced subgraphs, the size of a largest clique is equal to the chromatic number. A *hole* is a chordless cycle of length at least four. A hole is *odd* (*even*) if it contains an odd (even) number of nodes. A long standing conjecture of Berge [1] states that a graph G is perfect if and only if neither G nor its complement contains an odd hole. (The *complement* $\bar{G}$ of G has node set $V(G)$ and two nodes are adjacent in $\bar{G}$ if and only if they are not adjacent in G). Berge's conjecture is known as the Strong Perfect Graph Conjecture. It was proved recently by Chudnovsky, Robertson, Seymour and Thomas [3]. This conjecture was already known to hold for several special classes of perfect graphs.

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For example, Meyniel [13] showed that if every odd cycle of G is a triangle or contains at least two chords, then G is perfect. These graphs are known as *Meyniel graphs*.

Another well-known example is the following. A graph G is the *line graph* of a graph H if $V(G) = E(H)$ and $v_i, v_j \in V(G)$ are adjacent if $e_i, e_j \in E(H)$ have a common endnode. If G is the line graph of a bipartite graph H, then G is perfect. (Indeed, the maximum degree of a node in H is equal to the chromatic index of H and this implies that the chromatic number of G equals the size of its largest clique).

In this paper we introduce WP-free graphs (W stands for proper Wheel and P stands for Parachute: They will be defined later) and characterize the WP-free graphs that are perfect. Meyniel graphs and line graphs of bipartite graphs are perfect WP-free graphs.

WP-free graphs do not contain the complement of a hole H, $|H| \geq 7$. We show that if a WP-free graph contains no odd hole, then it is perfect. This is acheived by proving a structural theorem for even-signable WP-free graphs, a class of graphs that is larger than the class of WP-free graphs containing no odd hole. The proof of this theorem follows from two independent decomposition theorems, each interesting in its own right. The first is a generalization of the Burlet-Fonlupt decomposition of Meyniel graphs by clique cutsets and amalgams [2]. The second is a precursor of the recent decomposition theorem of Chudnovsky, Robertson, Seymour and Thomas [3] for Berge graphs that contain a line graph of a bipartite subdivision of a 3-connected graph.

1.1 Wheels, Parachutes and WP-Free Graphs.

A *wheel* (H, v) consists of a hole H together with a node v, called the *center*, that has at least three neighbors in H. If v has exactly k neighbors in H, the wheel is called a *k-wheel*.

Definition 1.1 *A* T-wheel *(or* twin wheel*) is a 3-wheel (H, v) such that the three neighbors of v in H are consecutive.*

A wheel (H, v) is a Δ-free wheel *(or* triangle-free wheel*) if the neighbors of v in H induce a stable set. That is, the graph induced by (H, v) is a triangle-free graph.*

A wheel (H, v) is a universal wheel *if v is adjacent to every node of H.*

A wheel (H, v) is an L-wheel *(or* line wheel*) if (H, v) is the line graph of a cycle C with a unique chord and $V(C)$ induces a triangle-free graph, i.e. the unique chord of C is not a triangular chord. So v has neighbors a_1, a_2, b_1 and b_2 in H, $H = a_1, P_1, b_1, b_2, P_2, a_2, a_1$ and P_1, P_2 are paths of length greater than 1.*

A wheel that is in none of the above four classes is called a proper wheel.

Definition 1.2 *An* L-parachute $LP(a_1b_1, a_2b_2, a_3, z)$ *is a graph induced by an L-wheel (H, a_3) where $H = a_1, b_1, \ldots, z, \ldots, b_2, a_2, \ldots, a_1$, where a_1, a_2, b_1, b_2 are the neighbors of a_3 in H, together with a chordless path $P = a_3, \ldots, z$ of length greater than 1. No node of $H \setminus \{z, b_1\}$ may be adjacent to an intermediate node of P.*

A T-parachute $TP(a_1, a_2, b_1, b_2, z)$ *is a graph induced by a T-wheel (H, a_2) where $H = b_1, a_1, b_2, \ldots, z, \ldots, b_1$, where b_1, a_1, b_2 are the neighbors of a_2 in H, together with a chordless path $P = a_2, \ldots, z$ of length greater than 1. No node of $H \setminus \{z, b_1\}$ may be adjacent to an intermediate node of P.*

A parachute *is either an L-parachute or a T-parachute.*

For an L-parachute or a T-parachute, let P_1, P_2 be respectively the b_1z-path and the b_2z-path in $H \setminus a_1$ and C_1, C_2 be the cycles induced by $P \cup P_1$ and $P \cup P_2$. Note that in a T-parachute or an L-parachute, the paths P_1 and P_2 may have length one.

In the definition below and throughout the rest of the paper, G *contains* G' if G' is an induced subgraph of G and G is G'*-free* if G does not contain G'.

Definition 1.3 *A graph is* WP-free *if it contains neither a proper wheel nor a parachute.*

Lemma 1.4 *Let G be an L-parachute $LP(a_1b_1, a_2b_2, a_3, z)$ with the property that no proper subgraph of G is a parachute or a proper wheel. Then G is of one of the following types, see Figure 1.*

type a) No intermediate node of P is adjacent to b_1 or b_2.

type b) An intermediate node of P is adjacent to b_1, (C_2, b_1) is a Δ-free wheel and b_1 is adjacent to z.

type c) An intermediate node of P is adjacent to b_1, (C_2, b_1) is a T-wheel and b_1 is adjacent to z.

type d) An intermediate node of P is adjacent to b_1, (C_2, b_1) is an L-wheel and b_1 is adjacent to z.

Proof: If no intermediate node of P is adjacent to b_1 or b_2, G is of type a). Suppose an intermediate node of P is adjacent to b_1, and b_1 is not adjacent to z. If the neighbor of a_3 in P is the only intermediate node of P that is adjacent to b_1, there is a smaller proper wheel with center a_3. Otherwise there is a smaller L-parachute. So b_1 must be adjacent to z and therefore (C_2, b_1) is a wheel, which is not proper by assumption and is not universal since b_1 and b_2 are nonadjacent. So (C_2, b_1) is either a Δ-free wheel or a T-wheel or an L-wheel and we have types b) or c) or d) in these three cases. □

Lemma 1.5 *Let G be a T-parachute $TP(a_1, a_2, b_1, b_2, z)$ that is not an L-parachute and such that no proper subgraph of G is a parachute or a proper wheel. Then G is one of the following graphs, see Figure 2.*

type a) No intermediate node of P is adjacent to b_1 or b_2.

type b) An intermediate node of P is adjacent to b_1, (C_2, b_1) is a Δ-free wheel and b_1 is adjacent to z.

type c) An intermediate node of P is adjacent to b_1, (C_2, b_1) is a T-wheel and b_1 is adjacent to z.

Proof: If no intermediate node of P is adjacent to b_1 or b_2 we have type a). Assume an intermediate node of P is adjacent to b_1. Since no proper induced subgraph of G is a parachute or a proper wheel, then b_1 is adjacent to z and therefore (C_2, b_1) is a wheel, which is not proper by assumption and is not universal since b_1 and b_2 are nonadjacent. If (C_2, b_1) is an L-wheel then G is also an L-parachute of type c. So (C_2, b_1) is either a Δ-free wheel or a T-wheel, and we have types b) or c). □

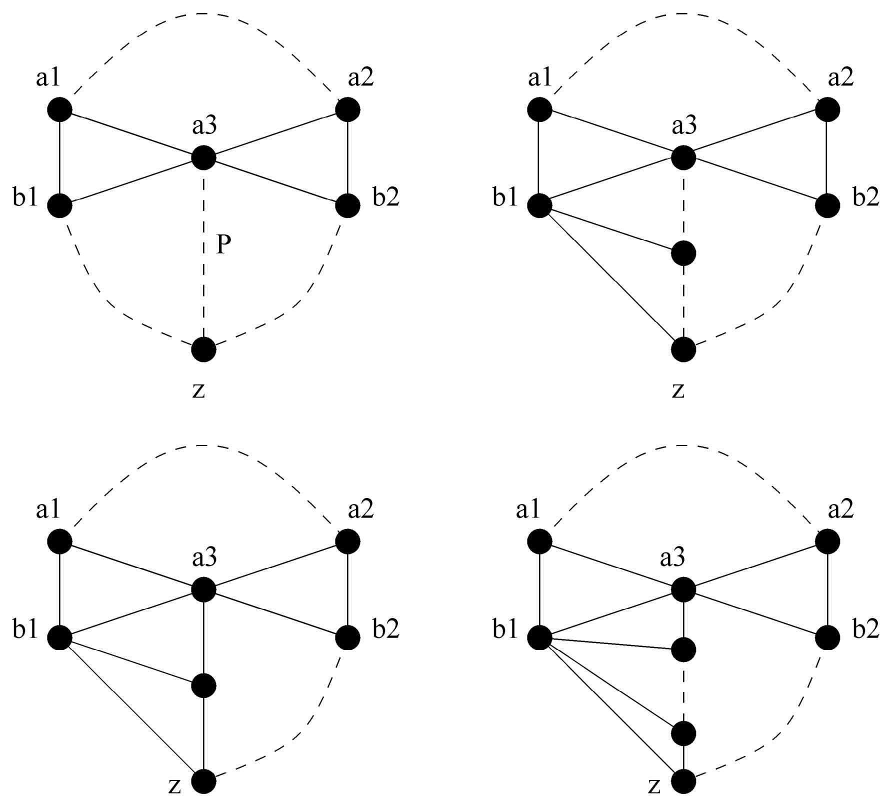

Figure 1: L-parachutes

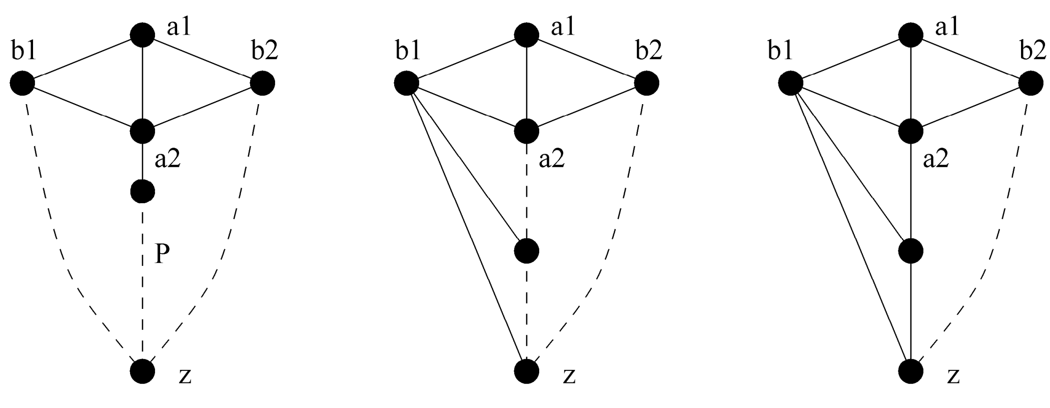

Figure 2: T-parachutes

A *cap* is a cycle C of length at least 5 with a unique chord that is a triangular chord of C. A cap is *odd* if C is odd.

Remark 1.6 *A graph G is Meyniel if and only if G contains no odd hole and no odd cap.*

Proof: G is not a Meyniel graph if and only if G contains an odd cycle that is not a triangle and has at most one chord. Let C be a smallest such cycle. C is either an odd hole or an odd cap. □

If G contains a cap, G contains an odd hole or an odd cap. So the class of cap-free graphs contains the class of Meyniel graphs. The structure of cap-free graphs is very similar to the structure of Meyniel graphs and was studied in [7]. Since every proper wheel and parachute contains a cap, the class of WP-free graphs contains the class of cap-free graphs.

A *diamond* is a cycle of length 4 with a unique chord. A *claw* is a graph on 4 nodes, one of them with degree 3 and the others with degree 1. The following characterization of the line graphs of bipartite graphs is due to Harary and Holtzmann [11]. It can be proven following the arguments of the proof of Remark 3.2.

Remark 1.7 *G is the line graph of a bipartite graph if and only if G contains no odd hole, no claw and no diamond.*

It is straightforward to check that if G is a proper wheel or a parachute, then G contains a claw or a diamond. This implies the following remark:

Remark 1.8 *The class of WP-free graphs containing no odd hole includes the class of Meyniel graphs and the class of line graphs of bipartite graphs.*

1.2 Even-Signable Graphs

We study even-signable WP-free graphs, a class of graphs that includes WP-free graphs containing no odd hole.

A graph G is *signed* if its edges are given *odd* or *even* labels. A subset of $E(G)$ is odd (resp. even) if it contains an odd (resp. even) number of edges labeled odd. A graph G is *even-signable* if there exists a signing of its edges such that every triangle is odd and every hole is even. These graphs were introduced in [6]. More results can be found in [7]. Note that, if G contains no odd hole, then G is even-signable since all its edges can be labeled odd. Also, if G is triangle-free, then G is even-signable since all its edges can be labeled even. It is shown in [7] that, if one can efficiently test whether G is even-signable, then one can also efficiently test whether G contains an odd hole.

The graphs in Figure 3 are relevant in this paper. Solid lines represent edges and dotted lines represent paths of length at least one. The first three graphs are referred to as *3-path configurations* ($3PC$'s). The first graph is called a $3PC(x,y)$ (or $3PC(\cdot,\cdot)$), where node x and node y are connected by three paths P_1, P_2 and P_3. The second is called a $3PC(xyz,u)$ (or $3PC(\Delta,\cdot)$), where xyz is a triangle and P_1, P_2 and P_3 are three paths with endnodes x, y and z respectively and a common endnode u. The third is called a $3PC(xyz,uvw)$ (or $3PC(\Delta,\Delta)$), consists of two node disjoint triangles xyz and uvw and paths P_1, P_2 and P_3

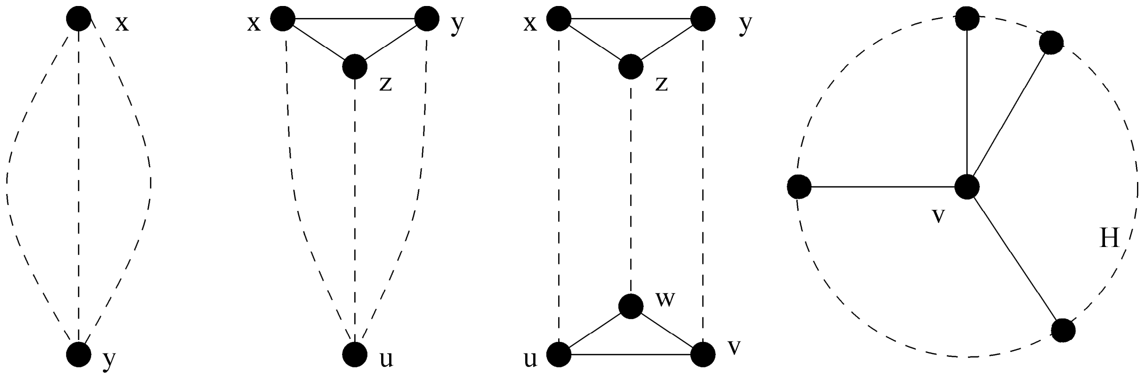

Figure 3: 3-path configurations and wheel

with endnodes x and u, y and v and z and w respectively. In all three cases, the nodes of $P_i \cup P_j$ induce a hole for $i \neq j$. This implies that all paths of a $3PC(\cdot,\cdot)$ have length greater than one, and at most one path of a $3PC(\Delta,\cdot)$ has length one.

A wheel (H, v) is an *odd wheel* if it contains an odd number of triangles: Since H is a hole, every triangle of (H, v) contains v and two adjacent nodes of H. So a wheel (H, v) is odd if the subgraph of H, induced by the neighbors of v, contains an odd number of edges.

A consequence of a theorem of Truemper [14] is the following co-NP characterization of even-signable graphs.

Theorem 1.9 *A graph is even-signable if and only if it contains no* $3PC(\Delta,\cdot)$ *and no odd wheel.*

A derivation of this result and a discussion of Truemper's theorem can be found in [7] and [8]. We find it convenient to work with even-signable graphs because the graphs of Theorem 1.9 are easy to spot when proving results.

1.3 The Main Theorem

In a graph G, a node set S is a *cutset* if the graph $G \setminus S$ is disconnected. A node set S is a *star* if it consists of a node x and neighbors of x. Chvátal [4] showed that a minimally imperfect graph cannot contain a star cutset.

A graph G has an *extended 2-join* if $V(G)$ can be partitioned into subsets V_A, V_B and U (U possibly empty), such that $A_1, A_2 \in V_A$, $B_1, B_2 \in V_B$ are nonempty disjoint sets with the following properties: (i) every node of A_1 is adjacent to every node of B_1, every node of A_2 is adjacent to every node of B_2 and these are the only adjacencies between V_A and V_B, (ii) every node of U is adjacent to $A_1 \cup A_2 \cup B_1 \cup B_2$ and possibly to other nodes in $V(G)$, (iii) the connected components of $G(V_A)$ meet both A_1 and A_2 and, if $|A_1| = |A_2| = 1$ then V_A does not induce a chordless path and, (iv) the connected components of $G(V_B)$ meet both B_1 and B_2 and, if $|B_1| = |B_2| = 1$ then V_B does not induce a chordless path.

An extended 2-join is called *extended strong 2-join* when, in addition, both $A_1 \cup B_1$, $A_2 \cup B_2$ induce cliques. When $U = \emptyset$, the extended 2-join reduces to the *2-join* introduced by Cornuéjols and Cunningham [9].

In this paper, we prove the following result.

Theorem 1.10 *Let G be an even-signable WP-free graph that is not a triangle-free graph nor the line graph of a triangle-free graph. Then G contains a star cutset or an extended strong 2-join or $\bar{G}$ is disconnected.*

Corollary 1.11 *Let G be a WP-free graph that contains no odd hole. Then G is a bipartite graph or the line graph of a bipartite graph or G contains a star cutset or an extended strong 2-join or $\bar{G}$ is disconnected.*

This result, together with the next two theorems, implies that the Strong Perfect Graph Conjecture holds for WP-free graphs.

Theorem 1.12 [4] *A minimally imperfect graph cannot contain a star cutset.*

The following theorem follows from a result of Conforti, Cornuéjols, Gasparyan and Vušković [5] on universal 2-amalgams.

Theorem 1.13 [5] *A minimally imperfect graph cannot contain an extended strong 2-join.*

Theorem 1.14 *A WP-free graph is perfect if and only if it contains no odd hole.*

Proof: The "if" part is obvious. We prove the "only if" statement. Let G be a minimally imperfect WP-free graph that contains no odd hole. Then G is even-signable. By Theorem 1.12, G does not contain a star cutset and by Theorem 1.13, G does not contain an extended strong 2-join. Furthermore, $\bar{G}$ is connected. Hence, by Corollary 1.11, G is a bipartite graph or the line graph of a bipartite graph. In both cases G is perfect, a contradiction. □

1.4 Proof Outline of the Main Theorem

A graph G has an *amalgam* if $V(G)$ can be partitioned into subsets V_A, V_B and U (U possibly empty), such that $A_1 \in V_A$, $B_1 \in V_B$ are nonempty sets with the following properties: (i) every node of A_1 is adjacent to every node of B_1 and these are the only adjacencies between V_A and V_B, (ii) U is a clique and every node of U is adjacent to $A_1 \cup B_1$ and possibly to other nodes in $V(G)$, (iii) $|V_A| \geq 2$ and $|V_B| \geq 2$.

The notion of *amalgam* was introduced by Burlet and Fonlupt [2]. The *join* introduced by Cunningham and Edmonds [10] is an amalgam with $U = \emptyset$.

A node u is *universal* for a graph H if u is adjacent to all the nodes in H.

Theorem 1.10 is in fact the consequence of the following stronger results.

Theorem 1.15 *Let G be an even-signable WP-free graph that does not contain an L-wheel nor a $3PC(\Delta, \Delta)$. Then either G is a triangle-free graph plus at most one universal node or G contains a clique cutset or an amalgam.*

This theorem is proved in Section 2.

Theorem 1.16 *Let G be an even-signable WP-free graph that contains an L-wheel or a $3PC(\Delta, \Delta)$. Then either G is the line graph of a triangle-free graph or G contains a star cutset or an extended strong 2-join or $\bar{G}$ is disconnected.*

This theorem is proved in Section 3.

2 GM-graphs

Definition 2.1 *A graph G is a GM-graph (Generalized Meyniel graph) if G is an even-signable WP-free graph and G does not contain an L-wheel or a $3PC(\Delta, \Delta)$.*

In this section we prove Theorem 1.15 which states that every GM-graph G is a triangle-free graph plus at most one universal node or G contains a clique cutset or an amalgam. This theorem is interesting in its own right. Indeed, when specialized to Meyniel graphs, this result is a famous theorem of Burlet and Fonlupt [2]: every Meyniel graph G is a bipartite graph plus at most one universal node or G contains a clique cutset or an amalgam. In addition, Theorem 1.15 has algorithmic consequences that we do not develop in this paper.

We first introduce some definitions.

For $S \subseteq V(G)$, we let $G(S)$ be the subgraph of G induced by the nodes in S. We let $N(S)$ denote the set of nodes with at least one neighbor in S. Two nodes u, v are *twins with respect to* S if u and v are adjacent and $N(u) \cap (S \setminus \{u, v\}) = N(v) \cap (S \setminus \{u, v\})$. If u and v are twins with respect to $V(G)$, we simply say that u and v are *twins*.

We denote a cap by (H, x) where H is a hole and x is a node adjacent to consecutive node a, b in H. The nodes a, b are called the *attachments* of the cap.

Given three disjoint node sets A, B and C such that no node of A is adjacent to a node of B, a *direct connection* between A and B is a minimal path P (in terms of its node set) between a node in A and a node in B. The direct connection P *avoids* the set C if no node of P is in C.

We will need the following technical lemma about caps in GM-graphs.

Lemma 2.2 *Let G be a GM-graph that contains no clique cutset but contains a cap (H, x) with attachments a, b. Then G has the following properties:*

(i) In every direct connection $P = x_1, \dots, x_n$ from x to $V(H) \setminus \{a, b\}$ in $G \setminus (V(H) \cup \{x\})$, node x_n is a universal node for H or is a twin of a or b with respect to H.

(ii) Let U be the set of universal nodes for H that are endnodes of some such direct connection and let T be the set of twins of a or b that are endnodes of some direct connection. Then T is a clique, every node of U is adjacent to every node of T and U contains two nonadjacent nodes u and u'.

(iii) There exists a node x' adjacent to u and u' such that (H, x') is a cap with attachments a and b.

Proof: Suppose that (i) does not hold. Among all caps (Q, y) with attachments $\{a, b\}$ and direct connection $P = x_1, \dots, x_n$ from y to $V(Q) \setminus \{a, b\}$ in $G \setminus (V(Q) \cup \{y\})$, such that x_n is neither a universal node for Q nor a twin of a or b with respect to Q, choose (Q, y) and P such that P is shortest possible. It follows from this choice of (Q, y) and P that no node x_j with $j \leq n-1$ is adjacent to both a and b. Also, at least one of the nodes a, b is not adjacent to any of the nodes x_j for $2 \leq j \leq n-1$ (otherwise Q can be modified, P shortened and (i) still does not hold). Assume w.l.o.g. that b is not adjacent to any of the nodes x_j for $2 \leq j \leq n-1$. By construction, x_n has at least one neighbor z in $V(Q) \setminus \{a, b\}$.

Assume first that x_n has one or two neighbors in Q. We only sketch the proof since checking the various cases is routine. If $n = 1$, there is a $3PC(\Delta, \cdot)$ or an odd wheel or a T-parachute or a $3PC(\Delta, \Delta)$. So $n \geq 2$. Since G does not contains an L-wheel or a $3PC(\Delta, \Delta)$ or a $3PC(\Delta, \cdot)$, it follows that a is adjacent to some node x_j for $j \leq n-1$ or b is adjacent to x_1. Let S be the hole containing $V(P) \cup \{b\}$ and possibly nodes of $(V(Q) \setminus \{a\}) \cup \{y\}$. Since (S, a) is neither a proper wheel nor an L-wheel, either a or b is adjacent to x_1. But now, there is a T-parachute or a $3PC(\Delta, \cdot)$ or a proper wheel or a $3PC(\Delta, \Delta)$, a contradiction.

So x_n has at least three neighbors in Q. Assume that x_n is adjacent to at most one of the nodes a, b, and let S denote the hole with nodes in $V(Q) \cup \{x_n\}$ that contains a, b and x_n. If $n \geq 2$, we have a contradiction to the choice of (Q, y) and P. If $n = 1$, we have a T-parachute if x_1 is adjacent to a or b and a proper wheel otherwise. So x_n is adjacent to both a and b and at least one other node of Q. Since (Q, x_n) is not a proper wheel nor a line wheel, x_n must be universal for Q or a twin of a or b with respect to Q. This completes the proof of (i).

Suppose that (ii) does not hold. Let x_n and x'_m be the last nodes of direct connections P and P' where $x_n \in T$ and $x'_m \in T \cup U$ are not adjacent. Assume w.l.o.g. that x_n is a twin of b with respect to H. If x'_m is a twin of a, then $V(H) \cup \{x_n, x'_m\}$ induces a T-parachute, a contradiction. So we can assume w.l.o.g. that both x_n and x'_m are adjacent to a, b and the neighbor b' of b in $V(H) \setminus \{a\}$.

If P and P' have no common node nor adjacent nodes, let C denote the hole induced by $V(P) \cup V(P') \cup \{b', x\}$. Now (C, b) is a proper wheel unless C is of length four, i.e. x is adjacent to x_n and x'_m. But then there is a T-parachute induced by $(V(H) \setminus \{b\}) \cup \{x, x_n, x'_m\}$.

So P and P' have a common node or adjacent nodes. Let Q be a shortest path from x_n to x'_m in $P \cup P'$. There is a T- parachute with top node b', side nodes x_n and x'_m and side paths contained in Q.

So x_n and x'_m are adjacent. This shows that T is a clique and every node of T is adjacent to every node of U. Since $U \cup T$ is not a clique cutset separating x from $V(H) \setminus \{a, b\}$, there must exist two nodes in U that are nonadjacent, say u and u'. This completes the proof of (ii).

Now we prove (iii). Let P and P' be direct connections from x to $V(H) \setminus \{a, b\}$ in $G \setminus (V(H) \cup \{x\})$ that end in u and u' respectively.

If P and P' have no common node nor adjacent nodes, let C denote the hole induced by $V(P) \cup V(P') \cup \{b', x\}$, where b' is the neighbor of b in $V(H) \setminus \{a\}$. Since (C, b) is not a proper wheel, b must be adjacent to every node of P and P'. By symmetry, a is adjacent to every node of P and P'. Since (C, a) is not a proper wheel, it follows that C has length four. So (iii) holds in this case.

Now assume that P and P' have a common node or adjacent nodes. Let Q be a shortest path from u to u' in $P \cup P'$, let C be the hole induced by $V(Q) \cup \{u, u', b'\}$ and C' the hole induced by $V(Q) \cup \{u, u', a'\}$ where a' is the neighbor of a in $V(H) \setminus \{b\}$. If Q contains an intermediate node adjacent to b, then Q has length two, otherwise (C, b) or (C', b) is a proper wheel. By symmetry, the same holds for a. Furthermore, when Q has length two, the claim holds if its intermediate node is adjacent to both a and b. So, whether Q has length two or not, we can assume w.l.o.g. that b is not adjacent to any intermediate node of Q. Let M be a shortest path from b to Q in $V(P) \cup V(P') \cup \{x\}$. Let m be the node of M adjacent to Q. By the choice of Q, m has at most three neighbors in Q. If m has two adjacent neighbors q_1, q_2 in Q, there is a $3PC(mq_1q_2, b)$. So we can assume w.l.o.g. that m has only one neighbor z in Q since, otherwise we can modify Q to get the desired property. Now there is a parachute with side nodes u and u', side paths Q_{uz} and $Q_{u'z}$, top node b', center node b and middle path M. This completes the proof of (iii). □

2.1 D-structures

Definition 2.3 *A* D-structure (C_1, C_2, K) *of* G *consists of disjoint sets of nodes* C_1, C_2 *and* K*, where* $|C_1| \geq 2$*,* $|C_2| \geq 2$ *and the nodes of* K *induce a clique of* G *(possibly* K *is empty). Furthermore, the subgraph* $G(C_1)$ *is connected and every node in* C_1 *is universal for* $C_2 \cup K$*, every node in* C_2 *is universal for* $C_1 \cup K$ *and there exists no node in* $V(G) \setminus (C_1 \cup C_2 \cup K)$ *adjacent to a node in* C_1 *and a node in* C_2*.*

This notion was introduced in [7], where it was shown that, if a cap-free graph G contains a D-structure, then G contains an amalgam. Here, we show the following result.

Theorem 2.4 *If* G *is a GM-graph that contains a D-structure, then* G *contains a clique cutset or an amalgam.*

Proof: Let U be the set of nodes in $V(G) \setminus (C_1 \cup C_2 \cup K)$ that are adjacent to C_1 and are connected to a node in C_2 by a path with nodes in $V(G) \setminus (C_1 \cup K)$.

Claim 1: *If* G *contains no clique cutset, every node in* U *is universal for* C_1*.*

Proof: Assume not and choose $u \in U$ contradicting the claim and $c_2 \in C_2$ connected by a shortest possible path with nodes in $V(G) \setminus (C_1 \cup K)$ and among all these paths, let $P = x_0 = u, x_1, \dots, x_n, x_{n+1} = c_2$ be one with the largest number of nodes adjacent to C_1. Since C_1 and C_2 belong to a D-structure, then $n \geq 1$. By our choice, intermediate nodes of P are either nonadjacent to C_1 or universal for C_1. Since u is adjacent but not universal to C_1 and $G(C_1)$ is connected, C_1 contains adjacent nodes a, b such that u is adjacent to a but not to b.

We now show that $G(V(P) \cup \{a, b\})$ contains a cap (H, x) where $H = a, x_i, P_{x_i x_j}, x_j, a$. Assume that P contains consecutive nodes that are both adjacent to a and let x_i, x_{i+1} be such nodes with highest index. Then $i < n$ by the definition of D-structure, so $P_{x_{i+2} x_{n+1}}$ contains a node adjacent to a. Let x_j be such a node, of lowest index and let $H = a, x_{i+1}, P_{x_{i+1} x_j}, x_j, a$. Now (H, x_i) is a cap. If P does not contain consecutive nodes that are both adjacent to a, let x_i be the node of lowest index $i \geq 1$ adjacent to a (and b) and let $H = a, x_0, P_{x_0 x_i}, x_i, a$. Now (H, b) is a cap.

Let (H, x) be a cap where $H = a, x_i, P_{x_i x_j}, x_j a$ and $j \geq i + 2$. Since G contains no clique cutset, by Lemma 2.2, G contains nonadjacent nodes z, z', universal for H (possibly adjacent to x) and, since K is a clique, at least one of these nodes, say z, is not in K. Now $z \notin C_1$, since otherwise x_{j-1} is adjacent to $z \in C_1$ but not $a \in C_1$ and so, if $j = n + 1$, the definition of D-structure is contradicted, and if $j \leq n$, the choice of u is contradicted. Furthermore $z \notin C_2$, since otherwise x_i is adjacent to $a \in C_1$ and $z \in C_2$, a contradiction to the definition of D-structure.

So $z \in V(G) \setminus (C_1 \cup C_2 \cup K)$. Now $j = i+2$ and z is universal for C_1, else the minimality of P is contradicted. Let P' be obtained from P by removing x_{i+1} and adding z. Now P and P' have the same length and P' contradicts our assumption that P has the largest number of neighbors in C_1. So this completes the proof of Claim 1.

Let K' contain the nodes in K that are not universal for U and $K'' = K \setminus K'$. Define $A = C_1$, $B = C_2 \cup K' \cup U$. We show that, if G contains no clique cutset, (A, B, K'') is an amalgam of G. Claim 1 shows that every node in B is universal for A and by definition of K'', every node in K'' is universal for U. Since (C_1, C_2, K) is a D-structure, every node in K'' is universal for $C_1 \cup C_2 \cup K'$.

Claim 2: *Let G' be the graph obtained from G by removing all edges with one endnode in A and the other in K'. If G contains no clique cutset, in $G'(V(G) \setminus (C_2 \cup K'' \cup U))$ no path connects a node of K' and a node of $C_1 = A$.*

Proof: Let $P = x, v_1, \ldots, v_p, k$ be a shortest path connecting $x \in C_1$ and $k \in K'$ and contradicting the claim. No intermediate node of P is adjacent to a node in C_2 else, by the definition of U, v_1 belongs to U. If $p \geq 2$, let c_2 be any node in C_2 and $H = k, x, v_1, \ldots, v_p, k$. Then (H, c_2) is a cap and since G contains no clique cutset, by Lemma 2.2, G contains two nonadjacent nodes universal for H and one of them, say z, is not in K. Since v_1 is adjacent to $x \in C_1$ and z, z is not in C_2. $z \notin C_1$, else v_p is adjacent to $z \in C_1$ and k and $P' = z, v_p, k$ contradicts the minimality of P. Now since $v_1 \notin U$ and v_1 is adjacent to z, z is also not in U. So $z \in V(G) \setminus (C_1 \cup C_2 \cup K \cup U)$ and $P' = x, z, k$ again contradicts the minimality of P.

So $P = x, v_1, k$. Since k is not universal for U, U contains a node not adjacent to k. Let u be such a node, connected in $G \setminus (C_1 \cup K)$ to a node of C_2, say c_2, by a shortest possible path and among these paths, let $Q = x_1 = u, \ldots, x_m = c_2$ have the largest number of neighbors of C_1. Note that Q may contain several nodes that are universal for C_1, so let $u_1, \ldots, u_n$ be such nodes of Q, with u_i closer to u than u_{i+1} ($u_1 = x_1 = u$ and $u_n = x_m = c_2$). Note that all nodes $u_1, \ldots, u_{n-1}$ belong to U.

We now show that no two consecutive nodes of Q are universal for C_1. For, let u_{i-1}, u_i, be consecutive nodes of highest index. Note that $i < n-1$ by the definition of D-structure. So let $H = x, u_i, Q_{u_i u_{i+1}}, u_{i+1}, x$, and (H, u_{i-1}) is a cap and again since G contains no clique cutset, by Lemma 2.2, there exists a node z not in K universal for H. Since u_i is adjacent to $x \in C_1$ and z, then $z \notin C_2$. Let x_j be the neighbor of u_{i+1} in $Q_{u_i u_{i+1}}$. Now $z \notin C_1$, else since x_j is adjacent to z, then $x_j \in U$ and, since x_j is not adjacent to x, Claim 1 is contradicted. So since z is adjacent to x and to x_j, then z is in U. Now $Q_{u_i u_{i+1}}$ has length 2 and z has no neighbor in $V(Q) \setminus V(Q_{u_i u_{i+1}})$ else the minimality of P is contradicted. Let P' be obtained from P by removing x_j and adding z. Now P and P' have the same length and P' contradicts the fact that P has the largest number of neighbors in C_1. So no two consecutive nodes of Q are universal for C_1.

Let x_i be the node of smallest index adjacent to k. Since by our choice, k is not adjacent to u_1 but is adjacent to all the nodes $u_2, \ldots, u_n = x_m$, such a node exists and it belongs to $Q_{x_2 u_2}$ (possibly $n = 2$). If $x_i = u_2$, let $H = x, u_1, Q_{u_1 u_2}, u_2, x$, and (H, k) is a cap. So by the same argument as above, there exists a node z not in K universal for H. Again, the above argument rules out the existence of such a node z and so x_i is an intermediate node of $Q_{u_1 u_2}$. Let $H = x, u_1, Q_{u_1 x_i}, x_i, k, x$. Since $v_1 \notin U$, v_1 is not adjacent to any node in

$Q_{u_1x_i}$. Now (H, v_1) is a cap and so there exists a node z not in K universal for H. Since $x_1 = u_1$ is adjacent to $x \in C_1$ and z, $z \notin C_2$. Since x_2 is adjacent to z but not to $x \in C_1$, by Claim 1, $z \notin C_1$. So the same argument as above rules out the existence of such a node z when z is adjacent to x_1, x_2 and x_3. So $i = 2$ and z is adjacent to x_1, x_2 but not x_3. By Lemma 2.2(i), $G \setminus \{x, k\}$ contains a chordless path $R = v_1 = r_1, \ldots, r_q = z$. Note that intermediate nodes of R may be adjacent to x or k but not to x_1 or x_2. At least one node of R belongs to $C_1 \cup K$, otherwise there exists a path from v_1 to C_2 whose intermediate nodes are in $V(R) \cup V(Q)$ and this path contains no node of $C_1 \cup K$, thus proving that $v_1 \in U$, a contradiction. So let r_i be the node of R with lowest index in $C_1 \cup K$. Then r_i is adjacent to c_2. So let $S = s_1 = v_1, \ldots, s_{n-1} = x_3, s_n = x_2$ be a shortest v_1x_2-path whose nodes are in $R_{r_1r_i} \cup Q_{x_2x_m}$. Since $S_{s_2s_{n-1}}$ is a direct connection from v_1 to H, avoiding x and k, by Lemma 2.2(i), x_3 must be a twin of node k with respect to H (indeed, x_3 is not adjacent to x_1, so it can be neither universal for H nor a twin of x). Now, by Lemma 2.2(ii), z is adjacent to x_3, a contradiction. This completes the proof of Claim 2.

The following claim shows that (A, B, K'') is an amalgam of G.

Claim 3: *Let G'' be obtained from G by removing all edges with one endnode in A and the other in B. Then in $G''(V(G) \setminus K'')$, no path connects a node in A and a node in B.*

Proof: Let $P = x_1, \ldots, x_n$ be a chordless path between x_1 in A and x_n in B and contradicting the claim. Claim 1 shows that if $x_n \in C_2$, then $x_2 \in U$, a contradiction. Claim 2 shows $x_n \notin K'$. So $x_n \in U$ and let P_{x_n} be a path with nodes in $V(G) \setminus (C_1 \cup K)$ connecting x_n and a node in C_2. Now there is a path with nodes in $V(G) \setminus (C_1 \cup K)$ between x_2 and a node in C_2 only using nodes of $V(P_{x_n}) \cup V(P)$. So x_2 must belong to U, a contradiction. □

2.2 M-structures

M-structures were first introduced by Burlet and Fonlupt [2] in their study of Meyniel graphs.

An induced subgraph $G(V_1)$ of G is called an *M-structure* (multipartite structure) if $\bar{G}(V_1)$ contains at least two connected components each with at least two nodes. Let $W_1, \ldots, W_k$ be the node sets of these connected components. The *proper subclasses* of $G(V_1)$ are the sets W_i of cardinality greater than or equal to 2. The *partition* of an M-structure is denoted by $(W_1, \ldots, W_r, K)$ where K is the union of all non-proper subclasses. Note that K induces a clique in G.

Lemma 2.5 *An M-structure $G(V_1)$ of G is maximal with respect to node inclusion, if and only if there exists no node $v \in V(G) \setminus V_1$ such that v is universal for a proper subclass of $G(V_1)$.*

Proof: Let $G(V_1 \cup \{u\})$ be an M-structure. Assume node u is not universal for any proper subclass of $G(V_1)$. In $\bar{G}(V_1 \cup \{u\})$ node u is adjacent to at least one node in each of the proper subclasses. Thus there exists only one proper subclass in $G(V_1 \cup \{u\})$, contradicting the assumption.

Conversely let node u be universal for some proper subclass W_i of $G(V_1)$. Then $\bar{G}(V_1 \cup \{u\})$ has at least two components with more than one node, the graph induced by W_i and at least one component with more than one node in $(V_1 \cup \{u\}) \setminus W_i$. □

The above proof yields the following:

Corollary 2.6 *Let $G(V_1)$ and $G(V_2)$ be M-structures with $V_1 \subseteq V_2$. Let W_i and Z_j be connected components of $\bar{G}(V_1)$ and $\bar{G}(V_2)$ respectively having nonempty intersection. Then $W_i \subseteq Z_j$.*

Lemma 2.7 *Let $G(V_1)$ be a maximal M-structure of a GM-graph G that has no clique cutset. Then no node in $V(G) \setminus V_1$ can be adjacent to two distinct proper subclasses of $G(V_1)$.*

Proof: Assume node $x' \in V(G) \setminus V_1$ is adjacent to two proper subclasses W_1 and W_2 of $G(V_1)$. Since $G(V_1)$ is maximal, by Lemma 2.5 node x' is not universal for either of the classes. Also since $\bar{G}(W_1)$ is connected, W_1 contains a pair of nonadjacent nodes x_1, y_1, such that x' is adjacent to x_1 but not to y_1. Similarly W_2 contains a pair of nonadjacent nodes x_2, y_2 such that x' is adjacent to x_2 but not to y_2. Let $H = x_1, x_2, y_1, y_2, x_1$. Then (H, x') is a cap. Since G has no clique cutset, by Lemma 2.2(iii), G contains a node x (possibly $x' = x$) adjacent to x_1, x_2 but not to y_1, y_2 and two nonadjacent nodes u, u' that are universal for the cap (H, x).

Claim 1: *Nodes u and u' are universal for W_1 and W_2 and neither u nor u' is in $W_1 \cup W_2$.*

Proof: Note first that the edges of $\bar{G}(V_1)$ that have their endnodes in $\{x_1, y_1, x_2, y_2, x, u, u'\}$ are x_1y_1, x_2y_2, xy_1, xy_2 and uu'. If the claim does not hold, then W_1 or W_2, say W_1, has the property that, in $\bar{G}$, u or u' has a neighbor in W_1, or u or u' is in W_1. In both cases, $\bar{G}(W_1 \cup \{x, u, u'\})$ is connected. Consider a shortest path in this graph between x and u, u'. W.l.o.g. let $P = u, z_1, \ldots, z_n, x$ be such a path. Now if $n = 1$ and u' is adjacent to z_1 in $\bar{G}$, then $\bar{G}$ contains a triangle z_1, u, u' together with a chordless path z_1, x, y_2, x_2 and no other edge connects the triangle and the path. This is the complement of a T-parachute on six nodes. Otherwise, if $n > 1$ or u' is not adjacent to z_1 in $\bar{G}$, then $u, z_1, \ldots, z_n, x, y_2, x_2$ contains a chordless path of length five. Again, this is the complement of a T-parachute on six nodes and the proof of Claim 1 is complete.

So by Lemma 2.5, since u, u' are nonadjacent, they must belong to the same proper subclass of $G(V_1)$, say W_3, which is distinct from W_1, W_2.

Claim 2: *Node x is universal for W_3.*

Proof: Assume not. Then $\bar{G}(W_3 \cup \{x\})$ is connected. Let $P = u, z_1, \ldots, z_n, x$ be a shortest path in this graph between u, u', say u, and x. Now the same proof as in Claim 1 shows the existence of a parachute.

So, by Lemma 2.5, x belongs to $G(V_1)$. However, in $\bar{G}(V_1)$, x is adjacent to $y_1 \in W_1$ and $y_2 \in W_2$, a contradiction to the fact that W_1, W_2 are distinct proper subclasses of $G(V_1)$. □

Theorem 2.8 *If G is a GM-graph containing an M-structure either with at least three proper subclasses, or with at least one proper subclass which is not a stable set, then G contains a clique cutset or an amalgam.*

Proof: If G contains a D-structure (C_1, C_2, K) then, by Lemma 2.4, G contains an amalgam. So the theorem follows from the proof of the following statement:

If G is a GM-graph containing an M-structure either with at least three proper subclasses, or with at least one proper subclass which is not a stable set, then G contains a D-structure (C_1, C_2, K).

Let $G(V_1)$ be an M-structure of G satisfying the above property and $G(V_2)$ a maximal M-structure with $V_1 \subseteq V_2$.

Claim 1: *The M-structure $G(V_2)$ either contains at least three proper subclasses or contains exactly two proper subclasses not both of which are stable sets.*

Proof: If $G(V_1)$ contains a proper subclass, say W_i, which is not a stable set, by Corollary 2.6, there exists a proper subclass, say Z_j of $G(V_2)$ such that $W_i \subseteq Z_j$. Then Z_j is not a stable set. If all proper subclasses of $G(V_1)$ are stable sets, then $G(V_1)$ has at least three proper subclasses say $W_1, W_2, \ldots, W_k$. If $G(V_2)$ has only two proper subclasses, say Z_1, Z_2, then by Corollary 2.6, we may assume w.l.o.g. that $W_1 \cup W_2 \subseteq Z_1$. Then Z_1 is not a stable set, since every node in W_1 is adjacent to a node in W_2. This completes the proof of Claim 1.

Claim 2: *Suppose that $G(V_2)$ is a maximal M-structure of G with partition (W_1, W_2, K), where W_1 is not a stable set. Then G contains a D-structure (C_1, C_2, K).*

Proof: Let C_1 be a connected component of $G(W_1)$ with more than one node. Let $C_2 = W_2$. Then (C_1, C_2, K) is a D-structure, since by Lemma 2.7 no node of $V(G) \setminus V_2$ is adjacent to a node in C_1 and a node in C_2, and $|C_2| \geq 2$, since W_2 is a proper subclass of $G(V_2)$. This completes the proof of Claim 2.

Claim 3: *Suppose that $G(V_2)$ is a maximal M-structure of G with at least three proper subclasses. Then G contains a D-structure (C_1, C_2, K).*

Proof: Let $W_1, W_2, \ldots, W_l$, $l \geq 3$ be the proper subclasses of $G(V_2)$ and let K be the collection of all non-proper subclasses. Let C_1 be the nodes in two proper subclasses of $G(V_2)$ (note that $G(C_1)$ is a connected graph), C_2 be the nodes in all the other proper subclasses of $G(V_2)$. Then (C_1, C_2, K) is a D-structure since $|C_1| \geq 2$, $|C_2| \geq 2$ and Lemma 2.7 shows that the only nodes having neighbors in both C_1 and C_2 belong to K. So the proof of Claim 3 is complete. □

Corollary 2.9 *Let G be a GM-graph that contains a cap. Then G contains a clique cutset or an amalgam.*

Proof: Assume G contain a cap but no clique cutset. By Lemma 2.2, G contains a cap (H, x) and nonadjacent nodes u, u' universal for (H, x). Since $\bar{G}(V(H) \cup x)$ is connected, G contains an M-structure with proper subclasses $W_1 = \{u, u'\}$ and $W_2 = \{V(H) \cup x\}$ and W_2 is not a stable set. By Theorem 2.8, G contains an amalgam. □

In [7], it was shown that, if G is a cap-free graph, then G contains an amalgam or G is triangulated or G is a triangle-free graph plus at most one universal node. Theorem 1.15 follows from this result and Corollary 2.9. Here, for the sake of completeness, we give a direct proof (without using [7]).

2.3 Expanded Holes

An *expanded hole* consists of nonempty sets of nodes $S_1, \ldots, S_n$, $n \geq 4$, not all singletons, such that, for all $1 \leq i \leq n$, the graphs $G(S_i)$ are connected. Furthermore, every $s_i \in S_i$ is adjacent to $s_j \in S_j$, $i \neq j$, if and only if $j = i + 1$ or $j = i - 1$ (modulo n).

Lemma 2.10 *Let G be a cap-free graph and let H be a hole of G. If s is a node having two adjacent neighbors in H, then either s is universal for H or s together with H induces an expanded hole.*

Proof: Let s be a node with two adjacent neighbors in H. If s has no other neighbors on H, then s induces a cap with H. Let $H = x_1, \dots, x_n, x_1$ with node s adjacent to x_1 and x_n. If s is not universal for H, and does not induce an expanded hole together with H, then let k be the smallest index for which s is not adjacent to x_k. Let l be the smallest index such that $l > k$ and s is adjacent to x_l. Now node x_{k-2} (x_n if $k = 2$) together with the hole $s, x_{k-1}, \dots, x_l, s$ forms a cap. □

Lemma 2.11 *Let G be a cap-free graph and let $S = \cup_{i=1}^{n} S_i$, $n > 4$, be a maximal expanded hole in G with respect to node inclusion. Either G contains an M-structure with a proper subclass that is not a stable set of G, or all nodes that are adjacent to a node in S_i and a node in S_{i+1} ($S_{n+1} = S_1$) for some i, are universal for S and induce a clique of G.*

Proof: Let u be a node adjacent to $s_1 \in S_1$ and $s_2 \in S_2$. By applying Lemma 2.10 to any hole that contains s_1 and s_2 and a node each from the sets S_j, $j > 2$, we have that u is adjacent to all nodes in $S \setminus (S_1 \cup S_2)$, else the maximality of S is contradicted. Now since node u is adjacent to s_1, s_2 and is universal for all sets S_j, $j > 2$, Lemma 2.10 shows that u is universal for S_1 and S_2, hence for S.

Let u and v be two nonadjacent nodes that are universal for S. Then u, v together with $s_1 \in S_1$, $s_2 \in S_2$ and $s_4 \in S_4$ induces an M-structure with proper sets $W_1 = \{u, v\}$ and $W_2 = \{s_1, s_2, s_4\}$. Furthermore W_2 is not a stable set of G. □

Theorem 2.12 *A cap-free graph that contains an expanded hole contains a clique cutset or an amalgam.*

Proof: Let $S = \cup_{i=1}^{n} S_i$ be a maximal expanded hole in G. First assume that $n = 4$. Then the node set S induces an M-structure with proper subclasses $S_1 \cup S_3$ and $S_2 \cup S_4$. $S_2 \cup S_4$ is not a stable set because, say, $|S_2| \geq 2$ and $G(S_2)$ is connected. Hence by Theorem 2.8 we are done. Now assume that $n > 4$. By Theorem 2.8 we may assume that G does not contain an M-structure with a proper subclass that is not a stable set of G. By Theorem 2.4, it is sufficient to show that G contains a D-structure (C_1, C_2, K). Assume w.l.o.g. that $|S_2| \geq 2$ and let K be the set of nodes that are universal for S. Lemma 2.11 shows that K is a clique of G. Let $C_1 = S_2$ and $C_2 = S_1 \cup S_3$. Lemma 2.11 shows that every node that is adjacent to a node of C_1 and a node of C_2 is universal for S and hence belongs to K. Therefore (C_1, C_2, K) is a D-structure. □

2.4 A Proof of Theorem 1.15

Now we are ready to prove Theorem 1.15.

Proof: If G contains a cap, by Corollary 2.9, G contains a clique cutset or an amalgam.

Assume that G is a connected cap-free graph. If G is a triangulated graph, G is either a clique or it contains a clique cutset. If G is a clique and contains at least four nodes, G

contains a join and if G contains less than four nodes, then G is a triangle-free graph plus at most one universal node.

Assume now that G is a connected cap-free graph that contains a hole. Let F be a maximal node set inducing a biconnected triangle-free subgraph of G. Assume that G does not have a clique cutset or an amalgam.

Claim 1: *Every node in $V(G) \setminus F$ that has at least two neighbors in F is universal for F.*

Proof: Let u be a node in $V(G) \setminus F$ having at least two neighbors in F. The graph induced by $F \cup \{u\}$ contains a triangle u, x, y else the maximality of F is contradicted. Let H be a hole in $G(F)$ containing x and y. (H exists since, by biconnectedness, x and y belong to a cycle and since $G(F)$ contains no triangle, a smallest cycle containing x and y is a hole). Lemma 2.10 shows that either u is universal for H or forms an expanded hole with H. Theorem 2.12 rules out the latter possibility. Let $F' \subseteq F$ be a maximal set of nodes such that $G(F')$ contains H, is biconnected and such that node u is universal for F'. If $F \neq F'$, then since $G(F)$ and $G(F')$ are biconnected, some $z \in F \setminus F'$ belongs to a hole that contains an edge of $G(F')$. Let H' be such a hole. By Lemma 2.10 and Theorem 2.12, node u is adjacent to all the nodes of H'. Let $F'' = F' \cup V(H')$. $G(F'')$ is biconnected, u is universal for F''. Hence F'' contradicts the maximality of F'. Hence u is universal for F and the proof of Claim 1 is complete.

Claim 2: *Let U be the set of universal nodes for F. Then the nodes in U induce a clique of G.*

Proof: Let $w, z \in U$ be two nonadjacent nodes of U and let $v_1, \ldots, v_n, v_1$ be a hole of $G(F)$. Then nodes w, z together with v_1, v_2, v_3 and v_4 induce an M-structure, either with two proper subclasses not both of which are stable if v_1 and v_4 are not adjacent, or with three proper subclasses. By Theorem 2.8, G contains an amalgam. This completes the proof of Claim 2.

Claim 3: *$V(G) = F \cup U$.*

Proof: Let $S = V(G) \setminus (F \cup U)$. By Claim 1, every node in S has at most one neighbor in F. Let C be a connected component of $G(S)$. By maximality of F, there is at most one node in F, say y, that has a neighbor in C. If such a node y exists, let $C_1, \ldots, C_l$ be the connected components of $G(S)$ adjacent to y. Let $V_1 = C_1 \cup \ldots C_l \cup \{y\}$, $A = \{y\}$, $K = U$, $V_2 = V(G) \setminus (V_1 \cup K)$ and B be the set of neighbors of y in F. Then (A, B, K) is an amalgam of G, separating V_1 from V_2.

If no component of $G(S)$ is adjacent to a node of F, let $V_1 = U \cup S$, $A = U$, $V_2 = B = F$. Then $(A, B, \emptyset)$ is an amalgam of G. This completes the proof of Claim 3.

If U contains at least two nodes, then let $V_1 = A = U$, $V_2 = B = F$ and $(A, B, \emptyset)$ is an amalgam of G. If U contains at most one node, then G is a triangle-free graph plus at most one universal node. □

3 Line graphs of triangle-free graphs and extensions

In this section, we prove Theorem 1.16.

3.1 L-graphs

If G is the line graph of a graph H, the nodes in a maximal clique of G correspond either to the edges in a triangle of H or to the edges incident with a node of H.

A graph G is *$L\Delta$-free* if G is the line graph of a triangle-free simple graph. In this case, there obviously is a one to one correspondence between maximal stars of H and maximal cliques of G.

Harary and Holtzmann [11] characterize the line graphs of bipartite simple graphs. In Remark 3.1 below we characterize $L\Delta$-free graphs in a similar way. The proof can be easily deduced from the proof of Remark 3.2.

Remark 3.1 *The following three conditions are equivalent.*

1) G is $L\Delta$-free.

2) G contains no claw and no diamond.

3) Every node $v \in V(G)$ belongs to at most two maximal cliques C_1 and C_2, and no node of $C_1 \setminus \{v\}$ is adjacent to a node in $C_2 \setminus \{v\}$.

Maffray and Reed [12] characterize the line graphs of bipartite multigraphs. The following remark has a similar proof and characterizes the line graphs of triangle-free multigraphs. A *gem* is a graph induced by a 5-cycle a, b, c, d, e, a with chords ac and ad.

Remark 3.2 *The following three conditions are equivalent.*

1) G is the line graph of a triangle-free multigraph H.

2) G contains no claw, no gem and no universal 4-wheel.

3) Every node $v \in V(G)$ belongs to at most two maximal cliques C_1 and C_2, and $C_1 \cap C_2$ consists of v and all its twins. No node of $C_1 \setminus C_2$ is adjacent to a node in $C_2 \setminus C_1$.

Proof: Assume G is the line graph of a triangle-free multigraph H. Since an edge of H has at most two endnodes, G is claw-free. Assume G contains a gem G' with $V(G') = \{a, b, c, d, e\}$ and $E(G') = \{ab, bc, cd, de, ea, ac, ad\}$. Since $\{b, c, a, d\}$ induce a diamond of G, the edges e_a, e_c of H corresponding to the nodes a and c of G are parallel edges with endnodes s, t, while e_b has s but not t as endnode and e_d has t but not s as endnode. By the same argument applied to the diamond induced by $\{a, c, d, e\}$, e_a, e_d are parallel, a contradiction. So G cannot contain a gem. The same argument shows that G cannot contain a universal 4-wheel and 1) $\rightarrow$ 2).

Assume that G satisfies 2) and suppose first that v belongs to three maximal cliques, C_1, C_2, C_3. Since every pair of cliques contains nonadjacent nodes, C_1 contains (possibly coincident) nodes a_2, a_3, C_2 contains (possibly coincident) nodes b_1, b_3 and C_3 contains (possibly coincident) nodes c_1, c_2 where b_1 and c_1, a_2 and c_2, a_3 and b_3 are nonadjacent. Together with v, these nodes induce a graph that contains a claw, a gem or an universal 4-wheel. For, choose a_2, a_3, b_3, b_1, c_1, c_2 so that they form the maximum number of coincident pairs. If all three pairs are concident, there is a claw. If two of the pairs are coincident, there is a gem. Otherwise there is a universal 4-wheel. So every node of G is in at most two maximal cliques. $C_1 \cap C_2$ obviously contains all twins of v. If a node in $C_1 \cap C_2$ is not a twin of v then it belongs to three maximal cliques, a contradiction. Finally, if a node of $C_1 \setminus C_2$

is adjacent to a node in $C_2 \setminus C_1$, v is in at least three maximal cliques, again a contradiction and 2) → 3).

Assume that G satisfies 3) and construct H as follows: $V(H)$ corresponds to the set of maximal cliques of G. For every node belonging to a unique maximal clique C_1, add to H a pendant edge attached to the node v_{C_1}. For every node belonging to maximal cliques C_1, C_2, add to H an edge with endnodes v_{C_1} and v_{C_2}, so that the nodes in $C_1 \cap C_2$ are associated to parallel edges. Since no node of $C_1 \setminus C_2$ is adjacent to a node of $C_2 \setminus C_1$, G is the line graph of H and H is a triangle-free multigraph. So 3) → 1). □

Let G be an $L\Delta$-free graph. Let G' be an induced subgraph of G and K' be a clique of G' with at least two nodes. Since G contains no diamond by Remark 3.1, there exists a unique clique K of G containing K'. We say that K is the *extension* of K'.

We say that a clique K of G is *big* if K has more than two nodes and K is *flat* if K contains exactly two nodes. Unless otherwise specified, all the cliques will be maximal.

A connected graph G has a *2-node cutset* $\{u, v\}$ if $G \setminus \{u, v\}$ is a disconnected graph.

Definition 3.3 *A graph G is an* L-graph *if it is an $L\Delta$-free graph and it satisfies the following properties.*

a) G is connected, contains a big clique and every node of G is in two cliques. (Equivalently, H is connected, contains a node of degree at least 3 and every node has degree at least 2).

b) G contains no join. (Equivalently, H contains no cutnode).

c) For every 2-node cutset of G, one of the components is an induced path. (Equivalently, if H contains two edges whose removal disconnects H, then one of the two components is a path).

It follows from this definition that if G is an L-graph, then G contains at least two big cliques. In fact, every hole of G has at least two edges belonging to big cliques.

A *segment* S of an L-graph G is a maximal induced connected subgraph of G such that no pair of nodes of S belongs to the same big clique of G. Note that a segment is a chordless path of G and may have length one or zero. Every node x of G is in exactly one segment, that we call S_x, so the segments of G partition $V(G)$. A segment S is *long* if $|V(S)| \geq 3$, *short* if $|V(S)| = 2$ and *atomic* if $|V(S)| = 1$. Furthemore if a segment S is short and K_x, K_y are the big cliques containing the endnodes of S, no atomic segment is in $K_x \cap K_y$ (i.e. $K_x \cap K_y$ is empty) since G contains no diamond.

Every L-graph has at least three segments. If G is a $3PC(\Delta, \Delta)$ or an L-wheel, then G is an L-graph and G is minimal with this property. These two graphs are called *elementary* L-graphs.

Lemma 3.4 *Let S_1, S_2, S_3 be three segments in an L-graph G. Then G contains an elementary L-graph B, such that S_1, S_2 and S_3 are all in B and S_1, S_2 are contained in distinct segments of B.*

Proof: By Definition 3.3 b), H is 2-connected and therefore H contains a cycle going through any two given edges. This implies the existence of a hole in G going through nodes $x_1 \in S_1$ and $x_2 \in S_2$. Since all cliques of S_1, S_2 are atomic or flat in G and S_1, S_2 are maximal with this property, it follows that, in every hole $C = P_1, S_1, P_2, S_2$ of G going through x_1 and x_2, at least one edge of P_1 and at least one edge of P_2 are extendable to big cliques of G.

Assume first that G contains a hole C going through x_1 and x_2 such that $S_3 \in C$. Let $C = x_1, Q_1, x_2, Q_2, x_1$, where $P_1 \subseteq Q_1$ and $P_2 \subseteq Q_2$. By Definition 3.3 c), $Q_1 \setminus \{x_1, x_2\}$ and $Q_2 \setminus \{x_1, x_2\}$ are in the same connected component of $G \setminus \{x_1, x_2\}$, so they are connected by a shortest path P in $G \setminus \{x_1, x_2\}$. Since every node of C is in two cliques and the cliques of S_1, S_2 are atomic or flat in G, then $P = y_1, \ldots, y_m$ (possibly $m = 1$) , where y_1 belongs to an extension of a clique in P_1 and y_m belongs to an extension of a clique in P_2. If $m = 1$, $C \cup P$ induce an L-wheel and if $m > 1$, $C \cup P$ induce a $3PC(\Delta, \Delta)$.

Assume now that no hole C going through x_1 and x_2 contains S_3. By Definition 3.3 b), c) S_3 belongs to a path $P = y_1, \ldots, y_m$ (possibly $m = 1$) , where y_1 belongs to an extension of a clique of P_1 and y_m belongs to an extension of a clique P_2. This shows that $C \cup P$ induce an elementary L-graph of G. □

Lemma 3.5 *Let C be a hole of an L-graph G. For every segment S_3 of $G \setminus C$, there is a path P in G containing S_3 such that $C \cup P$ is an elementary L-graph G_1 in G. The segments of G_1 are P and two subpaths of C.*

Proof: The proof is identical to the previous one. □

3.2 Tripods

A *triad* is a graph consisting of three internally node-disjoint paths $t, \ldots, x$; $t, \ldots, y$ and $t, \ldots, z$ of length greater than one, where t, x, y, z are distinct nodes. Furthermore, the graph induced by the nodes of the triad contains no other edge than those of the three paths. Node t is the *meet* of the triad.

A *fan* is a graph consisting of a path $P = x, \ldots, y$ of length greater that one, together with a node z not in P adjacent to at least one intermediate node in P and not adjacent to x and y. Node z is the *center* of the fan and the edges connecting z to P are the *spokes*. Furthermore, the graph induced by the nodes of the fan contains no other edge than those of P and the spokes.

A *stool* consists of a triangle $x'y'z'$ together with three node-disjoint paths $x', \ldots, x$; $y', \ldots, y$ and $z', \ldots, z$ of length at least one. Furthermore, the graph induced by the nodes of the stool contains no other edges than those of the triangle and of the three paths.

A *tripod* is a triad or a stool or a fan. Nodes x, y and z are called the *attachments* of the tripod.

Lemma 3.6 *Let G be a node-minimal graph with the following properties.*
(i) G contains nodes x, y, z such that no edge has both endnodes in $\{x, y, z\}$.
(ii) $V(G) \setminus \{x, y, z\}$ is nonempty.
(iii) G and $G \setminus \{x, y, z\}$ are both connected.
Then G is a tripod with attachments x, y and z.

Proof: Let G be a graph with the above properties and let $P_{xy} = x = y_1, \ldots, y_m = y$ be a shortest xy-path in $G \setminus \{z\}$, P_{xz} and P_{yz} similarly defined. Assume w.l.o.g. that P_{xy} is not shorter than any of the other two. If P_{xy} contains an intermediate node that is a neighbor of z, then by the minimality of G, $V(G) = V(P_{xy}) \cup \{z\}$ and G is a fan.

Otherwise let $P = z_1, \ldots, z_n$ be a direct connection between z and $V(P_{xy}) \setminus \{x, y\}$ (P exists since G and $G \setminus \{x, y, z\}$ are both connected), and let $P_z = z, z_1, \ldots, z_n$. By the minimality of G, $V(G) = V(P_{xy}) \cup V(P_z)$ and z_n either has a unique neighbor in P_{xy} or z_n has two neighbors in P_{xy} and these neighbors are adjacent.

By the minimality of G, at most one among x and y has a neighbor in P_z. Assume x has a neighbor in P_z. Then by the minimality of G, z_n is adjacent to the neighbor of x in P_{xy}, possibly to x and to no other node of P_{xy}. Now P_{zy} is longer than P_{xy}, contradicting our choice. So by symmetry, neither x nor y have a neighbor in P_z and therefore if z_n has two neighbors in P_{xy}, neither of these nodes is x or y and we have a stool in this case. Finally, if z_n has a unique neighbor in P_{xy}, say t, then t is not adjacent to x or y else our choice of P_{xy} is again contradicted and in this case we have a triad. □

3.3 Links

Let G be a graph that contains an L-wheel or a $3PC(\Delta, \Delta)$.

Let G' be an L-graph that is an induced subgraph of G. A *link* of G' is a chordless path $P = x_1, \ldots, x_n$ in $G \setminus G'$ (possibly $n = 1$) such that x_1 has a neighbor x_0 in G', x_n has a neighbor x_{n+1} in G', and x_0, x_{n+1} are *nonadjacent* nodes in *distinct* segments of G'. Furthermore P is minimal with the above property.

Lemma 3.7 *Let G' be an L-graph that is an induced subgraph of a graph G. Let U be a connected component of $G \setminus G'$ such that $N(U) \cap G'$ is not contained in a clique of G' and is not contained in a segment of G'. Then*

a) either U contains a link, or

b) $N(U) \cap V(G') = \{x,\ y,\ z\}$ where x and y are the distinct endnodes of a long segment and z is an atomic segment such that $z = K_x \cap K_y$, where K_x and K_y are the big cliques containing x and y.

If G is a WP-free graph, only a) can occur.

Proof: If $N(U) \cap G'$ contains two nodes that are nonadjacent and in distinct segments, then a) holds. So we may assume that this is not the case.

Since $N(U) \cap G'$ contains two nodes, say x and z, that are in distinct segments, then x and z belong to some big clique K_x of G'. Since $N(U) \cap G'$ contains a node y not in K_x, by Remark 3.1 3) we can assume that y is not adjacent to x. Since a) does not hold, the segments S_x, S_y coincide, so S_z, S_y are distinct. This implies that z and y belong to some big clique K_y and $z = K_x \cap K_y$ is an atomic segment of G'. Now all the other nodes of G' are readily seen to be nonadjacent to U. So b) holds.

Now, we show that if G is a WP-free graph, only a) can occur. Assume now that b) holds and let S_x be the long segment of G' containing x and y and S_z the atomic segment containing z. Since G' is an L-graph, by Lemma 3.4, G' contains an elementary L-graph G"

containing S_x and S_z in distinct segments and G" must be an L-wheel with atomic segment S_z.

Let U' be the subgraph of G induced by $V(U) \cup \{x, y, z\}$ with edges xz, yz removed. By Lemma 3.6 U' contains an induced subgraph T which is a tripod with attachments x, y, z. We show that the graph $G" \cup T$ contains a proper wheel or a parachute or a $3PC(\Delta, \cdot)$. Therefore if G is a WP-free graph, b) cannot occur and a) is the only possibility.

If T is a triad, then $G" \cup T$ contains an L-parachute $LP(x'x, y'y, z, t)$, where t is the meet of the triad.

If T is a stool with triangle abc, then $G" \cup T$ contains a $3PC(abc, z)$.

If T is a fan with center z, then $G" \cup T$ contains a proper wheel with center z.

If T is a fan with center x or y but not z, say x, let P_{yz} be a shortest yz-path in T and C_{yz} be the chordless cycle closed by edge yz with P_{yz}. If x has two neighbors in C_{yz}, then these neighbors are nonadjacent (since T is not a fan with center z). So, in this case, we have an L-parachute of type a) $LP(x'x, y'y, z, t)$, where t is the neighbor of x distinct from z. Now assume that x has at least three neighbors on P_{yz}. Since (C_{yz}, x) is a wheel which is not proper, (C_{yz}, x) is either a Δ-free wheel or a T-wheel or an L-wheel. We then have an L-parachute $LP(x'x, y'y, z, t)$ where t is the neighbor of x closest to y in P_{yz}. This L-parachute is of types b), c) or d). (Remark that in this proof we have not used the fact that G contains no T-parachute). □

So it is important to study the links of an L-graph G' of G. The following lemma gives a list of all possible links, when G' is an elementary L-graph.

Lemma 3.8 *Let G be an even-signable WP-free graph, G' an elementary L-graph in G and $P = x_1, \ldots, x_n$ be a link of G'. Then*

a) either $G' \cup P$ is an L-graph, or

b) $n = 1$ and x_1 is either universal for G' or the twin of an endnode of a segment of G'.

Proof: We use the following notation: The two big cliques of G' are $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. The segments of G' are $P_1 = a_1, \ldots, b_1$, $P_2 = a_2, \ldots, b_2$ and $P_3 = a_3, \ldots, b_3$. If G' is an L-wheel, then $a_3 = b_3$ and the segment P_3 is atomic, while P_1 and P_2 are long segments. Otherwise G' is a $3PC(\Delta, \Delta)$ and its segments are either long or short. The nodes in distinct segments P_i and P_j induce a hole of G', that we denote with H_{ij}.

Case 1 $n = 1$.

Let x_0, x_2 be neighbors of x_1 that are nonadjacent and in distinct segments, say P_i and P_j, of G'. Then either x_0, x_2 are the unique neighbors of x_1 in H_{ij} or (H_{ij}, x_1) is a wheel.

Case 1.1 x_0, x_2 are the unique neighbors of x_1 in H_{ij}, or (H_{ij}, x_1) is a Δ-free wheel.

Case 1.1.1 G' is a $3PC(\Delta, \Delta)$.

Assume w.l.o.g. that $i = 1$ and $j = 3$. Now x_1 has more that two neighbors in G', else we have a $3PC(A, x_0)$ or a $3PC(B, x_0)$. If x_1 has at most one neighbor in A and at most one neighbor in B, we have a $3PC(A, x_1)$ or a $3PC(B, x_1)$. Since no two neighbors of x_1 in H_{13} are adjacent, then x_1 cannot be adjacent to both a_1 and a_3 or both b_1 and b_3, so by symmetry we may assume that x_1 is adjacent to a_2, a_3 but not to a_1. Let x_0 be the neighbor of x_1, closest to a_1 in P_1. Then we have a T-parachute $TP(a_2, a_3, x_1, a_1, x_0)$ of type a or b.

Case 1.1.2 G' is an L-wheel.

Assume first that $i = 1$ and $j = 2$. If x_1 has at most one neighbor in A and at most one neighbor in B, we have a $3PC(A, x_1)$ when x_1 has at least three neighbors in G' and when x_1 has two neighbors in G' we have a $3PC(A, x_0)$ or a $3PC(B, x_0)$. So by symmetry, we may assume that x_1 is adjacent to $b_3 = a_3$, b_2 but not b_1. Let x_0 be the neighbor of x_1, closest to b_1 in P_1. Now x_0 is distinct from a_1, else let $C = a_1, P_1, b_1, b_2, x_1, a_1$ and (C, b_3) is an odd wheel. Now we have a T-parachute $TP(b_2, b_3, x_1, b_1, x_0)$ of type a or b.

Assume now that $i = 1$ and $j = 3$. If (H_{13}, x_1) is a Δ-free wheel and x_1 has no neighbor in P_2, then there is a proper wheel with center a_3 and if x_1 is adjacent to a_3, x_0 and to no node in P_2, then we have an L-parachute $LP(a_2a_1, b_2b_1, a_3, x_0)$. So x_1 has at least one neighbor in P_2. Now x_1 is adjacent to a_2 or b_2, say b_2, else we have a $3PC(A, x_1)$. Then we have a T-parachute $TP(b_2, b_3, x_1, b_1, x_0)$, where x_0 is the neighbor of x_1, closest to b_1 in P_1.

Case 1.2 (H_{ij}, x_1) is a universal wheel.

We may assume that x_1 is not universal for G', else b) holds.

Case 1.2.1 G' is a $3PC(\Delta, \Delta)$.

Assume w.l.o.g. that (H_{13}, x_1) is a universal wheel. Then both P_1 and P_3 have length less than three, otherwise (H_{12}, x_1) or (H_{23}, x_1) is a proper wheel. Furthermore, since (H_{13}, x_1) is an even wheel, either both P_1 and P_3 have length one or both P_1 and P_3 have length two. Assume $P_1 = a_1, t_1, b_1$ and $P_3 = a_3, t_3, b_3$. Then x_1 has no neighbor in P_2, else (H_{23}, x_1) is a proper wheel. Now the graph induced by $V(G') \setminus \{a_3, t_3\} \cup \{x_1\}$ is a T-parachute $TP(b_3, b_1, x_1, b_2, a_1)$ of type c.

Assume $P_1 = a_1, b_1$ and $P_3 = a_3, b_3$. Then x_1 has neighbors in P_2, else $V(G') \setminus \{a_1\} \cup \{x_1\}$ induces an odd wheel with center b_3. So (H_{12}, x_1) is an L-wheel or a T-wheel. If (H_{12}, x_1) is a L-wheel, let x_2, x_3 be the neighbors of x_1, where x_2 is closest to b_2 in P_2. Then $V(G') \setminus \{a_1\} \cup \{x_1\}$ induces an L-parachute of type c $LP(x_3x_2, a_3b_3, x_1, b_2)$. If (H_{12}, x_1) is a T-wheel, with x_1 adjacent to, say, b_2 we have a T-parachute $TP(b_3, x_1, a_3, b_2, a_2)$ of type c.

Case 1.2.2 G' is an L-wheel.

If (H_{12}, x_1) is a universal wheel and x_1 is not adjacent to $a_3 = b_3$, then P_1 and P_2 have both length 2, else (H_{13}, x_1) or (H_{23}, x_1) is a proper wheel. But then, we have a T-parachute $TP(a_2, a_1, x_1, a_3, b_1)$ of type c.

If (H_{13}, x_1) is a universal wheel, then x_1 has at least one neighbor in P_2 since otherwise (H, a_3) is a proper wheel where $H = a_2, P_2, b_2, b_1, x_1, a_1, a_2$. But now, since x_1 is not universal for G', (H_{12}, x_1) is a proper wheel.

Case 1.3 (H_{ij}, x_1) is an L-wheel or a T-wheel.

If x_1 has at most one neighbor in A and at most one neighbor in B, then (H_{ij}, x_1) is an L-wheel and x_1 has no neighbor in P_k, $k \neq i, j$, for otherwise we have a $3PC(A, x_1)$, and in this case a) holds. So by symmetry, we assume that x_1 has at least two neighbors in B. Furthermore x_1 is adjacent to both b_i and b_j, since otherwise, if x_1 is adjacent to b_i but not b_j, then (H_{ij}, x_1), (H_{ik}, x_1) or (H_{jk}, x_1), $k \neq i, j$, is a proper wheel or H_{jk} together with b_i and x_1 induces a T-parachute of type c.

Case 1.3.1 G' is an L-wheel.

Assume $i = 1$ and $j = 3$. Then x_1 is adjacent to b_1 and b_3.

If (H_{13}, x_1) is an L-wheel, then b_2 is the only neighbor of x_1 in P_2, else (H_{12}, x_1) is a proper wheel, and a) holds.

If (H_{13}, x_1) is a T-wheel, then either x_1 is adjacent to the neighbor b_1' of b_1 in P_1, or x_1 is adjacent to a_1.

If x_1 is adjacent to b_1', then x_1 has a neighbor in P_2, else we have an L-parachute $LP(a_2a_1, b_2b_1, a_3, b_1')$ of type c. Since x_1 is not adjacent to a_1, (H_{12}, x_1) is an L-wheel or a T-wheel. If (H_{12}, x_1) is a T-wheel, then b_2 is the only neighbor of x_1 in P_2, so x_1 is a twin of b_1 and b) holds. If (H_{12}, x_1) is an L-wheel, then the neighbors of x_1 in P_2 are a_2 and its neighbor in P_2, else (H_{23}, x_1) is a proper wheel. But now we have a T-parachute $TP(b_1, x_1, b_3, b_1', a_1)$ of type c.

If x_1 is adjacent to a_1, then x_1 has a neighbor in P_2, else we have a proper wheel with center b_3, so (H_{12}, x_1) must be an L-wheel, x_1 is a twin of a_3 and b) holds.

Assume $i = 1$ and $j = 2$. Then x_1 is adjacent to b_1 and b_2.

If (H_{12}, x_1) is an L-wheel then x_1 is adjacent to a_1, a_2, b_1, b_2 and no other node of H_{12}, else there is a proper wheel with center a_3. If x_1 is adjacent to b_3, it is a twin of b_3 and b) holds, and if x_1 is not adjacent to b_3, we have a T-parachute $TP(b_2, b_1, a_3, x_1, a_1)$ of type a. If (H_{12}, x_1) is a T-wheel then x_1 is w.l.o.g. adjacent to b_1, b_2, b_1' and no other node of H_{12}. If x_1 is adjacent to b_3, it is a twin of b_1 and b) holds, and if x_1 is not adjacent to b_3, we have an odd wheel with center b_3.

Case 1.3.2 G' is a $3PC(\Delta, \Delta)$.

Assume w.l.o.g. that $i = 1$ and $j = 3$. Then x_1 is adjacent to b_1 and b_3. If (H_{13}, x_1) is an L-wheel and x_1 has two neighbors in $P_1 \setminus \{b_1\}$ or $P_3 \setminus \{b_3\}$, say $P_1 \setminus \{b_1\}$, then b_2 is the only neighbor of x_1 in P_2, else (H_{12}, x_1) is a proper wheel, and a) holds. If x_1 is adjacent to a_1 and a_3, then x_1 has a neighbor in P_2, else we have a T-parachute $TP(a_1, a_3, a_2, x_1, b_3)$. If (H_{23}, x_1) is a Δ-free wheel, there is a T-parachute. So (H_{23}, x_1) must be an L-wheel, x_1 is adjacent to both a_2 and b_2 and a) holds.

If (H_{13}, x_1) is a T-wheel, then assume w.l.o.g. that x_1 is adjacent to b_1'. Node x_1 has a neighbor in P_2 since, otherwise, there is an odd wheel with center b_1. Now suppose that (H_{12}, x_1) is a universal wheel. Then, there is a T-parachute $TP(a_1, a_2, a_3, x_1, b_2)$. So (H_{12}, x_1) is an L-wheel or a T-wheel. (H_{12}, x_1) is a T-wheel, else (H_{23}, x_1) is a proper wheel. If P_1 has length one and x_1 is adjacent to a_1 and a_2, there is a T-parachute $TP(a_1, a_2, a_3, x_1, b_2)$. So b_2 is the only neighbor of x_1 in P_2 and b) holds.

Case 2 $n > 1$.

By Lemma 3.7, since P is a link, the neighbors of x_1 in G' are either contained in a big clique A or B or in a segment of G' and the same holds for x_n.

Claim 1 No intermediate node of P has a neighbor in G'.

Assume node y of G' is adjacent to an intermediate node x_j of P. Since P is a link, by Lemma 3.7, $(N(x_1) \cap G') \cup \{y\}$ is contained in a big clique or a segment of G' and the same holds for $(N(x_n) \cap G') \cup \{y\}$. So either $y = a_3 = b_3$, and the neighbors of one endnode of P, say x_1, are contained in A while the neighbors of x_n are contained in B or y is the endnode of a nonatomic segment say P_1, the neighbors of one endnode of P, say x_1, are contained in P_1 and the neighbors of x_n are contained in a big clique, say A. This shows that such a node y is unique.

Assume first that $y = a_3 = b_3$, so G' is an L-wheel. We also assume w.l.o.g. that x_1 is adjacent to a_1, while x_n is adjacent to b_2. Now x_1 is adjacent to a_2 but not to a_3, since otherwise (H, y) is a proper wheel where $H = a_2, a_1, x_1, P, x_n, b_2, P_2, a_2$. By symmetry, x_n is adjacent to b_1 and not to a_3. Let x_j be the node of lowest index adjacent to a_3. Now we have a T-parachute $TP(a_2, a_1, x_1, a_3, x_j)$.

Assume now that $y = a_1$, the neighbors of x_1 are contained in P_1 and the neighbors of x_n are contained in A, so x_n is adjacent to a_2 or a_3.

If x_n is adjacent to a_2, let Q be a shortest path between x_1 and b_1, whose intermediate nodes (if any) are in P_1. Let $H_1 = x_n, a_2, P_2, b_2, b_1, Q, x_1, P, x_n$. Now either (H_1, a_1) is a wheel or a_1 has two neighbors in H_1 and they are nonadjacent. Now x_n is also adjacent to a_3, else let $H_2 = x_n, a_2, a_3, P_3, b_3, b_1, Q, x_1, P, x_n$, then (H_2, a_1) is a proper wheel.

Finally x_n is also adjacent to a_1, else we have a T-parachute $TP(a_2, a_3, a_1, x_n, x_j)$, where x_j is the node of highest index adjacent to a_1. Let $H'_2 = x_n, a_3, P_3, b_3, b_1, Q, x_1, P, x_n$. Now (H_1, a_1) and (H'_2, a_1) are either both L-wheels or both T-wheels. If x_1 has a unique neighbor x_0 in P_1, we have either an L-parachute $LP(x_j x_{j-1}, x_n a_3, a_1, x_0)$ or a T-parachute $TP(x_n, a_1, x_{n-1}, a_3, x_0)$. If x_1 has several neighbors in P_1, we have a $3PC(\Delta, a_1)$ or an L-parachute $LP(x_j x_{j-1}, x_n a_3, a_1, x_1)$ or a T-parachute $TP(x_n, a_1, x_{n-1}, a_3, x_1)$.

If x_n is adjacent to a_3 and $a_3 \neq b_3$, the proof is identical. Finally, consider the case where x_n is adjacent to a_3 and $a_3 = b_3$. If x_1 has exactly two neighbors on P_1 and they are adjacent, there is a $3PC(\Delta, a_1)$ or a proper wheel with center a_1. Otherwise, there is an L-parachute $LP(a_2 a_1, b_2 b_1, a_3, x_0)$ or $LP(a_2 a_1, b_2 b_1, a_3, x_i)$ where x_i is the node of lowest index adjacent to a_1. This completes the proof of Claim 1.

Case 2.1 All the neighbors of x_1 in G' are contained in a big clique, say A, and all the neighbors of x_n are contained in B.

We assume w.l.o.g. that x_1 is adjacent to a_1 and x_n is adjacent to b_2.

Case 2.1.1 G' is an L-wheel.

Assume x_1 is adjacent to a_1 only. Then x_n is adjacent to $a_3 = b_3$, else we have an odd wheel with center a_3. Let $H = a_1, x_1, P, x_n, b_2, b_1, P_1, a_1$ if x_n is not adjacent to b_1, and $H = a_1, x_1, P, x_n, b_1, P_1, a_1$ if x_n is adjacent to b_1. Then (H, a_3) is a proper wheel.

Assume x_1 is adjacent to a_1, a_2 but not to a_3. If x_n is adjacent to a_3, we have an odd wheel with center a_3 and if x_n is not adjacent to a_3 we have a T-parachute $TP(a_1, a_2, x_1, a_3, b_2)$.

Assume x_1 is adjacent to a_1, a_3 but not to a_2. Let $H = a_1, x_1, P, x_n, b_2, P_2, a_2, a_1$. Then (H, a_3) is a proper wheel.

So x_1 is adjacent to a_1, a_2, a_3 and by symmetry, x_n is adjacent to b_1, b_2, and b_3 and a) holds in this case.

Case 2.1.2 G' is a $3PC(\Delta, \Delta)$.

Assume x_1 is adjacent to a_1 only. If x_n is adjacent to b_2 only, we have a $3PC(B, a_1)$. If x_n is adjacent to b_1, we have a $3PC(b_1 b_2 x_n, a_1)$. If x_n is adjacent to b_3 but not b_1, we have a T-parachute $TP(b_3, b_2, x_n, b_1, a_1)$.

Assume x_1 is adjacent to a_1, a_2 but not to a_3. If x_n is not adjacent to b_1, we have a $3PC(a_1 a_2 x_1, b_2)$. If x_n is adjacent to b_1, we have a T-parachute $TP(a_2, a_1, x_1, a_3, b_1)$ if x_n is not adjacent to b_3 and a $3PC(b_1 b_3 x_n, a_1)$ if x_n is adjacent to b_3.

Assume x_1 is adjacent to a_1, a_3 but not to a_2. By symmetry, we may assume that x_n is not adjacent to b_3 and we have a T-parachute $TP(a_1, a_3, x_1, a_2, b_2)$.

So x_1, is adjacent to a_1, a_2 and a_3. Again by symmetry, x_n is adjacent to b_1, b_2, and b_3 and a) holds in this case.

Case 2.2 All the neighbors of x_1 in G' are contained in a big clique, say A, and all the neighbors of x_n are contained in a segment, say P_1.

Then x_1 is adjacent to a_2 or a_3.

Case 2.2.1 G' is an L-wheel.

We first show that x_n has two neighbors in P_1 and these neighbors are adjacent. If not, either P_1 contains two neighbors of x_n and these neighbors are nonadjacent or P_1 has a unique neighbor of x_n. Assume the first possibility holds: If x_1 is adjacent to a_2 only, we have a $3PC(A, x_n)$. If x_1 is adjacent to a_3 only, we have an L-parachute $LP(a_2a_1, b_2b_1, a_3, x_n)$. If x_1 is adjacent to a_1, a_3 and possibly a_2 we have a $3PC(x_1a_1a_3, x_n)$ if x_n is not adjacent to both a_1 and x_1, and a T-parachute $TP(x_1, a_1, a_2, x_n, t)$, where t is the neighbor of x_n in P_1 that is closest to b_1, if x_n is adjacent to both a_1 and x_1. If x_1 is adjacent to a_1, a_2 but not a_3, there is a proper wheel with center a_3. Finally if x_1 is adjacent to a_2, a_3 but not a_1, we have a T-parachute $TP(a_2, a_3, a_1, x_1, x_n)$. So P_1 cannot contain two nonadjacent neighbors of x_n

The same proof rules out the case where P_1 has a unique neighbor of x_n, so x_n has two adjacent neighbors, say y and z, in P_1 and y is closer than z to a_1 in P_1.

Assume x_1 is adjacent to a_3. Then x_1 is adjacent to a_1, else we have a $3PC(x_nyz, a_3)$. Now x_1 is also adjacent to a_2, else we have a $3PC(x_nyz, a_1)$ when $a_1 \neq y$ and an L-parachute $LP(x_nz, x_1a_3, a_1, b_1)$ of type d when $y = a_1$ and $n > 2$. When $y = a_1$ and $n = 2$, we have an odd wheel with center a_1. So a) holds in this case.

Assume finally x_1 is adjacent to a_2 but not a_3. Then x_1 adjacent to a_1, else we have a $3PC(x_nyz, a_2)$. Now we have a $3PC(x_nyz, a_1)$ when $a_1 \neq y$ and when $y = a_1$, we have a proper wheel with center a_1.

Case 2.2.2 G' is a $3PC(\Delta, \Delta)$.

We assume w.l.o.g. that x_1 is adjacent to a_2. x_1 is adjacent to a_3 since, otherwise, there is a $3PC(B, a_2)$.

Assume that x_1 is not adjacent to a_1. If x_n has two nonadjacent neighbors in P_1, there is a T-parachute $TP(a_2, a_3, a_1, x_1, x_n)$. If x_n has a unique neighbor x_{n+1} in P_1, there is a T-parachute $TP(a_2, a_3, a_1, x_1, x_{n+1})$. If x_n has exactly two neighbors in P_1, say y, z, and they are adjacent, there is a $3PC(x_nyz, a_2)$.

So x_1 is adjacent to a_1, a_2 and a_3. If x_n has a unique neighbor x_{n+1} in P_1, there is a $3PC(x_1a_1a_2, x_{n+1})$. If x_n has two nonadjacent neighbors in P_1, there is a $3PC(x_1a_1a_2, x_n)$ if x_n is not adjacent to both x_1 and a_1 and a T-parachute $TP(x_1, a_1, a_2, x_n, t)$ otherwise, where t is the neighbor of x_n closest to b_1. So, x_n has exactly two neighbors in P_1, say y, z, and they are adjacent. Let y be the one that is closest to a_1 in P_1. If $y = a_1$, there is a proper wheel with center a_1. So $y \neq a_1$ and a) holds in this case.

Case 2.3 All the neighbors of x_1 are contained in a segment, say P_1 and all the neighbors of x_n are contained in a segment, say P_2.

Note that the choice of P_1 and P_2 is done w.l.o.g. by assuming that we are not in Case 2.2. We show that x_1 has exactly two neighbors and these two neighbors are adjacent. Assume x_1 has exactly one neighbor y in P_1. Since x_n has a neighbor in $P_2 \setminus \{b_2\}$, there is a $3PC(A, y)$.

If x_1 has two nonadjacent neighbors in P_1, replace P_1 by the chordless a_1b_1-path containing x_1 and nodes of P_1 and let $P' = x_2, \ldots, x_n$. The proof of Case 1 shows that P' has length bigger than 2 and x_2 now has x_1 as unique neighbor in P_1' and by the above argument, this is impossible. So x_1 has exactly two neighbors in P_1 and they are adjacent. By symmetry, x_n has exactly two neighbors in P_2 and they are adjacent. So a) holds in this case. □

Lemma 3.9 *Let G be an even-signable WP-free graph, G' be an L-graph in G and $P = x_1, \ldots, x_n$ be a link of G'. Then*

a) either $G' \cup P$ is an L-graph, or

b) $n = 1$ and x_1 is either universal for G' or the twin of an endnode of a segment of G'.

Proof: Since P is a link of G', x_1, x_n have neighbors x_0, x_{n+1} that are nonadjacent and in distinct segments S_{x_0} and $S_{x_{n+1}}$ of G'. Let S_3 be any other segment of G'. By Lemma 3.4, G' contains an elementary L-graph G_1, with S_{x_0} and $S_{x_{n+1}}$ in distinct segments of G_1 and containing S_3. So P is a link of G_1, and by Lemma 3.8, the statement holds when $G' = G_1$.

Case 1 $n = 1$.

Assume x_1 is a universal node for G_1 and x_1 is not adjacent to node y of G'. Let C be any hole of G_1. By Lemma 3.5, G' contains an elementary L-graph G_2, containing C and segment S_y. Since at least two nodes of C are nonadjacent and in distinct segments of G_2, x_1 is a link of G_2. Since (C, x_1) is a universal wheel but x_1 is not universal for G_2, Lemma 3.8 is contradicted.

Assume x_1 is a link of G_1 and is adjacent to all nodes in distinct cliques K_1', K_2' of G_1, not in the same segment of G_1 and to no other node of G_1. (This happens both when x_1 is a twin of an endnode of a segment of G_1 and when $G_1 \cup \{x_1\}$ is an L-graph). Let K_1, K_2 be the cliques of G', that extend K_1', K_2'.

Assume x_1 is adjacent to node y in $G' \setminus (K_1 \cup K_2)$ and let C be a hole of G_1 containing two nodes of K_1' and two nodes of K_2'. By Lemma 3.5, G' contains an elementary L-graph G_2, containing C and segment S_y. Now x_1 is a link of G_2, for x_1 is adjacent to y, and no node of C is in the same segment as y and at least one neighbor of x_1 in C is nonadjacent to y. Since (C, x_1) is an L-wheel or a T-wheel and x_1 is adjacent to y, Lemma 3.8 is contradicted in G_2.

Assume x_1 is not adjacent to node z in K_1. Let C be a hole containing two nodes of K_1' and two nodes of K_2'. By Lemma 3.5, G' contains an elementary L-graph G_2, containing C and segment S_z. Since G_2 contains at least three nodes of K_1, its restriction K_1^* to G_2 is a big clique of G_2 and since x_1 is adjacent to two nodes in K_1^*, two other nodes of C and no other node of G_2, x_1 is a link of G_2, violating Lemma 3.8.

So x_1 is adjacent to all nodes in $K_1 \cup K_2$ and is adjacent to no other node of G'. If K_1, K_2 have a common node in G', then x_1 is a twin of such a node and b) holds. Otherwise $G' \cup \{x_1\}$ is an L-graph and a) holds.

Case 2 $n > 1$.

Assume that node y of G' is adjacent to an intermediate node x_j of P. By Lemma 3.4, G' contains an elementary graph G_2 containing S_y, where S_{x_0} and $S_{x_{n+1}}$ are in distinct segments of G_2, So P is a link of G_2 (the minimality of P follows from the fact that P is a link of

G'). Since an intermediate node of P is adjacent to y, Lemma 3.8 is violated in G_2. So no intermediate node of P has a neighbor in G'.

Since P is a link of G_1, by Lemma 3.8, x_1 and x_n are adjacent to all the nodes in cliques K'_1, K'_2, not in the same segment of G_1. Let K_1, K_2 be the cliques of G' that extend K'_1, K'_2.

Assume x_1 is adjacent to node y in $G' \setminus K_1$. By Lemma 3.4, G' contains an elementary L-graph G_2, containing S_y, where S_{x_0} and $S_{x_{n+1}}$ are in distinct segments of G_2. So P is a link of G_2, contradicting Lemma 3.8. So all the neighbors of x_1 in G' are contained in K_1 and by symmetry, all the neighbors of x_n in G' are contained in K_2.

Assume x_1 is not adjacent to node z in K_1. By Lemma 3.4, G' contains an elementary L-graph G_2, containing S_z, where S_{x_0} and $S_{x_{n+1}}$ are in distinct segments of G_2. So P is a link of G_2, contradicting Lemma 3.8. So x_1 belongs to the extension of K_1 and by symmetry, x_n belongs to the extension of K_2 and a) holds in this case. □

3.4 A Proof of Theorem 1.16

A *twin class* of a graph G is a maximal subset of $V(G)$ with the property that every pair of nodes in it are twins. The twin classes of G partition $V(G)$ into cliques. A *restriction* of G is an induced subgraph H of G obtained by keeping exactly one node in each twin class. All the restrictions of G are obviously isomorphic graphs.

Let G be a graph. An induced subgraph G' of G is an *extended L-graph* if any restriction H of G' is an L-graph and every twin class of G' containing an intermediate node in a segment of H contains no other node. A *segment* of G' is a segment of one of its restrictions H together with all the nodes in the twin classes of its endnodes. A path P in $G \setminus G'$ is a *link* of G' if P is a link of some restriction H of G'. By Lemma 3.9, it follows that if P is a link of G', then P is a link of all the restrictions of G'.

Theorem 3.10 *Let G be an even-signable WP-free graph, G' a node-maximal extended L-graph in G and $P = x_1, \dots, x_n$ a link of G'. Then $n = 1$ and x_1 is a universal node for G'.*

Proof: Follows by applying Lemma 3.9 to all the possible restrictions of G' and the maximality of G'. □

We can now prove Theorem 1.16.

Proof: Let G be an even-signable WP-free graph that contains an L-wheel or a $3PC(\Delta, \Delta)$ as induced subgraph. Then G contains a node-maximal extended L-graph G' and let U be the set of nodes that are universal for G'.

Assume first that $V(G) = V(G') \cup U$. If $U \neq \emptyset$, then $\bar{G}$ is disconnected. If $U = \emptyset$ and at least one twin class of G contains at least two nodes, then G contains a star cutset. Finally, if $U = \emptyset$ and every twin class of G contains a single node, G is the line graph of a triangle-free graph.

Assume now $G \setminus (V(G') \cup U)$ is nonempty and let $C_1 \dots, C_n$ be the connected component of $G \setminus (V(G') \cup U)$. By Theorem 3.10 and Lemma 3.7, for every connected component C_i, $N(C_i) \cap G'$ is either contained in a clique of G' or in a segment of G' which is not atomic.

Assume first that a component, say C_1, has its neighbors in G' contained in a clique K. Then the removal of the nodes in $K \cup U$ separates C_1 from $G' \setminus K$ and we have a star cutset.

Assume now that no component has its neighbors in G' contained in a clique of G' and let K_1, K_2 be the two big cliques of G' that contain the endnodes of the nonatomic segment S containing the neighbors of C_1. Let A_1, A_2 be the subsets of K_1, K_2 that are the twin classes of the endnodes of S and let $B_1 = K_1 \setminus A_1$, $B_2 = K_2 \setminus A_2$. Since K_1, K_2 are cliques of an extended L-graph, A_1, A_2 are nonempty and disjoint, while $B_1 \setminus B_2$, $B_2 \setminus B_1$ are both nonempty. If $B_1 \cap B_2$ is empty, we have an extended strong 2-join separating $S \cup C_1$ from $G' \setminus S$, and if $B_1 \cap B_2$ is nonempty, we have a star cutset separating the same sets. □

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