Balanced Matrices

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dedicated to the memory of Claude Berge

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Abstract

A $0, \pm 1$ matrix is *balanced* if, in every submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. This definition was introduced by Truemper and generalizes the notion of balanced $0, 1$ matrix introduced by Berge. In this tutorial, we survey what is currently known about these matrices: polyhedral results, combinatorial and structural theorems, and recognition algorithms.

1 Introduction

A $0, \pm 1$ matrix H is a *hole matrix* if H contains two nonzero entries per row and per column and no proper submatrix of H has this property. A hole matrix H is square, say of order n, and its rows and columns can be permuted so that its nonzero entries are $h_{i,i}$, $1 \leq i \leq n$, $h_{i,i+1}$, $1 \leq i \leq n-1$, $h_{n,1}$ and no other. Note that $n \geq 2$ and the sum of the entries of H is even.

A hole matrix is *odd* if the sum of its entries is congruent to $2 \; mod \, 4$ and *even* if the sum of its entries is congruent to $0 \; mod \, 4$.

A $0, \pm 1$ matrix A is *balanced* if no submatrix of A is an odd hole matrix. This notion is due to Truemper [69] and it extends the definition of balanced $0, 1$ matrices introduced by Berge [2]. The class of balanced $0, \pm 1$ matrices includes balanced $0, 1$ matrices and totally unimodular $0, \pm 1$ matrices. (A matrix is *totally unimodular* if every square submatrix has determinant equal to $0, \pm 1$. The fact that total unimodularity implies balancedness follows, for example, from Camion's theorem [11] which states that a $0, \pm 1$ matrix A is totally unimodular if and only if A does not contain a square submatrix with an *even number* of nonzero entries per row and per column whose sum of the entries is congruent to $2 \; mod \, 4$.) In this tutorial, we survey what is currently known about balanced matrices: polyhedral results, combinatorial and structural theorems, and recognition algorithms. Previous surveys on this topic appear in [22], [18].

2 Integer Polytopes

A polytope is *integral* if all its vertices have only integer-valued components. The *set packing polytope*, defined by an $n \times m$ $0,1$ matrix A, is

$$P(A) = \{x \in R^n : Ax \leq \mathbf{1},\ 0 \leq x \leq \mathbf{1}\},$$

where $\mathbf{1}$ denotes a column vector of appropriate dimension whose entries are all equal to 1.

The next theorem characterizes a balanced $0,1$ matrix A in terms of the set packing polytope $P(A)$ as well as the set covering polytope $Q(A)$ and the set partitioning polytope $R(A)$:

$$Q(A) = \{x : Ax \geq \mathbf{1},\ 0 \leq x \leq \mathbf{1}\},$$
$$R(A) = \{x : Ax = \mathbf{1},\ 0 \leq x \leq \mathbf{1}\}.$$

Theorem 2.1 (Berge [3], Fulkerson, Hoffman, Oppenheim [41]) *Let M be a $0,1$ matrix. Then the following statements are equivalent:*

(i) M is balanced.

(ii) For each submatrix A of M, the set covering polytope $Q(A)$ is integral.

(iii) For each submatrix A of M, the set packing polytope $P(A)$ is integral.

(iv) For each submatrix A of M, the set partitioning polytope $R(A)$ is integral.

Given a $0,\pm1$ matrix A, let $p(A)$, $n(A)$ denote respectively the column vectors whose i^{th} components $p_i(A)$, $n_i(A)$ are the number of $+1s$ and the number of $-1s$ in the i^{th} row of matrix A. Theorem 2.1 extends to $0,\pm1$ matrices as follows.

Theorem 2.2 (Conforti, Cornuéjols [17]) *Let M be a $0,\pm1$ matrix. Then the following statements are equivalent:*

(i) M is balanced.

(ii) For each submatrix A of M, the generalized set covering polytope $Q(A) = \{x : Ax \geq \mathbf{1} - n(A),\ 0 \leq x \leq \mathbf{1}\}$ is integral.

(iii) *For each submatrix A of M, the generalized set packing polytope* $P(A) = \{x : Ax \leq \mathbf{1} - n(A),\ 0 \leq x \leq \mathbf{1}\}$ *is integral.*

(iv) *For each submatrix A of M, the generalized set partitioning polytope* $R(A) = \{x : Ax = \mathbf{1} - n(A),\ 0 \leq x \leq \mathbf{1}\}$ *is integral.*

To prove this theorem, we need the following two results. The first one is an easy application of computation of determinants by cofactor expansion.

Remark 2.3 *Let H be a $0, \pm 1$ hole matrix. If H is an even hole matrix, H is singular, and if H is an odd hole matrix, $det(H) = \pm 2$.*

Lemma 2.4 *If A is a balanced $0, \pm 1$ matrix, then the generalized set partitioning polytope $R(A)$ is integral.*

Proof: Assume that A contradicts the theorem and has the smallest size (number of rows plus number of columns). Then $R(A)$ is nonempty. Let $\bar{x}$ be a fractional vertex of $R(A)$. By the minimality of A, $0 < \bar{x}_j < 1$ for all j. It follows that A is square and nonsingular. So $\bar{x}$ is the unique vector in $R(A)$.

Let $a^1, \ldots, a^n$ denote the row vectors of A and let A_i be the $(n-1) \times n$ submatrix of A obtained by removing row a^i. By the minimality of A, the set partitioning polytope $R(A_i) = \{x \in R^n : A_i x = \mathbf{1} - n(A_i),\ 0 \leq x \leq \mathbf{1}\}$ is an integral polytope. Since A is square and nonsingular, the polytope $R(A_i)$ has exactly two vertices, say x^S, x^T. Since $\bar{x}$ is in $R(A_i)$, then $\bar{x} = \lambda x^S + (1-\lambda) x^T$. Since $0 < \bar{x}_j < 1$ for all j and x^S, x^T have 0,1 components, it follows that $x^S + x^T = \mathbf{1}$. Let k be any row of A_i. Since both x^S and x^T satisfy $a^k x = 1 - n(a^k)$, this implies that $a^k \mathbf{1} = 2(1 - n(a^k))$, i.e. row k contains exactly two nonzero entries. Applying this argument to two different matrices A_i, it follows that every row of A contains exactly two nonzero entries.

If A has a column j with only one nonzero entry a_{kj}, remove column j and row k. Since A is nonsingular, the resulting matrix is also nonsingular and the absolute value of the determinant is unchanged. Repeating this process, we get a square nonsingular matrix B of order at least 2, with exactly two nonzero entries in each row and column (possibly $B = A$). Now B can be put in block-diagonal form, where all the submatrices in the diagonal are hole matrices. Since B is nonsingular, all these submatrices are nonsingular and by Remark 2.3 they are odd hole matrices. Hence A is not balanced. $\square$

Theorem 2.5 *Let A be a balanced $0, \pm 1$ matrix with rows $a^i, i \in S$, and let S_1, S_2, S_3 be a partition of S. Then*

$$
\begin{aligned}
T(A) = \{x \in R^n : \ & a^i x \geq 1 - n(a^i) \quad for\ i \in S_1, \\
& a^i x = 1 - n(a^i) \quad for\ i \in S_2, \\
& a^i x \leq 1 - n(a^i) \quad for\ i \in S_3, \\
& 0 \leq x \leq \mathbf{1}\}
\end{aligned}
$$

is an integral polytope.

Proof: If $\bar{x}$ is a vertex of $T(A)$, it is a vertex of the polytope obtained from $T(A)$ by deleting the inequalities that are not satisfied with equality by $\bar{x}$. By Theorem 2.4, every vertex of this polytope has $0, 1$ components. □

Theorem 2.5 does not hold when the upper bound $x \leq \mathbf{1}$ is removed. To see this, consider the matrix

$$
A = \begin{pmatrix}
1 & 1 & 1 & 1 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Then $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 1, 1)$ is the unique solution of $Ax = 1 - n(A)$ and therefore it is a fractional vertex of the polyhedron $T(A)$ with $x \leq \mathbf{1}$ removed, for any partition of the rows of A into S_1, S_2 and S_3.

Proof of Theorem 2.2: Since balanced matrices are closed under taking submatrices, Theorem 2.5 shows that (i) implies (ii), (iii) and (iv).

Assume that A contains an odd hole submatrix H. By Remark 2.3, the vector $x = (\frac{1}{2}, \ldots, \frac{1}{2})$ is the unique solution of the system $Hx = \mathbf{1} - n(H)$. This proves all three reverse implications. □

2.1 Total Dual Integrality

A system of linear constraints is *totally dual integral* (TDI) if, for each integral objective function vector c, the dual linear program has an integral optimal solution (if an optimal solution exists). Edmonds and Giles [38] proved that, if a linear system $Ax \leq b$ is TDI and b is integral, then $\{x : Ax \leq b\}$ is an integral polyhedron.

Theorem 2.6 (Fulkerson, Hoffman, Oppenheim [41]) *Let* $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ *be a balanced* $0, 1$ *matrix. Then the following linear systemis TDI:*

$$\begin{aligned} A_1 x &\geq \mathbf{1} \\ A_2 x &\leq \mathbf{1} \\ A_3 x &= \mathbf{1} \\ x &\geq 0. \end{aligned} \tag{1}$$

Theorem 2.6 and the Edmonds-Giles theorem imply Theorem 2.1. In this section, we prove the following more general result.

Theorem 2.7 (Conforti, Cornuéjols [17]) *Let* $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ *be a balanced* $0, \pm 1$ *matrix. Then the following linear system is TDI:*

$$\begin{aligned} A_1 x &\geq \mathbf{1} - n(A_1) \\ A_2 x &\leq \mathbf{1} - n(A_2) \\ A_3 x &= \mathbf{1} - n(A_3) \\ 0 \leq x &\leq \mathbf{1}. \end{aligned} \tag{2}$$

The following transformation of a $0, \pm 1$ matrix A into a $0, 1$ matrix B is often seen in the literature: to every column a_j of A, $j = 1, \ldots, p$, associate two columns of B, say b_j^P and b_j^N, where $b_{ij}^P = 1$ if $a_{ij} = 1$, 0 otherwise, and $b_{ij}^N = 1$ if $a_{ij} = -1$, 0 otherwise. Let D be the $0, 1$ matrix with p rows and $2p$ columns d_j^P and d_j^N such that $d_{jj}^P = d_{jj}^N = 1$ and $d_{ij}^P = d_{ij}^N = 0$ for $i \neq j$.

Given a $0, \pm 1$ matrix $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ and the associated $0, 1$ matrix $B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$, define the following two linear systems:

$$\begin{aligned} A_1 x &\geq \mathbf{1} - n(A_1) \\ A_2 x &\leq \mathbf{1} - n(A_2) \\ A_3 x &= \mathbf{1} - n(A_3) \\ 0 \leq x &\leq \mathbf{1}, \end{aligned} \tag{3}$$

$$\begin{aligned} B_1 y &\geq \mathbf{1} \\ B_2 y &\leq \mathbf{1} \\ B_3 y &= \mathbf{1} \\ D y &= \mathbf{1} \\ y &\geq 0. \end{aligned} \tag{4}$$

A vector $x \in R^p$ satisfies (3) if and only if the vector $(y^P, y^N) = (x, \mathbf{1} - x)$ satisfies (4) and this transformation maps integer vectors into integer vectors. Hence the polytope defined by (3) is integral if and only if the polytope defined by (4) is integral. We show that, if A is a balanced $0, \pm 1$ matrix, then both (3) and (4) are TDI.

Lemma 2.8 *If* $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ *is a balanced* $0, \pm 1$ *matrix, the corresponding system (4) is TDI.*

Proof: The proof is by induction on the number m of rows of B. Let $c = (c^P, c^N) \in Z^{2p}$ denote an integral vector and R_1, R_2, R_3 the index sets of the rows of B_1, B_2, B_3 respectively. The dual of min $\{cy : \ y$ satisfies $(4)\}$ is the linear program

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} u_i + \sum_{j=1}^{p} v_j \\
& uB + vD \quad \leq c \\
& u_i \geq 0,\ i \in R_1 \\
& u_i \leq 0,\ i \in R_2.
\end{aligned} \tag{5}
$$

Since v_j only appears in two of the constraints $uB + vD \leq c$ and no constraint contains v_j and v_k, it follows that any optimal solution to (5) satisfies

$$v_j = \ \min\,(c_j^P - \sum_{i=1}^{m} b_{ij}^P u_i,\ c_j^N - \sum_{i=1}^{m} b_{ij}^N u_i). \tag{6}$$

Let $(\bar{u}, \bar{v})$ be an optimal solution of (5). If $\bar{u}$ is integral, then so is $\bar{v}$ by (6), and we are done. So assume that $\bar{u}_\ell$ is fractional. Let b^ℓ be the corresponding row of B, and let B_ℓ be the matrix obtained from B by removing row b^ℓ. By induction on the number of rows of B, the system (4) associated with B_ℓ is TDI. Hence the system

$$
\begin{aligned}
\max \quad & \sum_{i \neq \ell} u_i + \sum_{j=1}^{p} v_j \\
& u_\ell B_\ell + vD \quad \leq c - \lfloor \bar{u}_\ell \rfloor b^\ell \\
& u_i \geq 0, i \in R_1 \backslash \{\ell\} \\
& u_i \leq 0, i \in R_2 \backslash \{\ell\}
\end{aligned} \tag{7}
$$

has an integral optimal solution $(\tilde{u}, \tilde{v})$.

Since $(\bar{u}_1, \ldots, \bar{u}_{\ell-1}, \bar{u}_{\ell+1}, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_p)$ is a feasible solution to (7) and Theorem 2.5 shows that $\sum_{i=1}^{m} \bar{u}_i + \sum_{j=1}^{p} \bar{v}_j$ is an integer number,

$$\sum_{i \neq \ell} \tilde{u}_i + \sum_{j=1}^{p} \tilde{v}_j \geq \lceil \sum_{i \neq \ell} \bar{u}_i + \sum_{j=1}^{p} \bar{v}_j \rceil = \sum_{i=1}^{m} \bar{u}_i + \sum_{j=1}^{p} \bar{v}_j - \lfloor \bar{u}_\ell \rfloor.$$

Therefore the vector $(u^*, v^*) = (\tilde{u}_1, \ldots, \tilde{u}_{\ell-1}, \lfloor \bar{u}_\ell \rfloor, \tilde{u}_{\ell+1}, \ldots, \tilde{u}_m, \tilde{v}_1, \ldots, \tilde{v}_p)$ is integral, is feasible to (5) and has an objective function value not smaller than $(\bar{u}, \bar{v})$, proving that the system (4) is TDI. $\square$

Proof of Theorem 2.7: Let R_1, R_2, R_3 be the index sets of the rows of A_1, A_2, A_3. By Lemma 2.8, the linear system (4) associated with (3) is TDI. Let $d \in R^p$ be any integral vector. The dual of $\min\{dx : \ x \text{ satisfies } (3)\}$ is the linear program

$$\begin{aligned} \max \quad & w(\mathbf{1} - n(A)) - t\mathbf{1} \\ & wA - t \ \leq d \\ & w_i \geq 0, i \in R_1 \\ & w_i \leq 0, i \in R_2 \\ & t \geq 0. \end{aligned} \tag{8}$$

For every feasible solution $(\bar{u}, \bar{v})$ of (5) with $c = (c^P, c^N) = (d, 0)$, we construct a feasible solution $(\bar{w}, \bar{t})$ of (8) with the same objective function value as follows:

$$\begin{aligned} \bar{w} &= \bar{u} \\ \bar{t}_j &= \begin{cases} 0 & if \quad \bar{v}_j = -\sum_i b^N_{ij}\bar{u}_i \\ \sum_i b^P_{ij}\bar{u}_i - \sum_i b^N_{ij}\bar{u}_i - d_j & if \quad \bar{v}_j = d_j - \sum_i b^P_{ij}\bar{u}_i. \end{cases} \end{aligned} \tag{9}$$

When the vector $(\bar{u}, \bar{v})$ is integral, the above transformation yields an integral vector $(\bar{w}, \bar{t})$. Therefore (8) has an integral optimal solution and the linear system (3) is TDI. □

This theorem does not hold when the upper bound $x \leq \mathbf{1}$ is dropped from the linear system as shown by the example given after Theorem 2.5.

3 Colorings and Hypergraphs

3.1 Bicolorings

A *k-coloring* of a matrix A is a partition of columns of A into k sets or "colors" (some of them may be empty). In this section we consider 2-colorings.

Berge [2] introduced the following notion. A $0, 1$ matrix is *bicolorable* if its columns can be 2-colored into blue and red in such a way that every row with two or more 1*s* contains a 1 in a blue column and a 1 in a red column. Equivalently, for no row with at least two 1*s* all the 1*s* have the same color. This notion provides the following characterization of balanced $0, 1$ matrices.

Theorem 3.1 (Berge [2]) *A* $0,1$ *matrix* A *is balanced if and only if every submatrix of* A *is bicolorable.*

Ghouila-Houri [43] introduced the notion of *equitable bicoloring* for a $0,\pm1$ matrix A as follows. The columns of A are 2-colored into blue columns and red columns in such a way that, for every row of A, the sum of the entries in the blue columns differs from the sum of the entries in the red columns by at most one.

Theorem 3.2 (Ghouila-Houri [43]) *A* $0,\pm1$ *matrix* A *is totally unimodular if and only if every submatrix of* A *has an equitable bicoloring.*

This theorem generalizes a result of Heller and Tompkins [50] for matrices with at most two nonzero entries per row.

A $0,\pm1$ matrix A is *bicolorable* if its columns can be 2-colored into blue columns and red columns in such a way that every row with two or more nonzero entries either contains two entries of opposite sign in columns of the same color, or contains two entries of the same sign in columns of different colors. Equivalently, for no row with at least two nonzero entries all the $1s$ have the same color, say blue, and all the -1's are red. For a $0,1$ matrix, this definition coincides with Berge's notion of bicoloring. Clearly, if a $0,\pm1$ matrix has an equitable bicoloring as defined by Ghouila-Houri, then it is bicolorable. So the theorem below implies that every totally unimodular matrix is balanced.

Theorem 3.3 (Conforti, Cornuéjols [17]) *A* $0,\pm1$ *matrix* A *is balanced if and only if every submatrix of* A *is bicolorable.*

Proof: Assume first that A is balanced and let B be any submatrix of A. Remove from B any row with fewer than two nonzero entries. Since B is balanced, so is the matrix $(B,-B)$. It follows from Theorem 2.5 that the inequalities

$$\begin{aligned} Bx &\geq \mathbf{1}-n(B) \\ -Bx &\geq \mathbf{1}-n(-B) \\ 0\leq x &\leq \mathbf{1} \end{aligned} \tag{10}$$

define an integral polytope. Since it is nonempty (the vector $(\frac{1}{2}, \ldots, \frac{1}{2})$ is a solution), it contains a 0,1 vector $\bar{x}$. Color a column j of B red if $\bar{x}_j = 1$ and blue otherwise. By (10), this is a valid bicoloring of B.

Conversely, assume that A contains an odd hole matrix H. We claim that H is not bicolorable. Suppose otherwise. Since H contains exactly 2 nonzero entries per row, the bicoloring condition shows that the vector of all zeroes can be obtained by adding the blue columns and subtracting the red columns. So H is singular, a contradiction to Remark 2.3. □

In Section 4.1, we prove a bicoloring theorem that extends all the above results (Theorem 4.3).

Cameron and Edmonds [10] showed that the following simple algorithm finds a bicoloring of a balanced matrix.

Algorithm (Cameron and Edmonds [10])

Input: A $0, \pm 1$ matrix A.

Output: A bicoloring of A or a proof that the matrix A is not balanced.

Stop if all columns are colored or if some row is incorrectly colored. Otherwise, color a new column red or blue as follows.

If some row of A forces the color of a column, color this column accordingly.

If no row of A forces the color of a column, arbitrarily color one of the uncolored columns.

In the above algorithm, a row a^i *forces* the color of a column when all the columns corresponding to the nonzero entries of a^i have been colored except one, say column k, and row a^i, restricted to the colored columns, violates the bicoloring condition. In this case, the bicoloring rule dictates the color of column k.

When the algorithm fails to find a bicoloring, the sequence of forcings that resulted in an incorrectly colored row identifies an odd hole submatrix of A.

Note that a matrix A may be bicolorable even if A is not balanced. In fact, the algorithm may find a bicoloring of A even if A is not balanced. For example, if $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$, the algorithm may color the first two columns blue and the last two red, which is a bicoloring of A. For this reason, the algorithm cannot be used as a recognition of balancedness.

3.2 k-Colorings

A $0, 1$ matrix A is *k-colorable* if there exists a k-coloring of its columns such that for every row i that has at least two 1s in colors $J \cup L$ there are entries $a_{ij} = a_{il} = 1$ where $j \in J$ and $l \in L$. This is equivalent to saying that every pair of colors J, L constitutes a bicoloring (as defined in the previous section) of the submatrix A_{JL} of A, induced by columns $J \cup L$.

Theorem 3.4 (Berge [4]) *A $0, 1$ matrix A is balanced if and only if every submatrix of A is k-colorable for every k.*

Proof: The "if" part follows from Theorem 3.1. We now show that if every column submatrix of A is bicolorable, then A is k-colorable for every k. By Theorem 3.1 this proves the result. For a given k-coloring of A, let $r(i)$ be the number of colors that are represented in row i, i.e. the number of colors J for which $a_{ij} = 1$ for some $j \in J$. Consider a k-coloring of A such that the sum of $r(i)$ over all rows i of A is maximized. Suppose that this k-coloring of A does not satisfy the above definition. Then there are colors J, L that do not give a bicoloring of the matrix A_{JL}. Let J', L' be a bicoloring of A_{JL}, and consider a new k-coloring of A where J and L are replaced by J' and L' and all the other colors stay the same. In this new coloring the sum of $r(i)$ over all rows i of A has increased, in comparison to the original one, a contradiction. $\square$

The above proof shows that if A is a balanced matrix one can efficiently construct a k-coloring of A, that satisfies the above condition, using the algorithm of Cameron and Edmonds.

Similarly the notion of equitable bicoloring is extended to the notion of equitable k-coloring. A k-coloring of a $0, \pm 1$ matrix A is *equitable* if every pair of colors J, L constitutes an equitable bicoloring of A_{JL}. A similar argument as in the proof above, gives the following result.

Theorem 3.5 (de Werra [36]) *A $0, \pm 1$ matrix A is totally unimodular if and only if every submatrix of A has an equitable k-coloring for every k.*

A $0,\pm 1$ matrix A is *k-colorable* if there exists a k-coloring of its columns so that every pair of colors J, L constitutes a bicoloring of A_{JL}.

Conjecture 3.6 (Conforti and Zambelli) *A $0,\pm 1$ matrix A is balanced if and only if every submatrix of A is k-colorable for every k.*

For $k = 2$ the conjecture is equivalent to Theorem 3.3. This conjecture is open for $k = 3$. Note that the conjecture holds for every totally unimodular matrix A since every equitable k-coloring of A is a k-coloring that satisfies the above condition. De Werra [37] gives a weaker notion of k-colorability of a $0,\pm 1$ matrix and proves that balanced matrices satisfy it.

3.3 Balanced Hypergraphs

A $0,1$ matrix A can be represented by a hypergraph. Then the definition of balancedness for $0,1$ matrices is a natural extension of the property of not containing odd cycles for graphs. In fact, this is the motivation that led Berge [2] to introduce the notion of balancedness: A hypergraph $\mathcal{H} = (V, \mathcal{E})$, where V represents the column set and $\mathcal{E}$ represents the row set of A, is *balanced* if every odd cycle C of $\mathcal{H}$ has an edge containing at least three nodes of C. Equivalently, $\mathcal{H}$ is balanced if the associated $0,1$ matrix A is balanced. We refer to Berge [6] for an introduction to the theory of hypergraphs. Several results on bipartite graphs generalize to balanced hypergraphs, such as König's bipartite matching theorem, as stated in the next theorem. In a hypergraph, a *matching* is a set of pairwise nonintersecting edges and a *transversal* is a node set intersecting all the edges.

Theorem 3.7 (Berge, Las Vergnas [7]) *In a balanced hypergraph, the maximum cardinality of a matching equals the minimum cardinality of a transversal.*

Proof: Follows from Theorem 2.6 applied with $A_1 = A_3 = \emptyset$ and the primal objective function $\max \sum_j x_j$. $\square$

The next result generalizes a theorem of Gupta [47] on bipartite multigraphs.

Theorem 3.8 (Berge [5]) *In a balanced hypergraph $\mathcal{H} = (V, \mathcal{E})$, the minimum number of nodes in an edge equals the maximum cardinality of a family of disjoint transversals.*

Proof: Let ϵ_{min} be the minimum cardinality of an edge in $\mathcal{H}$, and let A be the incidence matrix of $\mathcal{H}$. Since A is balanced, by Theorem 3.4, A is ϵ_{min}-colorable and this coloring induces a partition of V in ϵ_{min} colors. Let J be a color. We show that J is a transversal of $\mathcal{H}$. Assume not; then there is an edge e that does not contain any node colored J. Since $|e| \geq \epsilon_{min}$, there exists a color, say L, that contains at least two nodes of e. This shows that the submatrix A_{JL} is not bicolored, a contradiction. □

The *chromatic number* of a hypergraph is the minimum number of colors needed to color its nodes so that no edge contains two nodes of the same color.

Theorem 3.9 (Berge [5]) *In a balanced hypergraph $\mathcal{H} = (V, \mathcal{E})$, the maximum number of nodes in an edge equals the chromatic number of $\mathcal{H}$.*

Proof: Let ϵ_{max} be the maximum number of nodes in an edge of $\mathcal{H}$, and let A be the incidence matrix of $\mathcal{H}$. Since A is balanced, it is ϵ_{max}-colorable by Theorem 3.4. By the same argument as before, such a coloring provides a coloring of $\mathcal{H}$. □

One of the first results on matchings in graphs is the following celebrated theorem of Hall.

Theorem 3.10 (Hall [49]) *A bipartite graph has no perfect matching if and only if there exist disjoint node sets R and B such that $|B| > |R|$ and every edge having one endnode in B has the other in R.*

The following result generalizes Hall's theorem to balanced hypergraphs.

Theorem 3.11 (Conforti, Cornuéjols, Kapoor, Vušković [20]) *A balanced hypergraph $\mathcal{H} = (V, \mathcal{E})$ has no perfect matching if and only if there exist disjoint node sets R and B such that $|B| > |R|$ and every edge contains at least as many nodes in R as in B.*

We give a short polyhedral proof of Theorem 3.11, due to Schrijver [65]. Huck and Triesh [55] give a combinatorial proof.

Proof: Assume $\mathcal{H}$ admits a perfect matching M. Then for every disjoint subsets R, B of V such that $|B \cap e| \leq |R \cap e|$ for every $e \in \mathcal{E}$, we have:

$$|B| = \sum_{e \in M} |B \cap e| \leq \sum_{e \in M} |R \cap e| = |R|.$$

So the condition is necessary.

We prove sufficiency: Assume $\mathcal{H}$ admits no perfect matching and let A be the node-edge incidence matrix of $\mathcal{H}$. Then by Theorem 2.1, the system $Ay = \mathbf{1},\ y \geq 0$ defines an integral polytope. Therefore, since $\mathcal{H}$ has no perfect matching, this system has no solution. Hence, by Farkas' lemma, there is a vector x such that $A^T x \geq 0$ and $\mathbf{1}^T x < 0$. We can assume $-\mathbf{1} \leq x \leq \mathbf{1}$. Let $z = \mathbf{1} - x$. Then $0 \leq z \leq \mathbf{2}$, $A^T z \leq A^T \mathbf{1}$ and $\mathbf{1}^T z > \mathbf{1}^T \mathbf{1} = |V|$. Consider the linear program:

$$\begin{array}{ll} min & (A^T \mathbf{1})^T u + \mathbf{2}^T v \\ & Au + Iv \geq \mathbf{1} \\ & u, v \geq 0. \end{array}$$

By Theorem 2.6 its constraints form a TDI system. Since the system satisfied by z corresponds to the dual of the above linear program, it follows that it has an integral solution z. So there is an integral vector x such that $A^T x \geq 0,\ \mathbf{1}^T x < 0,\ -\mathbf{1} \leq x \leq \mathbf{1}$. Now set $B = \{v \in V | x_v = -1\}$ and $R = \{v \in V | x_v = 1\}$. Then B, R satisfy the conditions of the theorem. □

It is well known that a bipartite graph with maximum degree Δ contains Δ edge-disjoint matchings. The same property holds for balanced hypergraphs. The following result is equivalent to Theorem 3.9. We provide a proof based on Theorem 3.11.

Corollary 3.12 *The edges of a balanced hypergraph $\mathcal{H}$ with maximum degree Δ can be partitioned into Δ matchings.*

Proof: By adding edges containing a unique node, we can assume that $\mathcal{H}$ is Δ-regular. (This operation does not destroy the property of being balanced). We now show that $\mathcal{H}$ has a perfect matching. Assume not. By Theorem 3.11, there exist disjoint node sets R and B such that $|B| > |R|$ and $|R \cap e| \geq |B \cap e|$ for every edge e of $\mathcal{H}$. Adding these inequalities over all edges, we get $|R| \geq |B|$ since $\mathcal{H}$ is Δ-regular, a contradiction. So $\mathcal{H}$ contains a perfect matching M. Removing the edges of M, the result now follows by induction. □

3.4 2-Section Graphs and Clique-Hypergraphs

The main result of this section was found by Prisner [63].

The *2-section graph* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is the simple undirected graph $G = (V, E)$ having the same node set as $\mathcal{H}$; two of its nodes are adjacent if and only if they belong to the same edge of $\mathcal{H}$.

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a *clique-hypergraph* if $\mathcal{E}$ is the family of all the maximal cliques of its 2-section graph G. Obviously, if $\mathcal{H}$ is a clique-hypergraph, $\mathcal{H}$ does not contain any repeated or dominated edges. In [48] an algorithm is given, to list the set $\mathcal{K}$ of all maximal cliques of a graph $G = (V, E)$. Its running time is $O(|V| \times |E| \times |\mathcal{K}|)$. So the clique-hypergraph of a graph G can be efficiently constructed.

Lemma 3.13 *A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a clique-hypergraph if and only if $\mathcal{H}$ contains no dominated or repeated edge, and for every triple of edges, say e_1, e_2, e_3, the set of nodes $V_{123} = (e_1 \cap e_2) \cup (e_2 \cap e_3) \cup (e_1 \cap e_3)$ is contained in some edge of $\mathcal{H}$.*

Proof: Let G be the 2-section graph of $\mathcal{H}$. Since V_{123} is contained in a clique of G, the condition is obviously necessary. We now prove sufficiency. If $\mathcal{H}$ is not a clique-hypergraph, then some set of nodes pairwise adjacent in G is not contained in and edge of $\mathcal{H}$; let V' be a minimal such set. Clearly $|V'| \geq 3$. By the minimality of V', for every $v \in V'$, the set $V' \backslash v$ is contained in an edge e_v of $\mathcal{H}$. Assume $\{v_1, v_2, v_3\} \subseteq V'$. Now $V' \subseteq (e_{v_1} \cap e_{v_2}) \cup (e_{v_1} \cap e_{v_3}) \cup (e_{v_2} \cap e_{v_3})$ and $e_{v_1}, e_{v_2}, e_{v_3}$ satisfy the above conditions. □

Let us define a hypergraph to be *semi-balanced* if its incidence matrix contains no 3×3 hole matrix. Balanced hypergraphs are obviously semi-balanced.

Given hypergraph $\mathcal{H} = (V, \mathcal{E})$, let $\mathcal{E}_{max}$ be the subset of $\mathcal{E}$ consisting of one copy of every maximal edge of $\mathcal{H}$, and let $\mathcal{H}_{max} = (V, \mathcal{E}_{max})$.

Lemma 3.14 *Let $\mathcal{H} = (V, \mathcal{E})$ be a semi-balanced hypergraph. Then $\mathcal{H}_{max} = (V, \mathcal{E}_{max})$ is a clique-hypergraph.*

Proof: By construction, $\mathcal{H}_{max}$ contains no dominated or repeated edge. So assume $\mathcal{H}_{max}$ is not a clique-hypergraph. By Lemma 3.13, $\mathcal{H}_{max}$ contains edges e_1, e_2, e_3 such that the set V_{123} is not contained in any other edge of $\mathcal{H}_{max}$. In particular, there exist nodes $v_{12} \in (e_1 \cap e_2) \setminus e_3$ and v_{13}, v_{23} similarly defined. Let A be the incidence matrix of $\mathcal{H}$. Now the rows and columns of A corresponding to v_{12}, v_{13}, v_{23} and e_1, e_2, e_3 induce a 3×3 hole matrix. $\square$

Lemma 3.15 *Let $\mathcal{H} = (V, \mathcal{E})$ be a semi-balanced hypergraph not containing any repeated edges. Then every edge of $\mathcal{H}_{max}$ contains two vertices that do not belong to any other edge of $\mathcal{H}$.*

Proof: Obviously $\mathcal{H}$ and $\mathcal{H}_{max}$ have the same 2-section graph G. Furthermore, since $\mathcal{H}$ is semi-balanced, so is $\mathcal{H}_{max}$. So by Lemma 3.14, $\mathcal{H}_{max}$ is the clique-hypergraph of G. Assume the lemma is false, and let $e \in \mathcal{E}_{max}$ be an edge violating the above condition. Obviously, e contains at least three nodes. Since every pair of nodes in e belong to some other edge of $\mathcal{E}_{max}$, G is also the 2-section graph of the hypergraph $\mathcal{H}_{max} \setminus e$. However, since e is missing, $\mathcal{H}_{max} \setminus e$ is not the clique-hypergraph of G. By Lemma 3.14, $\mathcal{H}_{max} \setminus e$ is not semi-balanced and hence both $\mathcal{H}_{max}, \mathcal{H}$, are not semi-balanced, a contradiction. $\square$

Corollary 3.16 (Prisner [63]) *Let $\mathcal{H}$ be a balanced hypergraph that is the clique-hypergraph of G. Then the number of edges of $\mathcal{H}$ is bounded by the number of edges of G.*

Proof: By Lemma 3.15, every edge of $\mathcal{H}$ contains an edge of G that belongs to no other edge of $\mathcal{H}$. $\square$

4 Related Integer Polytopes

4.1 k-Balanced Matrices

We introduce a hierarchy of balanced $0, \pm 1$ matrices that contains as its two extreme cases the balanced and totally unimodular matrices. The following well known result of Camion will be used.

A $0, \pm 1$ matrix which is not totally unimodular but whose proper submatrices are all totally unimodular is said to be *almost totally unimodular*. Camion [12] proved the following:

Theorem 4.1 (Camion [12] and Gomory [cited in [12]]) *Let A be an almost totally unimodular $0, \pm 1$ matrix. Then A is square, $\det A = \pm 2$ and A^{-1} has only $\pm\frac{1}{2}$ entries. Furthermore, each row and each column of A has an even number of nonzero entries and the sum of all entries in A equals 2 mod 4.*

Proof: Clearly A is square, say $n \times n$. If $n = 2$, then indeed, $\det A = \pm 2$. Now assume $n \geq 3$. Since A is nonsingular, it contains an $(n-2) \times (n-2)$ nonsingular submatrix B. Let $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ and $U = \begin{pmatrix} B^{-1} & 0 \\ -DB^{-1} & I \end{pmatrix}$. Then $\det U = \pm 1$ and $UA = \begin{pmatrix} I & B^{-1}C \\ 0 & E - DB^{-1}C \end{pmatrix}$. We claim that the 2×2 matrix $E - DB^{-1}C$ has all entries equal to $0, \pm 1$. Suppose to the contrary that $E - DB^{-1}C$ has an entry different from $0, \pm 1$ in row i and column j. Denoting the corresponding entry of E by e_{ij}, the corresponding column of C by c^j and row of D by d^i,

$$\begin{pmatrix} B^{-1} & 0 \\ -d^iB^{-1} & 1 \end{pmatrix} \begin{pmatrix} B & c^j \\ d^i & e_{ij} \end{pmatrix} = \begin{pmatrix} I & B^{-1}c^j \\ 0 & e_{ij} - d^iB^{-1}c^j \end{pmatrix}$$

and consequently A has an $(n-1) \times (n-1)$ submatrix with a determinant different from $0, \pm 1$, a contradiction.

Consequently, $\det A = \pm \det UA = \pm \det(E - DB^{-1}C) = \pm 2$.

So, every entry of A^{-1} is equal to $0, \pm\frac{1}{2}$. Suppose A^{-1} has an entry equal to 0, say in row i and column j. Let $\bar{A}$ be the matrix obtained from A by removing column i and let h^j be the j^{th} column of A^{-1} with row i removed. Then $\bar{A}h^j = u^j$, where u^j denotes the j^{th} unit vector. Since $\bar{A}$ has rank $n-1$, this linear system of equations has a unique solution h^j. Since $\bar{A}$ is totally unimodular and u^j is integral, this solution h^j is integral. Since $h^j \neq 0$, this contradicts the fact that every entry of h^j is equal to $0, \pm\frac{1}{2}$. So A^{-1} has only $\pm\frac{1}{2}$ entries.

This property and the fact that AA^{-1} and $A^{-1}A$ are integral, imply that A has an even number of nonzero entries in each row and column.

Finally, let α denote a column of A^{-1} and $S = \{i : \alpha_i = +\frac{1}{2}\}$ and $\bar{S} = \{i : \alpha_i = -\frac{1}{2}\}$. Let k denote the sum of all entries in the columns of A

indexed by $\bar{S}$. Since $A\alpha$ is a unit vector, the sum of all entries in the columns of A indexed by S equals $k+2$. Since every column of A has an even number of nonzero entries, k is even, say $k = 2p$ for some integer p. Therefore, the sum of all entries in A equals $4p+2$. $\square$

For any positive integer k, we say that a $0, \pm 1$ matrix A is *k-balanced* if A does not contain any almost totally unimodular submatrix with at most $2k$ nonzero entries in each row. Truemper [70] gives a construction of all the minimal matrices that are not k-balanced.

Note that every almost totally unimodular matrix contains at least 2 nonzero entries per row and per column. So the odd hole matrices are the almost totally unimodular matrices with at most 2 nonzero entries per row. Therefore the balanced matrices are the 1-balanced matrices and the totally unimodular matrices with n columns are the k-balanced matrices for $k \geq \lfloor n/2 \rfloor$. The class of k-balanced matrices was introduced by Truemper and Chandrasekaran [72] for 0,1 matrices and by Conforti, Cornuéjols and Truemper [24] for $0, \pm 1$ matrices. Let $\mathbf{k}$ denote a column vector whose entries are all equal to k.

Theorem 4.2 (Conforti, Cornuéjols and Truemper [24]) *Let A be an $m \times n$ k-balanced $0, \pm 1$ matrix with rows a^i, $i \in [m]$, b be a vector with entries b_i, $i \in [m]$, and let S_1, S_2, S_3 be a partition of $[m]$. Then*

$$P(A,b) = \{x \in R^n \; : \; \begin{array}{l} a^i x \leq b_i \text{ for } i \in S_1 \\ a^i x = b_i \text{ for } i \in S_2 \\ a^i x \geq b_i \text{ for } i \in S_3 \\ 0 \leq x \leq \mathbf{1}\} \end{array}$$

is an integral polytope for all integral vectors b such that $-n(A) \leq b \leq \mathbf{k} - n(A)$.

Proof: Assume the contrary and let A be a k-balanced matrix of smallest order such that $P(A,b)$ has a fractional vertex $\bar{x}$ for some vector b such that $-n(A) \leq b \leq \mathbf{k} - n(A)$ and some partition S_1, S_2, S_3 of $[m]$. Then by the minimality of A, $\bar{x}$ satisfies all the constraints in $S_1 \cup S_2 \cup S_3$ at equality. So we may assume $S_1 = S_3 = \emptyset$. Furthermore all the components of x are fractional, otherwise let A^f be the column submatrix of A corresponding to the fractional components of $\bar{x}$ and A^p be the column submatrix of A corresponding to the components of $\bar{x}$ that are equal to 1. Let $b^f = b - p(A^p) + n(A^p)$. Then $-n(A^f) \leq b^f \leq \mathbf{k} - n(A^f)$ since $b^f = b - p(A^p) + n(A^p) = A^f\bar{x} \geq n(A^f)$ and because $b^f = b - p(A^p) + n(A^p) \leq b + n(A^p) \leq k - n(A) + n(A^p) \leq k - n(A^f)$.

Since the restriction of $\bar{x}$ to is fractional components is a vertex of $P(A^f, b^f)$ with $S_1 = S_3 = \emptyset$, the minimality of A is contradicted. So A is a square nonsingular matrix which is not totally unimodular. Let G be an almost totally unimodular submatrix of A. Since A is not k-balanced, G contains a row i such that $p_i(G) + n_i(G) > 2k$. Let A^i be the submatrix of A obtained by removing row i and let b^i be the corresponding subvector of b. By the minimality of A, $P(A^i, b^i)$ with $S_1 = S_3 = \emptyset$ is an integral polytope and since A is nonsingular, $P(A^i, b^i)$ has exactly two vertices, say z^1 and z^2. Since $\bar{x}$ is a vector whose components are all fractional and $\bar{x}$ can be written as the convex combination of the $0,1$ vectors z^1 and z^2, then $z^1 + z^2 = \mathbf{1}$. For $\ell = 1, 2$, define

$$L(\ell) = \{j;\ \textit{either } g_{ij} = 1 \textit{ and } z_i^\ell = 1 \textit{ or } g_{ij} = -1 \textit{ and } z_i^\ell = 0\}.$$

Since $z^1 + z^2 = \mathbf{1}$, it follows that $|L(1)| + |L(2)| = p_i(G) + n_i(G) > 2k$. Assume w.l.o.g. that $|L(1)| > k$. Now this contradicts

$$|L(1)| = \sum_j g_{ij} z_j^1 + n_i(G) \leq b_i + n_i(A) \leq k$$

where the first inequality follows from $A^i z^1 = b^i$. □

This theorem generalizes previous results by Hoffman and Kruskal [51] for totally unimodular matrices, Berge [3] for $0,1$ balanced matrices, Conforti and Cornuéjols [17] for $0, \pm 1$ balanced matrices, and Truemper and Chandrasekaran [72] for k-balanced 0,1 matrices.

A $0, \pm 1$ matrix A has a *k-equitable bicoloring* if its columns can be partitioned into blue columns and red columns so that:

- the bicoloring is equitable for the row submatrix A' determined by the rows of A with at most $2k$ nonzero entries,
- every row with more than $2k$ nonzero entries contains k pairwise disjoint pairs of nonzero entries such that each pair contains either entries of

opposite sign in columns of the same color or entries of the same sign in columns of different colors.

Obviously, an $m \times n$ $0, \pm 1$ matrix A is bicolorable if and only if A has a 1-equitable bicoloring, while A has an equitable bicoloring if and only if A has a k-equitable bicoloring for $k \geq \lfloor n/2 \rfloor$. The following theorem provides a new characterization of the class of k-balanced matrices, which generalizes the bicoloring results of Section 3.1 for balanced and totally unimodular matrices.

Theorem 4.3 (Conforti, Cornuéjols and Zambelli [26]) *A $0, \pm 1$ matrix A is k-balanced if and only if every submatrix of A has a k-equitable bicoloring.*

Proof: Assume first that A is k-balanced and let B be any submatrix of A. Assume, up to row permutation, that

$$B = \begin{pmatrix} B' \\ B'' \end{pmatrix}$$

where B' is the row submatrix of B determined by the rows of B with $2k$ or fewer nonzero entries. Consider the system

$$\begin{array}{rcl} B'x & \geq & \left\lfloor \dfrac{B'\mathbf{1}}{2} \right\rfloor \\ -B'x & \geq & -\left\lceil \dfrac{B'\mathbf{1}}{2} \right\rceil \\ B''x & \geq & \mathbf{k} - n(B'') \\ -B''x & \geq & \mathbf{k} - n(-B'') \\ 0 \leq x & \leq & \mathbf{1} \end{array} \tag{11}$$

Since B is k-balanced, also $\begin{pmatrix} B \\ -B \end{pmatrix}$ is k-balanced. Therefore the constraint matrix of system (11) above is k-balanced. One can readily verify that $-n(B') \leq \left\lfloor \frac{B'\mathbf{1}}{2} \right\rfloor \leq \mathbf{k} - n(B')$ and $-n(-B') \leq -\left\lceil \frac{B'\mathbf{1}}{2} \right\rceil \leq \mathbf{k} - n(-B')$. Therefore, by Theorem 4.2 applied with $S_1 = S_2 = \emptyset$, system (11) defines an integral polytope. Since the vector $(\frac{1}{2}, ..., \frac{1}{2})$ is a solution for (11), the polytope is nonempty and contains a $0, 1$ point $\bar{x}$. Color a column i of B

blue if $\bar{x}_i = 1$, red otherwise. It can be easily verified that such a bicoloring is, in fact, k-equitable.

Conversely, assume that A is not k-balanced. Then A contains an almost totally unimodular matrix B with at most $2k$ nonzero elements per row. Suppose that B has a k-equitable bicoloring, then such a bicoloring must be equitable since each row has, at most, $2k$ nonzero elements. By Theorem 4.1, B has an even number of nonzero elements in each row. Therefore the sum of the columns colored blue equals the sum of the columns colored red, therefore B is a singular matrix, a contradiction. $\square$

Given a $0, \pm 1$ matrix A and positive integer k, one can find in polynomial time a k-equitable bicoloring of A or a certificate that A is not k-balanced as follows:

Find a basic feasible solution of (11). If the solution is not integral, A is not k-balanced by Theorem 4.2. If the solution is a 0,1 vector, it yields a k-equitable bicoloring as in the proof of Theorem 4.3.

Note that, as with the algorithm of Cameron and Edmonds [10] discussed in Section 3.1, a 0,1 vector may be found even when the matrix A is not k-balanced.

Using the fact that the vector $(\frac{1}{2}, ..., \frac{1}{2})$ is a feasible solution of (11), a basic feasible solution of (11) can actually be derived in strongly polynomial time using an algorithm of Megiddo [59].

4.2 Perfect and Ideal $0, \pm 1$ Matrices

A $0,1$ matrix A is said to be *perfect* if the set packing polytope $P(A)$ is integral. A $0,1$ matrix A is *ideal* if the set covering polytope $Q(A)$ is integral. The study of perfect and ideal $0,1$ matrices is a central topic in polyhedral combinatorics. Theorem 2.1 shows that every balanced $0,1$ matrix is both perfect and ideal.

The integrality of the set packing polytope associated with a $(0,1)$ matrix A is related to the notion of perfect graph. A graph G is *perfect* if, for every induced subgraph H of G, the chromatic number of H equals the size of its largest clique. The fundamental connection between the theory of perfect graphs and integer programming was established by Fulkerson [40], Lovász [57] and Chvátal [14]. The *clique-node matrix* of a graph G is a $0,1$

matrix whose columns are indexed by the nodes of G and whose rows are the incidence vectors of the maximal cliques of G.

Theorem 4.4 (Lovász [57], Fulkerson [40], Chvátal [14]) *Let A be a 0,1 matrix. The set packing polytope $P(A)$ is integral if and only if the rows of A of maximal support form the clique-node matrix of a perfect graph.*

Now we extend the definition of perfect and ideal $0, 1$ matrices to $0, \pm 1$ matrices. A $0, \pm 1$ matrix A is *ideal* if the generalized set covering polytope $Q(A) = \{x : Ax \geq \mathbf{1} - n(A),\ 0 \leq x \leq \mathbf{1}\}$ is integral. A $0, \pm 1$ matrix A is *perfect* if the generalized set packing polytope $P(A) = \{x : Ax \leq \mathbf{1} - n(A),\ 0 \leq x \leq \mathbf{1}\}$ is integral. By Theorem 2.2, balanced $0, \pm 1$ matrices are both perfect and ideal.

Hooker [54] was the first to relate idealness of a $0, \pm 1$ matrix to that of a family of 0,1 matrices. A similar result for perfection was obtained in [19]. These results were strengthened by Guenin [46] and by Boros, Čepek [8] for perfection, and by Nobili, Sassano [61] for idealness. The key tool for these results is the following:

Given a $0, \pm 1$ matrix A, let P and R be $0, 1$ matrices of the same dimension as A, with entries $p_{ij} = 1$ if and only if $a_{ij} = 1$, and $r_{ij} = 1$ if and only if $a_{ij} = -1$. The matrix $D_A = \begin{pmatrix} P & R \\ I & I \end{pmatrix}$ is the $0, 1$ *extension* of A. Note that the transformation $x^+ = x$ and $x^- = \mathbf{1} - x$ maps every vector x in $P(A)$ into a vector in $\{(x^+, x^-) \geq 0 :\ Px^+ + Rx^- \leq \mathbf{1},\ x^+ + x^- = \mathbf{1}\}$ and every vector x in $Q(A)$ into a vector in $\{(x^+, x^-) \geq 0 :\ Px^+ + Rx^- \geq \mathbf{1},\ x^+ + x^- = \mathbf{1}\}$. So $P(A)$ and $Q(A)$ are respectively the faces of $P(D_A)$ and $Q(D_A)$, obtained by setting the inequalites $x^+ + x^- \leq \mathbf{1}$ and $x^+ + x^- \geq \mathbf{1}$ at equality. Thus, if $P(D_A)$ is an integral polytope, then so is $P(A)$. Similarly $Q(D_A)$ integral implies $Q(A)$ integral. To get a converse, we introduce the following notion.

Consider a $0, \pm 1$ matrix A with two rows a^1 and a^2 such that there is one index k such that $a^1_k a^2_k = -1$ and, for all $j \neq k$, $a^1_j a^2_j = 0$. A *disjoint implication* of A is the $0, \pm 1$ vector $a^1 + a^2$. For a $0, \pm 1$ matrix A, the matrix A^+ obtained by recursively adding all disjoint implications and removing all dominated rows (those whose support is not maximal in the packing case; those whose support is not minimal in the covering case) is called the *disjoint completion* of A. Note that $P(A) = P(A^+)$ and $Q(A) = Q(A^+)$.

Theorem 4.5 (Nobili, Sassano [61]) *Let A be a $0, \pm 1$ matrix. Then A is ideal if and only if the $0, 1$ matrix D_{A^+} is ideal.*

Furthermore A is ideal if and only if $\min\{cx : x \in Q(A)\}$ has an integral optimum x for every vector $c \in \{0, \pm 1, \pm\infty\}^n$.

Theorem 4.6 (Guenin [46]) *Let A be a $0, \pm 1$ matrix such that $P(A)$ is not contained in any of the hyperplanes $\{x : x_j = 0\}$ or $\{x : x_j = 1\}$. Then A is perfect if and only if the $0, 1$ matrix D_{A^+} is perfect.*

Note that this result does not hold when the assumption on the hyperplanes $\{x : x_j = 0\}$ and $\{x : x_j = 1\}$ is dropped. For example, consider $A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$. Then $P(A)$ is an integral polytope since it only contains the point 0, whereas $P(D_{A^+})$ is not an integral polytope since $A^+ = A$ and $P(D_A)$ has the fractional vertex (x^+, x^-) where $x^+ = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $x^- = 0$.

Theorem 4.7 (Guenin [46]) *Let A be a $0, \pm 1$ matrix such that $P(A)$ is not contained in any of the hyperplanes $\{x : x_j = 0\}$ or $\{x : x_j = 1\}$. Then A is perfect if and only if $\max\{cx : x \in P(A)\}$ admits an integral optimal solution for every $c \in \{0, \pm 1\}^n$. Moreover, if A is perfect, the linear system $Ax \leq \mathbf{1} - n(A)$, $0 \leq x \leq \mathbf{1}$ is TDI.*

This is the natural extension of the Lovász's theorem for perfect $0, 1$ matrices. The next theorem characterizes perfect $0, \pm 1$ matrices in terms of excluded submatrices. A row of a $0, \pm 1$ matrix A is *trivial* if it contains at most one nonzero entry. Note that trivial rows can be removed without changing $P(A)$.

Theorem 4.8 (Guenin [46]) *Let A be a $0, \pm 1$ matrix such that $P(A)$ is not contained in any of the hyperplanes $\{x : x_j = 0\}$ or $\{x : x_j = 1\}$. Then A is perfect if and only if A^+ does not contain*

1) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$ *as a submatrix, or*

2) a column submatrix which, without its trivial rows, is obtained from a minimally imperfect 0,1 matrix B by switching signs of all entries in a subset of the columns of B.

For ideal $0, \pm 1$ matrices, a similar characterization was obtained in terms of excluded "weak minors" by Nobili and Sassano [61].

4.3 Propositional Logic

In propositional logic, *atomic propositions* $x_1, \ldots, x_j, \ldots, x_n$ can be either *true* or *false.* A *truth assignment* is an assignment of "true" or "false" to every atomic proposition. A *literal* is an atomic proposition x_j or its negation $\neg x_j$. A *clause* is a disjunction of literals and is *satisfied* by a given truth assignment if at least one of its literals is true.

A survey of the connections between propositional logic and integer programming can be found in [53].

A truth assignment satisfies a set of m clauses

$$\bigvee_{j \in P_i} x_j \vee (\bigvee_{j \in N_i} \neg x_j) \quad \text{for } i = 1 \ldots, m$$

if and only if the corresponding $0, 1$ vector satisfies the system of inequalities

$$\sum_{j \in P_i} x_j - \sum_{j \in N_i} x_j \geq 1 - |N_i| \quad \text{for } i = 1 \ldots, m.$$

The above system of inequalities is of the form

$$Ax \geq \mathbf{1} - n(A), \tag{12}$$

where A is an $m \times n$ $0, \pm 1$ matrix.

We consider three classical problems in logic. The *satisfiability problem* (SAT) associated to a set S of clauses, consists of finding a truth assignment that satisfies all the clauses in S or showing that none exists. Equivalently, SAT consists of finding a $0, 1$ solution x to (12) or showing that none exists.

The *weighted maximum satisfiability problem* (MAXSAT) asociated to a set S of clauses and a weight vector w whose components are indexed by the clauses in S consists of finding a truth assignment that maximizes the total weight of the satisfied clauses. MAXSAT can be formulated as the integer program:

$$\begin{array}{ll} min & \sum_{i=1}^{m} w_i s_i \\ & Ax + s \geq \mathbf{1} - n(A) \\ & x \in \{0,1\}^n, s \in \{0,1\}^m. \end{array}$$

Logical inference in propositional logic is associated to a set S of clauses (the premises) and a clause C (the conclusion), and consists of deciding whether every truth assignment that satisfies all the premises in S also satisfies the conclusion C.

Let $Ax \geq \mathbf{1} - n(A)$ be the system of inequalities associated with the set S of premises. The conclusion $C = (\bigvee_{j \in P(C)} x_j) \vee (\bigvee_{j \in N(C)} \neg x_j)$ cannot be deduced from S if and only if there exists a $0,1$ vector satisfying the following system:

$$\begin{array}{l} Ax \geq \mathbf{1} - n(A), \\ x_j = 0 \quad \text{for all } j \in P(C), \\ x_j = 1 \quad \text{for all } j \in N(C). \end{array}$$

Equivalently, the conclusion C can be represented by the inequality

$$\sum_{j \in P(C)} x_j - \sum_{j \in N(C)} x_j \geq 1 - |N(C)|,$$

or, more compactly, $cx \geq 1 - |N(C)|$ where c denotes the n-vector with components $c_j = 1$ for $j \in P(C)$, $c_j = -1$ for $j \in N(C)$ and $c_j = 0$ otherwise. Then C cannot be deduced from S if and only if the integer program

$$min \; \{cx : Ax \geq \mathbf{1} - n(A), \; x \in \{0,1\}^n\} \tag{13}$$

has a solution with value $-|N(C)|$.

These three problems are NP-hard in general but SAT and logical inference can be solved efficiently for Horn clauses, clauses with at most two literals and several related classes [9], [13], [71]. MAXSAT remains NP-hard for Horn clauses with at most two literals [42]. A set S of clauses is *balanced* if the corresponding $0, \pm 1$ matrix A defined in (12) is balanced. Similarly, a set of clauses is *ideal* if A is ideal. By Theorem 2.2, every balanced set of clauses is ideal. The vertices of (12) are integral for an ideal set of clauses, which implies that the underlying integer program can be solved as a linear program in that case:

Theorem 4.9 *Let S be an ideal set of clauses. Then SAT, MAXSAT and logical inference can be solved in polynomial time by linear programming.*

This has consequences for probabilistic logic as defined by Nilsson [60]. Being able to solve MAXSAT in polynomial time provides a polynomial time separation algorithm for probabilistic logic via the ellipsoid method, as observed by Georgakopoulos, Kavvadias and Papadimitriou [42]. Hence probabilistic logic is solvable in polynomial time for ideal sets of clauses.

Lemma 4.10 *Let S be an ideal set of clauses. If every clause of S contains more than one literal then, for every atomic proposition x_j, there exist at least two truth assignments satisfying S, one in which x_j is true and one in which x_j is false.*

Proof: Since the point $x_j = 1/2,\ j = 1, \ldots, n$ belongs to the polytope $Q(A) = \{x : Ax \geq \mathbf{1} - n(A),\ 0 \leq x \leq \mathbf{1}\}$ and $Q(A)$ is an integral polytope, then the above point can be expressed as a convex combination of $0, 1$ vectors in $Q(A)$. Clearly, for every index j, there exists in the convex combination a $0, 1$ vector with $x_j = 0$ and another with $x_j = 1$. $\Box$

A consequence of Lemma 4.10 is that, for an ideal set of clauses, SAT can be solved more efficiently than by general linear programming.

Theorem 4.11 (Conforti, Cornuéjols [16]) *Let S be an ideal set of clauses. Then S is satisfiable if and only if a recursive application of the following procedure stops with an empty set of clauses.*

Recursive Step

If $S = \emptyset$ then S is satisfiable.

If S contains a clause C with a single literal (unit clause)*, set the corresponding atomic proposition x_j so that C is satisfied. Eliminate from S all clauses that become satisfied and remove x_j from all the other clauses. If a clause becomes empty, then S is not satisfiable* (unit resolution)*.*

If every clause in S contains at least two literals, choose any atomic proposition x_j appearing in a clause of S and add to S one of the clauses x_j and $\neg x_j$.

The above algorithm for SAT can also be used to solve the logical inference problem when S is an ideal set of clauses, see [16]. For balanced (or ideal) sets of clauses, it is an open problem to solve MAXSAT in polynomial time by a direct method, without appealing to polynomial time algorithms for general linear programming.

4.4 Nonlinear $0, 1$ Optimization

Consider the nonlinear $0, 1$ maximization problem

$$\begin{array}{rcl} max & & \sum_k a_k \prod_{j \in T_k} x_j \prod_{j \in R_k} (1 - x_j) \\ x & \in & \{0,1\}^n \end{array}$$

where, w.l.o.g., all ordered pairs (T_k, R_k) are distinct and $T_k \cap R_k = \emptyset$. This is an NP-hard problem. A standard linearization of this problem was proposed by Fortet [39]:

$$\begin{array}{rcl} max \ \sum a_k y_k & & \\ y_k - x_j & \leq & 0 \ \textit{for all } k \ \textit{s.t.} \ a_k > 0, \ \textit{for all } j \in T_k \\ y_k + x_j & \leq & 1 \ \textit{for all } k \ \textit{s.t.} \ a_k > 0, \ \textit{for all } j \in R_k \\ y_k - \sum_{j \in T_k} x_j + \sum_{j \in R_k} x_j & \geq & 1 - |T_k| \ \textit{for all } k \ \textit{s.t.} \ a_k < 0 \\ y_k, \ x_j & \in & \{0,1\} \ \textit{for all } k \ \textit{and } j. \end{array}$$

When the constraint matrix is balanced, this integer program can be solved as a linear program, as a consequence of Theorem 2.7. Therefore, in this case, the nonlinear $0, 1$ maximization problem can be solved in polynomial time. The relevance of balancedness in this context was pointed out by Crama [33].

5 The Structure of Balanced Matrices

5.1 Bipartite Representation of a $0, \pm 1$ Matrix

In an undirected graph, a *hole* is a chordless cycle of length greater than 3. A cycle is *balanced* if its length is a multiple of 4. A graph is *balanced* if all its chordless cycles are balanced. Clearly, a balanced graph is simple and bipartite.

The *bipartite representation of a* $0,1$ *matrix* A is the bipartite graph $G(A) = (V^r \cup V^c, E)$ having a node in V^r for every row of A, a node in V^c for every column of A and an edge ij joining nodes $i \in V^r$ and $j \in V^c$ if and only if the entry a_{ij} of A equals 1.

Note that a $0,1$ matrix is balanced if and only if its bipartite representation is a balanced graph.

The *bipartite representation of a* $0,\pm1$ *matrix* A is the signed bipartite graph $G(A) = (V^r \cup V^c, E)$ having a node in V^r for every row of A, a node in V^c for every column of A and an edge ij joining nodes $i \in V^r$ and $j \in V^c$ if and only if the entry a_{ij} is nonzero. Furthermore a_{ij} is the sign of the edge ij. This concept extends the one introduced above. Conversely, for a bipartite graph $G = (V^r \cup V^c, E)$, with signs ±1 on its edges, there is a unique matrix A for which $G = G(A)$ (up to transposition of the matrix, permutation of rows and permutation of columns).

5.2 Signing 0,1 Matrices: Camion's Algorithm and Truemper's Theorem

A $0,1$ matrix is *balanceable* if its nonzero entries can be signed +1 or -1 so that the resulting $0,\pm1$ matrix is balanced. A bipartite graph G is *balanceable* if $G = G(A)$ and A is a balanceable matrix.

Camion [12] observed that the signing of a balanceable matrix into a balanced matrix is unique up to multiplying rows or columns by -1, and he gave a simple algorithm to obtain this signing. We present Camion's result next.

Let A be a $0,\pm1$ matrix and let A' be obtained from A by multiplying a set S of rows and columns by -1. A is balanced if and only if A' is. Note that, in the bipartite representation of A, this corresponds to switching signs on all edges of the cut $\delta(S)$. Now let R be a 0,1 matrix and $G(R)$ its bipartite representation. Since every edge of a maximal forest F of $G(R)$ is contained in a cut that does not contain any other edge of F, it follows that if R is balanceable, there exists a balanced signing of R in which the edges of F have any specified (arbitrary) signing.

This implies that, if a 0,1 matrix A is balanceable, one can find a balanced signing of A as follows.

CAMION'S SIGNING ALGORITHM

Input: *A 0,1 matrix A and its bipartite representation G, a maximal forest F of G and an arbitrary signing of the edges of F.*

Output: *A signing of G in which the edges of F are signed as specified in the input, and if A is balanceable then the signing is balanced.*

Index the edges of G $e_1, \dots, e_n$, so that the edges of F come first, and every edge e_j, $j \geq |F|+1$, together with edges having smaller indices, closes a hole H_j of G. For $j = |F|+1, \dots, n$, sign e_j so that the sum of the weights of H_j is congruent to 0 mod 4.

Note that the rows and columns corresponding to the nodes of H_j define a hole submatrix of A.

The fact that there exists an indexing of the edges of G as required in the signing algorithm follows from the following observation. For $j \geq |F|+1$, we can select e_j so that the path connecting the endnodes of e_j in the subgraph $(V(G), \{e_1, \dots, e_{j-1}\})$ is shortest possible. The hole H_j identified this way is also a hole in G. This forces the signing of e_j, since all the other edges of H_j are signed already. So, once the (arbitrary) signing of F has been chosen, the signing of G is unique. Therefore we have the following result.

Theorem 5.1 *If the input matrix A is a balanceable 0,1 matrix, Camion's signing algorithm produces a balanced $0, \pm 1$ matrix B. Furthermore every balanced $0, \pm 1$ matrix that arises from A by signing its nonzero entries either $+1$ or -1, can be obtained by switching signs on rows and columns of B.*

If one applies Camion's algorithm to the bipartite representation of the following matrix, the signing produced would leave one of the four holes unbalanced, proving that the matrix is not balanceable.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Assume that we have an algorithm to check if a bipartite graph is balanceable. Then, we can check whether a signed bipartite graph G is balanced as follows. Let G' be a copy of G that is not signed. Test whether G' is balanceable. If it is not, then G is not balanced. Otherwise, let F be a maximal forest of G'. Run the signing algorithm on G' with the edges of F signed as they are in G. Then G is balanced if and only if the signing of G' coincides with the signing of G.

We now give a characterization due to Truemper [71] of the bipartite graphs that are balanceable.

In a bipartite graph, a *wheel* (H, v) consists of a hole H and a node v having at least three neighbors in H. The wheel (H, v) is *odd* if v has an odd number of neighbors in H. A *3-path configuration* is an induced subgraph consisting of three internally node-disjoint paths connecting two nonadjacent nodes u and v and containing no edge other than those of the paths. If u and v are in opposite sides of the bipartition, i.e. the three paths have an odd number of edges, the 3-path configuration is called a *3-odd-path configuration*. In Figure 1, solid lines represent edges and dotted lines represent paths with at least one edge.

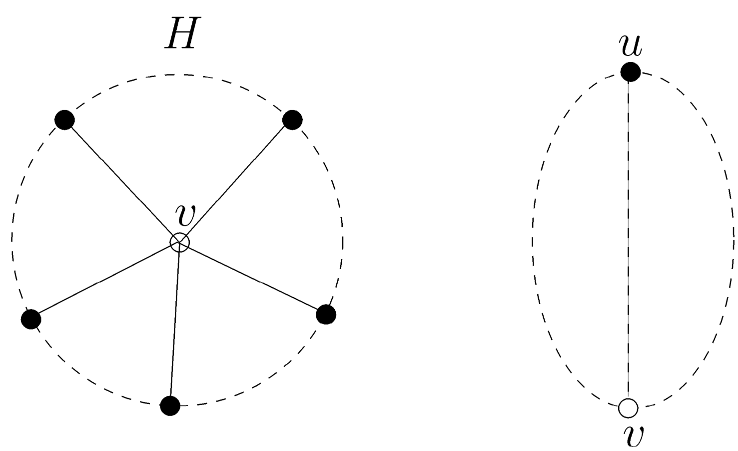

Figure 1: An odd wheel and a 3-odd-path configuration

Both a 3-odd-path configuration and an odd wheel have the following properties: each edge belongs to exactly two holes and the total number of edges is odd. Therefore in any signing, the sum of the labels of all holes is equal to 2 *mod* 4. This implies that at least one of the holes is not balanced, showing that neither 3-odd-path configurations nor odd wheels are balanceable. These are in fact the only minimal bipartite graphs that are not balanceable, as shown by the following theorem.

Theorem 5.2 (Truemper [71]) *A bipartite graph is balanceable if and only if it does not contain an odd wheel or a 3-odd-path configuration as an induced subgraph.*

We prove Theorem 5.2 following Conforti, Gerards and Kapoor [27].

For a connected bipartite graph G that contains a clique cutset K_t with t nodes, let $G'_1, \ldots, G'_n$ be the connected components of $G \setminus K_t$. The *blocks* of G are the subgraphs G_i induced by $V(G'_i) \cup K_t$ for $i = 1, \ldots, n$.

Lemma 5.3 *If a connected bipartite graph G contains a K_1 or K_2 cutset, then G is balanceable if and only if each block is balanceable.*

Proof: If G is balanceable, then so are the blocks. Therefore we only have to prove the converse. Assume that all the blocks are balanceable. Give each block a balanced signing. If the cutset is a K_1 cutset, this yields a balanced signing of G. If the cutset is a K_2 cutset, re-sign each block so that the edge of that K_2 has the sign $+1$. Now take the union of these signings. This yields a balanced signing of G again. $\Box$

Thus, in the remainder of the proof, we can assume that G is a connected bipartite graph with no K_1 or K_2 cutset.

Lemma 5.4 *Let H be a hole of G. If $G \neq H$, then H is contained in a 3-path configuration or a wheel of G.*

Proof: Choose two nonadjacent nodes u and w in H and a uw-path $P = u, x, \ldots, z, w$ whose intermediate nodes are in $G \setminus H$ such that P is as short as possible. Such a pair of nodes u, w exists since $G \neq H$ and G has no K_1 or K_2 cutset. If $x = z$, then H is contained in a 3-path configuration or a wheel. So assume $x \neq z$. By our choice of P, u is the only neighbor of x in H and w is the only neighbor of z in H.

Let Y be the set of nodes in $V(H) - \{u, w\}$ that have a neighbor in P. If Y is empty, H is contained in a 3-path configuration. So assume Y is nonempty. By the minimality of P, the nodes of Y are pairwise adjacent and they are adjacent to u and w. This implies that Y contains a single node y and that y is adjacent to u and w. But then $V(H) \cup V(P)$ induces a wheel with center y. □

For $e \in E(G)$, let G^e denote the graph with a node v_H for each hole H of G containing e and an edge $v_{H_i}v_{H_j}$ if and only if there exists a wheel or a 3-path configuration containing both holes H_i and H_j.

Lemma 5.5 *G^e is a connected graph.*

Proof: Suppose not. Let $e = uw$. Choose two holes H_1 and H_2 of G with v_{H_1} and v_{H_2} in different connected components of G^e, with the minimum distance $d(H_1, H_2)$ in $G \setminus \{u, v\}$ between $V(H_1) - \{u, w\}$ and $V(H_2) - \{u, w\}$ and, subject to this, with the smallest $|V(H_1) \cup V(H_2)|$.

Let T be a shortest path from $V(H_1) - \{u, v\}$ to $V(H_2) - \{u, v\}$ in $G \setminus \{u, v\}$. Note that T is just a node of $V(H_1) \cap V(H_2) \setminus \{u, v\}$ when this set is nonempty. The graph G' induced by the nodes in H_1, H_2 and T has no K_1 or K_2 cutset. By Lemma 5.4, H_1 is contained in a 3-path configuration or a wheel of G'. Since each edge of a 3-path configuration or a wheel belongs to two holes, there exists a hole $H_3 \neq H_1$ containing edge e in G'. Since v_{H_1} and v_{H_3} are adjacent in G^e, it follows that v_{H_2} and v_{H_3} are in different components of G^e. Since H_1 and H_3 are distinct holes, H_3 contains a node in $V(H_2) \cup V(T) \setminus V(H_1)$. If H_3 contains a node in $V(T) \setminus (V(H_1) \cup V(H_2))$, then $V(H_1) \cap V(H_2) = \{u, v\}$ and $d(H_3, H_2) < d(H_1, H_2)$ a contradiction to the choice of H_1, H_2.

Therefore H_3 contains a node x in $V(H_2) \setminus V(H_1)$. By our choice of H_1, H_2, we have that $V(H_1) \cap V(H_2) \setminus \{u, v\}$ is nonempty. Let $P_1 = H_1 \setminus e$ and $P_2 = H_2 \setminus e$ and let s, t be the nodes in $V(H_1) \cap V(H_2)$ such that the st-subpath P_2^{st} of P_2 contains x and is shortest. Let P_1^{st} be the st-subpath of P_1. Since H_2 is a hole, P_1^{st} contains an intermediate node $z \in V(H_1) \setminus V(H_2)$. Now $V(H_3) \cup V(H_2)$ is contained in $V(H_1) \cup V(H_2) \setminus z$, a contradiction to our choice of H_1, H_2. □

Proof of Theorem 5.2: We showed already that odd wheels and 3-odd-path configurations are not balanceable. It remains to show that, conversely, if G contains no odd wheel or 3-odd-path configuration, then G is balanceable. Suppose G is a counterexample with the smallest number of nodes. By

Lemma 5.3, G is connected and has no K_1 or K_2 cutset. Let $e = uv$ be an edge of G. Since $G \setminus \{u, v\}$ is connected, there exists a spanning tree F of G where u and v are leaves. Arbitrarily sign F and use Camion's signing algorithm in $G \setminus \{u\}$ and $G \setminus \{v\}$. By the minimality of G, these two graphs are balanceable and therefore Camion's algorithm yields a unique signing of all the edges except e. Furthermore, all holes not going through edge e are balanced. Since G is not balanceable, any signing of e yields some holes going through e that are balanced and some that are not. By Lemma 5.5, there exists a wheel or a 3-path configuration C containing an unbalanced hole H_1 and a balanced hole H_2 both going through edge e. Now we use the fact that each edge of C belongs to exactly two holes of C. Since the holes of C distinct from H_1 and H_2 do not go through e, they are balanced. Furthermore, applying the above fact to all edges of C, the sum of all labels in C is 1 mod 2, which implies that C has an odd number of edges. Thus C is an odd wheel or a 3-odd-path configuration, a contradiction. $\Box$

5.3 Decomposition Theorems

In this section, we present decomposition theorems for balanced $0, 1$ matrices due to Conforti, Cornuéjols and Rao [23] and balanceable $0, 1$ matrices due to Conforti, Cornuéjols, Kapoor and Vušković [21]. We state the decomposition theorems in terms of the bipartite representation of such matrices, as defined in Section 5.1.

5.3.1 Cutsets

A set S of nodes (edges) of a connected graph G is a *node (edge) cutset* if the subgraph of G obtained by removing the nodes (edges) in S, is disconnected.

For a node x, let $N(x)$ denote the set of all neighbors of x. In a bipartite graph, an *extended star* is defined by disjoint subsets T, A, N of $V(G)$ and a node $x \in T$ such that

(i) $N \subset N(x)$,

(ii) every node of A is adjacent to every node of T,

(iii) $A \neq \emptyset$ and if $|T| \geq 2$, then $|A| \geq 2$.

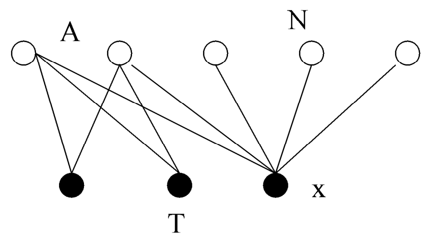

Figure 2: Extended star

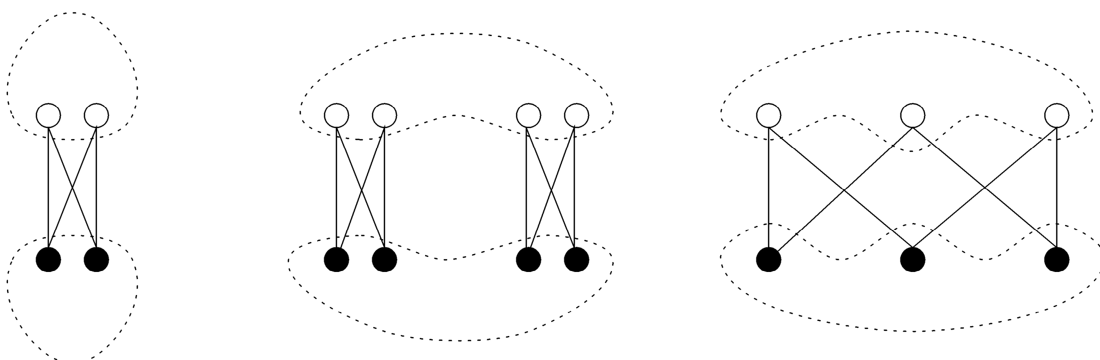

Figure 3: A 1-join, a 2-join and a 6-join

This concept was introduced by Conforti, Cornuéjols and Rao [23] and is illustrated in Figure 2. An *extended star cutset* is one where $T \cup A \cup N$ is a node cutset. An extended star cutset with $N = \emptyset$ is called a *biclique cutset*. An extended star cutset having $T = \{x\}$ is called a *star cutset*. Note that a star cutset is a special case of a biclique cutset.

A graph G has a *1-join* if its nodes can be partitioned into sets H_1 and H_2, with $|H_1| \geq 2$ and $|H_2| \geq 2$, so that $A_1 \subseteq H_1$, $A_2 \subseteq H_2$ are nonempty, all nodes of A_1 are adjacent to all nodes of A_2 and these are the only adjacencies between H_1 and H_2. This concept was introduced by Cunningham and Edmonds [35].

A graph G has a *2-join* if its nodes can be partitioned into sets H_1 and H_2 so that $A_1, B_1 \subseteq H_1$, $A_2, B_2 \subseteq H_2$ where A_1, B_1, A_2, B_2 are nonempty and disjoint, all nodes of A_1 are adjacent to all nodes of A_2, all nodes of B_1

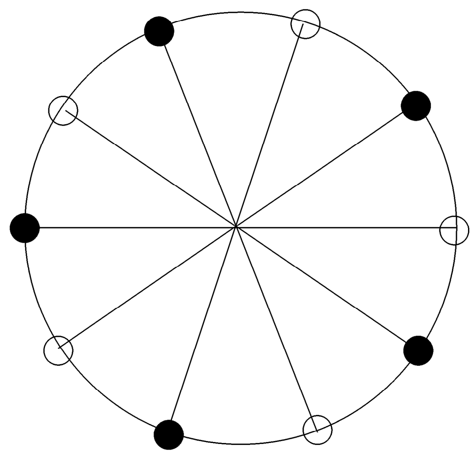

Figure 4: R_{10}

are adjacent to all nodes of B_2 and these are the only adjacencies between H_1 and H_2. Also, for $i = 1, 2$, H_i has at least one path from A_i to B_i and if A_i and B_i are both of cardinality 1, then the graph induced by H_i is not a chordless path. We also say that $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ is a 2-join of G. This concept was introduced by Cornuéjols and Cunningham [32].

In a connected bipartite graph G, let A_i, $i = 1, \ldots, 6$, be disjoint nonempty node sets such that, for each i, every node in A_i is adjacent to every node in $A_{i-1} \cup A_{i+1}$ (indices are taken modulo 6), and these are the only edges in the subgraph A induced by the node set $\cup_{i=1}^{6} A_i$. Assume that $E(A)$ is an edge cutset but that no subset of its edges forms a 1-join or a 2-join. Furthermore assume that no connected component of $G \setminus E(A)$ contains a node in $A_1 \cup A_3 \cup A_5$ and a node in $A_2 \cup A_4 \cup A_6$. Let G_{135} be the union of the components of $G \setminus E(A)$ containing a node in $A_1 \cup A_3 \cup A_5$ and G_{246} be the union of components containing a node in $A_2 \cup A_4 \cup A_6$. The set $E(A)$ constitutes a *6-join* if the graphs G_{135} and G_{246} contain at least four nodes each. This concept was introduced by Conforti, Cornuéjols, Kapoor and Vušković [21].

5.3.2 Main Theorem

A graph is *strongly balanceable* if it is balanceable and contains no cycle with exactly one chord. This class of bipartite graphs is well studied in the literature, see [28]. We discuss it in Section 5.5.2. The following graph, which is not strongly balanceable, plays an important role: R_{10} is the bipartite graph on ten nodes defined by the cycle $C = x_1, \ldots, x_{10}, x_1$ of length ten with chords $x_i x_{i+5}$, $1 \leq i \leq 5$, see Figure 4. Equivalently, R_{10} is the bipartite representation of the matrix $\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$, which appears in Seymour's decomposition of totally unimodular matrices [66]. Note that the signing of R_{10} that assigns $+1$ to the edges of C and -1 to all the other edges is a balanced signing of R_{10}. The corresponding $0, \pm 1$ matrix is actually totally unimodular.

Theorem 5.6 (Conforti, Cornuéjols, Kapoor and Vušković [21]) *A balanceable bipartite graph that is not strongly balanceable is either R_{10} or contains a 2-join, a 6-join or an extended star cutset.*

Figure 5 exhibits examples showing that none of the three kinds of cutsets can be dropped from Theorem 5.6.

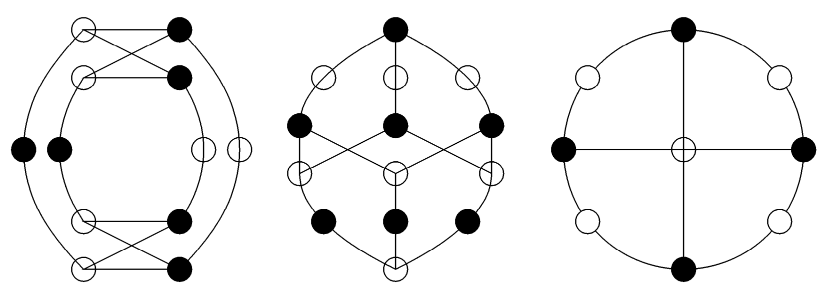

Figure 5: Examples showing that no cutset can be dropped in the theorem

Connected 6-Holes

A *triad* consists of three internally node-disjoint paths $t, \dots, u$; $t, \dots, v$ and $t, \dots, w$, where t, u, v, w are distinct nodes and u, v, w belong to the same side of the bipartition. Furthermore, the graph induced by the nodes of the triad contains no other edges than those of the three paths. Nodes u, v and w are called the *attachments* of the triad.

A *fan* consists of a chordless path $x, \dots, y$ together with a node z adjacent to at least one node of the path, where x, y and z are distinct nodes all belonging to the same side of the bipartition. Nodes x, y and z are called the *attachments* of the fan.

A *connected 6-hole* Σ is a graph induced by two disjoint node sets $T(\Sigma)$ and $B(\Sigma)$ such that each induces either a triad or a fan, the attachments of $T(\Sigma)$ and $B(\Sigma)$ induce a 6-hole and there are no other adjacencies between the nodes of $T(\Sigma)$ and $B(\Sigma)$. Figure 6 depicts the four types of connected 6-holes.

The following theorem concerns the class of balanceable bipartite graphs that do not contain a connected 6-hole or R_{10} as induced subgraph.

Theorem 5.7 (Conforti, Cornuéjols and Rao [23]) *A balanceable bipartite graph not containing R_{10} or a connected 6-hole as induced subgraph either is strongly balanceable or contains a 2-join or an extended star cutset.*

So it remains to find a decomposition of balanceable bipartite graphs that contain R_{10} or connected 6-holes as induced subgraph. This is accomplished as follows.

Theorem 5.8 (Conforti, Cornuéjols, Kapoor and Vušković [21]) *A balanceable bipartite graph containing R_{10} as a proper induced subgraph has a biclique cutset.*

Theorem 5.9 ([21]) *A balanceable bipartite graph that contains a connected 6-hole as induced subgraph, has an extended star cutset or a 6-join.*

Now Theorem 5.6 follows from Theorems 5.7, 5.8 and 5.9.

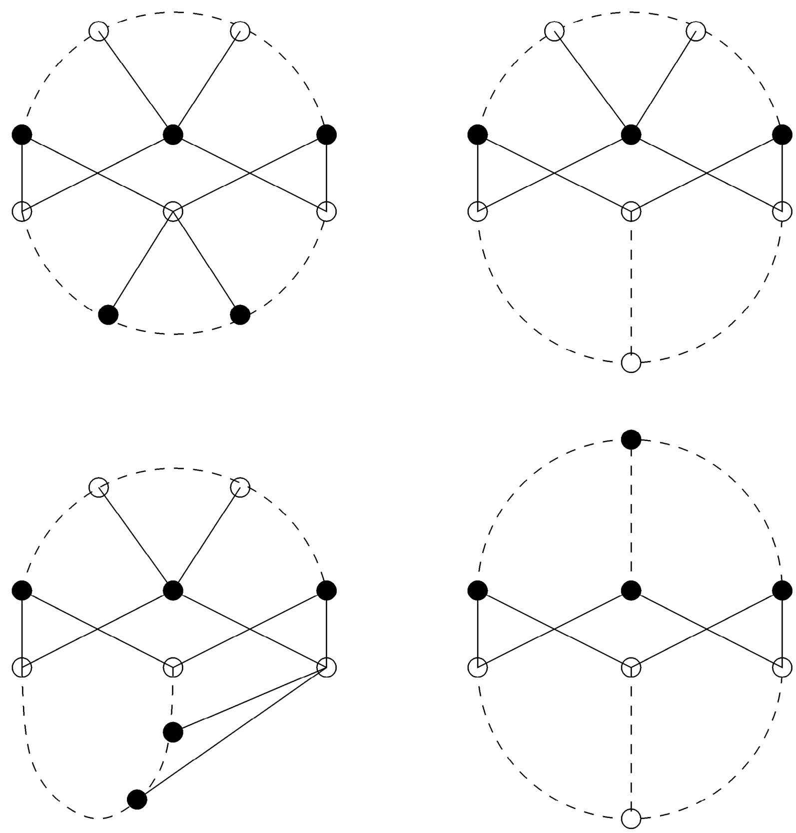

Figure 6: The four types of connected 6-holes

5.4 Recognition Algorithm

Conforti, Cornuéjols, Kapoor and Vušković [21] give a polynomial time algorithm to check whether a $0, \pm 1$ matrix A is balanced. The algorithm works on the bipartite representation $G(A)$ introduced. Since each edge of $G(A)$ is signed $+1$ or -1 according to the corresponding entry in the matrix A, we call G a *signed bipartite graph*.

Let G be a connected signed bipartite graph. The removal of a node cutset or edge cutset disconnects G into two or more connected components. From these components we construct *blocks of decomposition* by adding some new nodes and signed edges. We say that a decomposition is *balancedness preserving* when it has the following property: all the blocks are balanced if and only if G itself is balanced. The central idea in the algorithm is to decompose G using balancedness preserving decompositions into a polynomial number of basic blocks that can be checked for balancedness in polynomial time.

For the 2-join and 6-join, the blocks can be defined so that the decompositions are balancedness preserving. For the extended star cutset it is not known how to construct blocks of decomposition that are balancedness preserving and generate a polynomial decomposition tree. To overcome this problem, the algorithm uses the idea of cleaning, first introduced by Conforti and Rao [29], [30]. An input graph G is first transformed into a *clean* graph G' (to be defined later), and then G' is decomposed, the decompositions in G' being balancedness preserving.

Recently Zambelli [74], based on an idea introduced by Chudnovsky and Seymour for recognizing Berge graphs [15], has given a polynomial algorithm to test balancedness in a signed bipartite graph that does not use the decomposition theorem: it uses cleaning and shortest paths techniques. We summarize here the ideas behind his algorithm.

The algorithm first detects whether the input graph has a 3-odd-path configuration (as defined in Section 5.2), based on the following result:

In a bipartite graph G, consider a 3-odd-path configuration with the smallest number of nodes, induced by paths P_1, P_2, P_3 connecting nodes u and v. Let m_i be a middle node of path P_i. In a subgraph obtained from G by removing some neighbors of u and v, any shortest path from m_i to u and v can be substituted for P_i yielding another smallest 3-odd-path configuration.

This result yields a polynomial time algorithm to detect whether a bipartite graph contains a 3-odd-path configuartion.

A *detectable 3-wheel* is a wheel (H, v) where v has three neighbors in H and two of the neighbors of v in H have distance two in H. By an analogous method Zambelli shows the following:

There exists a polynomial time algorithm that checks whether a bipartite graph that does not contain a 3-odd-path configuration, contains a detectable 3-wheel.

By Theorem 5.2, if a bipartite graph contains a 3-odd-path configuration or a detectable 3-wheel, it is not balanceable.

A node v is *major* for a hole H if v has at least three neighbors in H. The following result is proved by Conforti, Cornuéjols, Kapoor and Vušković [21].

Theorem 5.10 *Let H be a smallest unbalanced hole in a signed bipartite graph. Then H contains two edges such that every major node for H is adjacent to at least one of the endnodes of these two edges.*

A signed bipartite graph is *clean* if it is either balanced or contains a smallest unbalanced hole H with no major vertices for H.

Based on the above theorem a polynomial time algorithm is constructed in [21], that takes as input a signed bipartite graph G and outputs a clean graph G', such that G is balanced if and only if G' is balanced.

Let G be a signed bipartite graph that does not contain a 3-odd path configuration nor a detectable 3-wheel. The last step of Zambelli's algorithm is based on the following:

Let G be a clean signed bipartite graph that does not contain a 3-odd-path configuration or a detectable 3-wheel. There exists a polynomial time algorithm, based on shortest path methods, that checks whether G is balanced.

The algorithms outlined in this section recognize in polynomial time whether a signed bipartite graph contains an unbalanced hole. Interestingly Kapoor [56] has shown that it is NP-complete to recognize whether a signed bipartite graph contains an unbalanced hole going through a prespecified node.

5.5 More Decomposition Theorems

Several subclasses of balanced matrices have beautiful decomposition properties of their own. Totally unimodular matrices for example can be decomposed using a deep theorem of Seymour [66]. This result is surveyed in [64], [62] or [31] and we do not review it here. We review instead the structure and properties of several other classes of balanced matrices.

5.5.1 Totally Balanced $0, 1$ Matrices

A $0, 1$ matrix A is *totally balanced* if every hole submatrix of A is the 2×2 submatrix of all $1s$. Equivalently, a bipartite graph G is *totally balanced* if every hole of G has length 4. Totally balanced matrices arise in location theory. Several authors (Golumbic and Goss [45], Anstee and Farber [1], Hoffman, Kolen and Sakarovitch [52] and Lubiw [58] among others) have given properties of these matrices.

A *biclique* is a complete bipartite graph with at least one node from each side of the bipartition. For a node u, let $N(u)$ denote the set of all neighbors of u. An edge uv is *bisimplicial* if the node set $N(u) \cup N(v)$ induces a biclique. The following theorem of Golumbic and Goss [45] characterizes totally balanced bipartite graphs.

Theorem 5.11 (Golumbic, Goss, [45]) *A totally balanced bipartite graph has a bisimplicial edge.*

This theorem yields a polynomial time algorithm to test whether a bipartite graph G is totally balanced: for if e is a bisimplicial edge of G, then G is totally balanced if and only if $G \setminus e$ is totally balanced.

A $0, 1$ matrix A is in *standard greedy form* if it contains no 2×2 submatrix of the form $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, where the order of the rows and columns in the submatrix is the same as in the matrix A. This name comes from the fact that the linear program

$$\begin{array}{rcccl} max & \sum y_i & & & \\ & yA & \leq & c & \\ 0 & \leq & y & \leq & p \end{array} \tag{14}$$

can be solved by a greedy algorithm. Namely, given $y_1, \ldots, y_{k-1}$ such that $\sum_{i=1}^{k-1} a_{ij} y_i \leq c_j, \quad j = 1, \ldots, n$ and $0 \leq y_i \leq p_i, \quad i = 1, \ldots, k-1$, set y_k to the largest value such that $\sum_{i=1}^{k} a_{ij} y_i \leq c_j, \quad j = 1, \ldots, n$ and $0 \leq y_k \leq p_k$. The resulting greedy solution is an optimum solution to this linear program. What does this have to do with totally balanced matrices? The answer is in the next theorem.

Theorem 5.12 (Anstee, Farber [1], Hoffman, Kolen, Sakarovitch [52], Lubiw [58]) *A $0,1$ matrix is totally balanced if and only if its rows and columns can be permuted into standard greedy form.*

This transformation can be performed in time $O(nm^2)$ [52].

Totally balanced $0,1$ matrices come up in various ways in the context of facility location problems on trees. For example, the *covering problem*

$$\begin{array}{rcl} min \ \sum_1^n c_j x_j + \sum_1^m p_i z_i & & \\ \sum_j a_{ij} x_j + z_i & \geq & 1, \quad i = 1, \ldots, m \\ x_j, z_i & \in & \{0,1\} \end{array} \tag{15}$$

can be interpreted as follows: c_j is the set up cost of establishing a facility at site j, p_i is the penalty if client i is not served by any facility, and $a_{ij} = 1$ if a facility at site j can serve client i, 0 otherwise.

When the underlying network is a tree and the facilities and clients are located at nodes of the tree, it is customary to assume that a facility at site j can serve all the clients in a *neighborhood subtree* of j, namely, all the clients within distance r_j from node j.

An *intersection matrix* of the set $\{S_1, \ldots, S_m\}$ versus $\{R_1, \ldots, R_n\}$, where S_i, $i = 1, \ldots, m$, and R_j, $j = 1, \ldots, n$, are subsets of a given set, is defined to be the $m \times n$ $0,1$ matrix $A = (a_{ij})$ where $a_{ij} = 1$ if and only if $S_i \cap R_j \neq \emptyset$.

Theorem 5.13 (Giles [44]) *The intersection matrix of neighborhood subtrees versus nodes of a tree is totally balanced.*

It follows that the above location problem on trees (15) can be solved as a linear program (by Theorem 2.1 and the fact that totally balanced matrices are balanced). In fact, by using the standard greedy form of the neighborhood subtrees versus nodes matrix, and by noting that (15) is the dual of (14), the greedy solution described earlier for (14) can be used, in conjunction with complementary slackness, to obtain an elegant solution of the covering problem. The above theorem of Giles has been generalized as follows.

Theorem 5.14 (Tamir [67]) *The intersection matrix of neighborhood subtrees versus neighborhood subtrees of a tree is totally balanced.*

Other classes of totally balanced $0, 1$ matrices arising from location problems on trees can be found in [68].

5.5.2 Restricted and Strongly Balanced Matrices

A signed bipartite graph G is *restricted balanced* if the weight of every cycle of G is congruent to 0 *mod* 4. A signed bipartite graph is *strongly balanced* if every cycle of weight 2 *mod* 4 has at least two chords. Restricted (strongly, resp.) balanced $0, \pm 1$ matrices are defined to be the matrices whose bipartite representation is a restricted (strongly, resp.) balanced bipartite graph. It follows from the definition that restricted balanced $0, \pm 1$ matrices are strongly balanced, and it can be shown that strongly balanced $0, \pm 1$ matrices are totally unimodular, see [28]. Restricted (strongly, resp.) balanceable 0,1 matrices are those where the nonzero entries can be signed $+1$ or -1 so that the resulting $0, \pm 1$ matrix is restricted (strongly, resp.) balanced.

Theorem 5.15 (Conforti, Rao [28]) *A strongly balanceable bipartite graph either is restricted balanceable or contains a 1-join.*

Crama, Hammer and Ibaraki [34] define a $0, \pm 1$ matrix A to be *strongly unimodular* if every basis of (A, I) can be put in triangular form by permutation of rows and columns.

Theorem 5.16 (Crama, Hammer, Ibaraki [34]) *A* $0, \pm 1$ *matrix is strongly unimodular if and only if it is strongly balanced.*

Yannakakis [73] has shown that a restricted balanceable $0, 1$ matrix having both a row and a column with more than two nonzero entries has a very special 3-separation: the bipartite graph representation has a 2-join consisting of two single edges. A bipartite graph is *2-bipartite* if all the nodes in one side of the bipartition have degree at most 2.

Theorem 5.17 (Yannakakis [73]) *A restricted balanceable bipartite graph either is 2-bipartite or contains a cutnode or contains a 2-join consisting of two edges.*

Based on this theorem, Yannakakis designed a linear time algorithm for checking whether a $0, \pm 1$ matrix is restricted balanced. A different algorithm for this recognition problem was given by Conforti and Rao [28]:

Construct a spanning forest in the bipartite graph and check if there exists a cycle of weight 2 mod 4 which is either fundamental or is the symmetric difference of fundamental cycles. If no such cycle exists, the signed bipartite graph is restricted balanced.

A bipartite graph is *linear* if it does not contain a cycle of length 4. Note that an extended star cutset in a linear bipartite graph is always a star cutset, due to Condition (ii) in the definition of extended star cutsets. Conforti and Rao [29] proved the following theorem for linear balanced bipartite graphs:

Theorem 5.18 (Conforti, Rao [29]) *A linear balanced bipartite graph either is restricted balanced or contains a star cutset.*

A cycle C in a signed bipartite graph G is *unbalanced* if the sum of the weights of the edges in C is congruent to 2 *mod* 4. It is easy to see that a signed bipartite graph has a balanced cycle if and only if it has a balanced hole. It follows that the following two classes of graphs are equivalent: signed bipartite graphs in which all cycles are unbalanced, and signed bipartite graphs in which all holes are unbalanced. These graphs are characterized by Conforti, Cornuéjols and Vušković in [25], where a linear algorithm for testing membership in this class is given.

5.6 Some Conjectures and Open Questions

5.6.1 Eliminating Edges

Conjecture 5.19 (Conforti, Cornuéjols, Kapoor, Vušković [21]) *In a balanced signed bipartite graph G, either every edge belongs to some R_{10}, or some edge can be removed from G so that the resulting signed bipartite graph is still balanced.*

The condition on R_{10} is necessary since removing any edge from R_{10} yields a wheel with three spokes or a 3-odd-path configuration as induced subgraph. This conjecture implies that given a $0, \pm 1$ balanced matrix we can sequentially turn the nonzero entries to zero until every nonzero belongs to some R_{10} matrix, while maintaining balanced $0, \pm 1$ matrices at each steps. For $0, 1$ matrices, the above conjecture reduces to the following:

Conjecture 5.20 (Conforti, Rao [29]) *Every balanced bipartite graph contains an edge which is not the unique chord of a cycle.*

It follows from the definition that restricted balanced signed bipartite graphs are exactly the ones such that the removal of *any* subset of edges leaves a restricted balanced signed bipartite graph.

Conjecture 5.19 holds for signed bipartite graphs that are strongly balanced since, by definition, the removal of any edge leaves a chord in every unbalanced cycle.

Theorem 5.11 shows that the graph obtained by eliminating a bisimplicial edge in a totally balanced bipartite graph is totally balanced. Hence Conjecture 5.20 holds for totally balanced bipartite graphs.

5.6.2 Strengthening the Decomposition Theorems

The extended star decomposition is not balancedness preserving. This heavily affects the running time of the recognition algorithm for balancedness. Therefore it would be desirable to find strengthenings of Theorem 5.6 that only use operations that preserve balancedness. We have been unable to obtain these results even for linear balanced bipartite graphs [30].

Another direction in which the main theorem might be strengthened is as follows.

Conjecture 5.21 ([21]) *Every balanceable bipartite graph G which is not signable to be totally unimodular has an extended star cutset.*

This conjecture was shown to hold when G is the bipartite representation of a balanced $0, 1$ matrix [23].

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References

[1] R. Anstee and M. Farber, Characterizations of totally balanced matrices, *Journal of Algorithms 5* (1984) 215-230.

[2] C. Berge, Sur certains hypergraphes généralisant les graphes bipartites, in: *Combinatorial Theory and its Applications I* (P. Erdös, A. Rényi and V. Sós eds.), *Colloq. Math. Soc. János Bolyai 4*, North Holland, Amsterdam (1970) 119-133.

[3] C. Berge, Balanced matrices, *Mathematical Programming 2* (1972) 19-31.

[4] C. Berge, Notes sur les bonnes colorations d'un hypergraphe, *Cahiers du Centre d'etudes de Recherche Operationnelle 15* (1973) 219-223.

[5] C. Berge, Balanced matrices and the property G, *Mathematical Programming Study 12* (1980) 163-175.

[6] C. Berge, *Hypergraphs,* North Holland (1989).

[7] C. Berge and M. Las Vergnas, Sur un théorème du type König pour hypergraphes, in *International Conference on Combinatorial Mathematics, Annals of the New York Academy of Sciences 175* (1970) 32-40.

[8] E.O. Boros and O. Čepek, On perfect $0, \pm 1$ matrices, *Discrete Mathematics 165* (1997) 81-100.

[9] E. Boros, Y. Crama and P.L. Hammer, Polynomial-time inference of all valid implications for Horn and related formulae, *Annals of Mathematics and Artificial Intelligence 1* (1990) 21-32.

[10] K. Cameron and J. Edmonds, Existentially polytime theorems, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science 1,* American Mathematical Society, Providence, R.I. (1990) 83-100.

[11] P. Camion, Caractérisation des matrices unimodulaires, *Cahiers du Centre d' Études de Recherche Opérationelle 5* (1963) 181-190.

[12] P. Camion, Characterization of totally unimodular matrices, *Proceedings of the American Mathematical Society 16* (1965) 1068-1073.

[13] V. Chandru and J.N. Hooker, Extended Horn sets in propositional logic, *Journal of the ACM 38* (1991) 205-221.

[14] V. Chvátal, On certain polytopes associated with graphs, *Journal of Combinatorial Theory B 18* (1975) 138-154.

[15] M. Chudnovsky and P. Seymour, Recognizing Berge graphs, preprint (2003).

[16] M. Conforti and G. Cornuéjols, A class of logic problems solvable by linear programming, *Journal of the ACM 42* (1995) 1107-1113.

[17] M. Conforti and G. Cornuéjols, Balanced $0, \pm 1$ matrices, bicoloring and total dual integrality, *Mathematical Programming 71* (1995) 249-258.

[18] M. Conforti and G. Cornuéjols, Balanced matrices, to appear in: *"Handbooks in Operations Research and Management Science"* , (K. Aardal, G.L. Nemhauser, and R. Weismantel eds.), *Elsevier, Amsterdam.* (2004).

[19] M. Conforti, G. Cornuéjols and C. De Francesco, Perfect $0, \pm 1$ matrices, *Linear Algebra and its Applications 43* (1997) 299-309.

[20] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, Perfect matchings in balanced hypergraphs, *Combinatorica 16* (1996) 325-329.

[21] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, Balanced $0, \pm 1$ matrices, Parts I-II, *Journal of Combinatorial Theory B 81* (2001) 243-306.

[22] M. Conforti, G. Cornuéjols, A. Kapoor, M. R. Rao and K. Vušković, Balanced matrices, in: *Mathematical Programming, State of the Art 1994*, (J. R. Birge and K. G. Murty eds.), University of Michigan Press (1994) 1-33.

[23] M. Conforti, G. Cornuéjols and M. R. Rao, Decomposition of balanced matrices, *Journal of Combinatorial Theory B 77* (1999) 292-406.

[24] M. Conforti, G. Cornuéjols and K. Truemper, From totally unimodular to balanced $0, \pm 1$ matrices: A family of integer polytopes, *Mathematics of Operations Research 19* (1994) 21-23.

[25] M. Conforti, G. Cornuéjols and K. Vušković, Balanced cycles and holes in bipartite graphs, *Discrete Mathematics* 199 (1999) 27-33.

[26] M. Conforti, G. Cornuéjols and G. Zambelli, Bicolorings and equitable bicolorings of matrices, preprint (2001).

[27] M. Conforti, A.M.H. Gerards and A. Kapoor, A theorem of Truemper, *Combinatorica 20* (2000) 15-26.

[28] M. Conforti and M. R. Rao, Structural properties and recognition of restricted and strongly unimodular matrices, *Mathematical Programming 38* (1987) 17-27.

[29] M. Conforti and M. R. Rao, Structural properties and decomposition of linear balanced matrices, *Mathematical Programming 55* (1992) 129-168.

[30] M. Conforti and M. R. Rao, Testing balancedness and perfection of linear matrices, *Mathematical Programming 61* (1993) 1-18.

[31] G. Cornuéjols, *Combinatorial Optimization: Packing and Covering,* SIAM (2001).

[32] G. Cornuéjols and W.H. Cunningham, Compositions for perfect graphs, *Discrete Mathematics 55* (1985) 245-254.

[33] Y. Crama, Concave extensions for nonlinear $0-1$ maximization problems, *Mathematical Programming 61* (1993) 53-60.

[34] Y. Crama, P.L. Hammer and T. Ibaraki, Strong unimodularity for matrices and hypergraphs, *Discrete Applied Mathematics 15* (1986) 221-239.

[35] W. H. Cunningham and J. Edmonds, A combinatorial decomposition theory, *Canadian Journal of Mathematics 32* (1980) 734-765.

[36] D. de Werra, On some characterizations of totally unimodular matrices, *Mathematical Programming 20* (1981) 14-21.

[37] D. de Werra, Good and nice colorings of balanced hypergraphs, manuscript, (2002).

[38] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, *Annals of Discrete Mathematics 1* (1977) 185-204.

[39] R. Fortet, Applications de l'algèbre de Boole en recherche opérationelle, *Revue Française de Recherche Opérationelle 4* (1976) 251-259.

[40] D. R. Fulkerson, Anti-blocking polyhedra, *Journal of Combinatorial Theory B 12* (1972) 50-71.

[41] D. R. Fulkerson, A. Hoffman and R. Oppenheim, On balanced matrices, *Mathematical Programming Study 1* (1974) 120-132.

[42] G. Georgakopoulos, D. Kavvadias and C. H. Papadimitriou, Probabilistic satisfiability, *Journal of Complexity 4* (1988) 1-11.

[43] A. Ghouila-Houri, Charactérisations des matrices totalement unimodulaires, *C. R. Acad. Sc. Paris 254* (1962) 1192-1193.

[44] R. Giles, A balanced hypergraph defined by subtrees of a tree, *ARS Combinatorica 6* (1978) 179-183.

[45] M. C. Golumbic and C. F. Goss, Perfect elimination and chordal bipartite graphs, *Journal of Graph Theory 2* (1978) 155-163.

[46] B. Guenin, Perfect and ideal $0, \pm 1$ matrices, *Mathematics of Operations Research 23* (1998) 322-338.

[47] R. P. Gupta, An edge-coloration theorem for bipartite graphs of paths in trees, *Discrete Mathematics 23* (1978) 229-233.

[48] V. Guruswami, C. Pandu Rangan, Algorithmic aspects of clique-transversal and clique-independent sets, *Discrete Applied Mathematics 100* (2000) 183-202.

[49] P. Hall, On representatives of subsets *J. London Math. Soc.* (1935) 26-30.

[50] I. Heller, C.B. Tompkins, An extension of a theorem of Dantzig's, in: *Linear Inequalities and Related Systems* (H.W. Kuhn and A.W. Tucker eds.), Princeton University Press (1956) 247-254.

[51] A.J. Hoffman and J.K. Kruskal, Integral boundary points of convex polyhedra, in *Linear Inequalities and Related Systems* (H.W. Kuhn and A.W. Tucker eds.), Princeton University Press (1956) 223-246.

[52] A. J. Hoffman, A. Kolen and M. Sakarovitch, Characterizations of totally balanced and greedy matrices, *SIAM Journal of Algebraic and Discrete Methods 6* (1985) 721-730.

[53] J.N. Hooker, A quantitative approach to logical inference, *Decision Support Systems 4* (1988) 45-69.

[54] J.N. Hooker, Resolution and the integrality of satisfiability polytopes, *Mathematical Programming 74* (1996) 1-10.

[55] A. Huck and E. Triesh, Perfect matchings in balanced hypergraphs, a combinatorial approach, *Combinatorica 22* (2002) 409-416.

[56] A. Kapoor, On the complexity of finding holes in bipartite graphs, preprint, Carnegie Mellon University (1993).

[57] L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Mathematics 2* (1972) 253-267.

[58] A. Lubiw, Doubly lexical ordering of matrices, *SIAM Journal of Computing 16* (1987) 854-879.

[59] N. Megiddo, On finding primal- and dual-optimal bases, *Journal of Computing 3* (1991) 63-65.

[60] N. J. Nilsson, Probabilistic logic, *Artificial Intelligence 28* (1986) 71-87.

[61] P. Nobili and A. Sassano, $(0, \pm 1)$ Ideal matrices, *Mathematical Programming 80* (1998) 265-281.

[62] J. G. Oxley, *Matroid Theory,* Oxford University Press (1992).

[63] E. Prisner, Hereditary clique-Helly graphs, *J. Combinatorial Mathematics and Combinatorial Computing 14* (1993) 216-220.

[64] A. Schrijver, *Theory of Linear and Integer Programming,* Wiley (1986).

[65] A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency,* Springer-Verlag (2003).

[66] P. D. Seymour, Decomposition of regular matroids, *Journal of Combinatorial Theory B 28* (1980) 305-359.

[67] A. Tamir, A class of balanced matrices arising from location problems, *SIAM Journal on Algebraic and Discrete Methods 4* (1983) 363-370.

[68] A. Tamir, Totally balanced and totally unimodular matrices defined by center location problems, *Discrete Applied Mathematics 16* (1987) 245-263.

[69] K. Truemper, Alpha-balanced graphs and matrices and GF(3)-representability of matroids, *Journal of Combinatorial Theory B 32* (1982) 112-139.

[70] K. Truemper, A decomposition theory for matroids. VII. Analysis of minimal violation matrices, *Journal of Combinatorial Theory B 55* (1992) 302-335.

[71] K. Truemper, *Effective Logic Computation*, Wiley (1998).

[72] K. Truemper and R. Chandrasekaran, Local unimodularity of matrix-vector pairs, *Linear Algebra and its Applications 22* (1978) 65-78.

[73] M. Yannakakis, On a class of totally unimodular matrices, *Mathematics of Operations Research 10* (1985) 280-304.

[74] G. Zambelli, A polynomial recognition algorithm for balanced matrices, preprint (2003).