A Class of Perfect Graphs Containing P_6

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Abstract

Let P_6 denote the induced path on six nodes. We prove that if a perfect graph G contains P_6 as an induced subgraph but not two families introduced by Conforti and Cornuéjols then G is bipartite or disconnected, or $\bar{G}$ has a star cutset.

Keywords: perfect graph, decomposition, star cutset

1 Introduction

1.1 Main Result

In this paper, we follow the definitions and notation in West [11]. A graph G is *perfect* if, for any $W \subseteq V(G)$, the chromatic number of $G(W)$ is equal to the clique number of $G(W)$. Otherwise, it is imperfect. A minimally imperfect graph is an imperfect graph whose proper induced subgraphs are perfect. A well-known result about perfect graphs, which was conjectured by Berge [1] and proved by Lovász [8], is that a graph G is perfect if and only if its complement $\bar{G}$ is perfect. A *hole* is a chordless cycle of length at least four, and a hole is *odd* if it has an odd number of edges. The *strong perfect graph conjecture* (SPGC), also proposed by Berge [1] in 1960, states that a graph is minimally imperfect if and only if it is an odd hole or the complement of an odd hole. This conjecture was proved recently by Chudnovsky, Robertson, Seymour and Thomas [2]. We say that G *contains* H if H is isomorphic to an induced subgraph of G. We say that G is H*-free* if G does not contain H.

A *star cutset* is a node cutset such that one node of the cutset is adjacent to all the other nodes of the cutset. Chvátal [3] showed their importance in the study of perfect graphs. Conforti and Cornuéjols [4] considered a class of perfect graphs that can be decomposed into bipartite graphs and line graphs of bipartite graphs using star cutsets and another decomposition called extended strong 2-joins. These graphs are called WP-free and are defined by excluding two families of induced subgraphs which we will define later. WP-free graphs do not contain $\bar{P}_6$. In this paper, we extend the class of WP-free graphs by allowing

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$\bar{P}_6$ and another family as induced subgraphs. Graphs in this larger class will be called WP′-free. This class of graphs contains all bipartite graphs (and, more generally, all Meyniel graphs [9]), all line graphs of bipartite graphs and all complements of bipartite graphs. The main result of this paper is the following decomposition theorem.

Theorem 1 *Let G be a WP′-free perfect graph. If G contains $\bar{P}_6$ as an induced subgraph, then G has a star cutset, or $\bar{G}$ is a bipartite graph or is disconnected.*

1.2 Notation and Definitions

A node u is *adjacent* to a node set S (or S is adjacent to u) if u is adjacent to at least one node in S. A node u is *not adjacent* to a node set S (or S is not adjacent to u) if u is adjacent to none of S. Node u is *universal* for S if u is adjacent to every node in S. Let S_1 and S_2 be disjoint node sets in G. A path $P(v_1, v_2, \dots, v_n)$ in $G \setminus (S_1 \cup S_2)$ *minimally* connects S_1 and S_2 if P is a chordless path, only v_1 in P is adjacent to S_1 and only v_n in P is adjacent to S_2.

A *wheel* $(H; v)$ consists of a hole H (a chordless cycle with at least four nodes) and a center v such that v has at least three neighbors in H. A wheel is an *odd wheel* if it contains an odd number of triangles. It is easy to check that an odd wheel contains an odd hole. So a perfect graph cannot contain an odd wheel.

A wheel $(H; v)$ is called a *twin wheel* or *T-wheel* if v has exactly three neighbors in H and these three neighbors induce a path. A wheel $(H; v)$ is called a Δ-*free* wheel if $(H; v)$ induces a triangle-free graph. A wheel $(H; v)$ is called a *universal* wheel if v is adjacent to every node in H. A wheel $(H; v)$ is called a *line wheel* or *L-wheel* if it contains exactly two triangles and these two triangles have only the center v in common. A wheel is called a *proper wheel* if it is in none of the above four classes.

An *L-parachute* $LP(a_1, b_1, a_2, b_2, a_3, z)$ is a graph induced by an L-wheel $(H; a_3)$ where $H = a_1, b_1, \dots, z, \dots, b_2, a_2, \dots, a_1$, and a_1, a_2, b_1 and b_2 are the neighbors of a_3 in H, together with a chordless path $P(a_3, \dots, z)$ of length greater than 1 (i.e. with at least two edges). No node of $H \setminus \{z, b_1\}$ may be adjacent to an intermediate node of P.

A *T-parachute* $TP(a_1, a_2, b_1, b_2, z)$ (see Fig.1) is a graph induced by an T-wheel $(H; a_2)$ where $H = b_1, a_1, b_2, \dots, z, \dots, b_1$, and a_1, b_1 and b_2 are the neighbors of a_2 in H, together with a chordless path $P(a_2, \dots, z)$ of length greater than 1. No node of $H \setminus \{z, b_1\}$ may be adjacent to an intermediate node of P. Sometimes, we use $TP(a_1, a_2, b_1, b_2, z, u_1, u_2, \dots, u_m)$ to denote the T-parachute where $u_1, u_2, \dots, u_m$ are all the other nodes in the T-parachute.

A *parachute* is either an L-parachute or a T-parachute. A graph is *WP-free* if it contains neither a proper wheel nor a parachute. All these definitions were introduced by Conforti and Cornuéjols [4]. A graph is *WP′-free* if it contains neither a proper wheel nor a T-parachute other than $\bar{P}_6$. Notice that the class of WP′-free graphs contains bipartite graphs, complements of bipartite graphs and line graphs of bipartite graphs (since any proper wheel and T-parachute other than $\bar{P}_6$ contains a triangle, a stable set of size 3 and either a claw or a diamond).

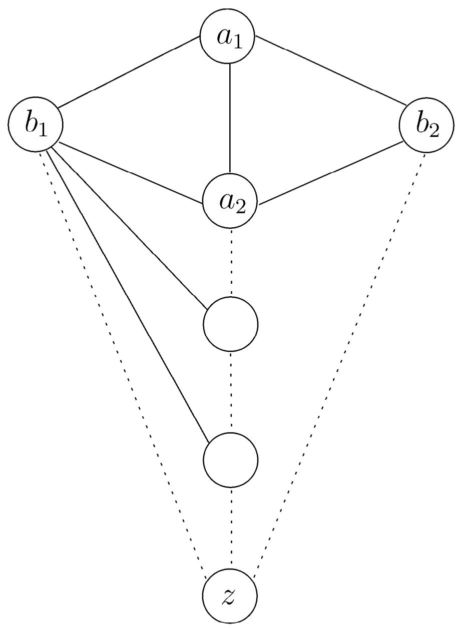


Figure 1: a T-parachute

1.3 Motivation

Bipartite graphs, line graphs of bipartite graphs and the complements of these graphs are perfect graphs. Let us call *basic graph* a graph in one of these four classes. What is the structure of nonbasic perfect graphs? Chudnovsky, Robertson, Seymour and Thomas [2] showed that nonbasic perfect graphs contain a skew partition, a 2-join or its complement or a homogeneous pair. In this paper, we focus on a subclass of perfect graphs that have a finer structure. Conforti and Cornuéjols [4] showed that, if a nonbasic perfect graph G contains no proper wheel or parachute, then G a star cutset or an extended strong 2-join or $\bar{G}$ is disconnected. Perfect graphs with proper wheels are not basic, and perfect graphs with big parachutes (all parachutes other than the two graphs with 6 nodes in Fig.2) are not basic, either. The graph in Fig.2(a), called L_6, is the complement of the line graph of a bipartite graph, and the graph in Fig.2(b) is a co-bipartite graph. So both are in basic classes. Our motivation is to generalize the result in Conforti and Cornuéjols [4] by allowing $\bar{P}_6$ as an induced subgraph. It follows from Theorem 1 and the results in [4] that all perfect graphs G that contain no proper wheel, no big parachute and no L_6 can be decomposed into bipartite graphs, line graphs of bipartite graphs and complements of bipartite graphs using star cutsets and extended strong 2-joins, or $\bar{G}$ is disconnected. Recently, Conforti, Cornuéjols and Zambelli [5] proved a decomposition theorem for perfect graphs G such that neither G nor $\bar{G}$ contains a proper wheel or a long prism. This class does not contain the class studied in this paper and vice versa.

We call the graph of Fig.2(b) a $\bar{P}_6$ because its complement is a chordless path with 6 nodes. Notice that in this graph, nodes 6 and 2 are symmetric, nodes 5 and 3 are symmetric, and

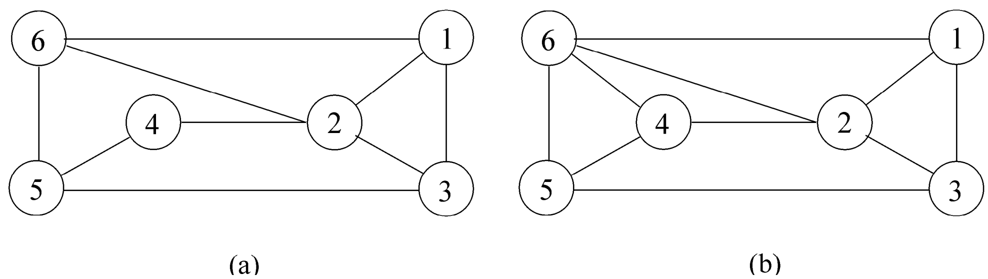


Figure 2: L_6 and $\bar{P}_6$

nodes 4 and 1 are symmetric. This symmetry will be used in the proofs.

1.4 Proof of Theorem 1

In this paper, we consider a WP$'$-free perfect graph G that contains $\bar{P}_6$. *In the rest of this paper, when we refer to a T-parachute, it will always be a T-parachute other than* $\bar{P}_6$.
To prove Theorem 1, we first prove the following result.

Theorem 2 *Let G be a WP$'$-free perfect graph that contains $\bar{P}_6$. Let Σ be the node set of a $\bar{P}_6$ of G and let x be a node in $G \setminus \Sigma$ adjacent to Σ. Then G has a star cutset or $\Sigma \cup \{x\}$ induces a co-bipartite graph.*

The proof of this theorem is given in Section 2. Using this result, we then prove the following theorem in Section 3, which implies Theorem 1.
Given a graph F, define an auxiliary graph H as follows. The nodes of H correspond to the P_6's of F. Two nodes of H are adjacent if and only if the corresponding P_6's have at least one edge in common. We say that an induced subgraph B of F is *maximally P_6-connected* if B contains at least one P_6 and the nodes of H corresponding to the P_6's of B induce a connected component of H.

Theorem 3 *Let G be a WP$'$-free perfect graph that contains a $\bar{P}_6$ and let B be a maximally P_6-connected induced subgraph of $\bar{G}$. If G has no star cutset then B is a bipartite connected component of $\bar{G}$.*

2 Proof of Theorem 2

In this section, we prove Theorem 2. Let G be a WP$'$-free perfect graph that contains a $\bar{P}_6$. Let Σ denote the node set of a $\bar{P}_6$ of G. We label the $\bar{P}_6$ as in Fig.2. We prove Theorem 2 by first enumerating the possible adjacencies of a node x in $G \setminus \Sigma$ to the node set Σ. As shown in Lemma 4, the possible adjacencies can be divided into four classes. Then we prove that G has a star cutset in the last three cases (Lemmas 6 and 7). Theorem 2 follows.

Lemma 4 *If a node x in $G \setminus \Sigma$ is adjacent to Σ then only one of the following cases occurs.*

1) $\Sigma \cup \{x\}$ induces a co-bipartite graph.

2) The subgraph induced by $\Sigma \cup \{x\}$ is one of the graphs in Fig.3. These two graphs are isomorphic.

3) The subgraph induced by $\Sigma \cup \{x\}$ is one of the graphs in Fig.4. Each of them is the complement of the line graph of a bipartite graph. The graph in Fig.4(a) is isomorphic to the graph in Fig.4(b).

4) $N(x) \cap \Sigma$ induces a clique distinct from $\{1, 2, 3\}$ or $\{4, 5, 6\}$.

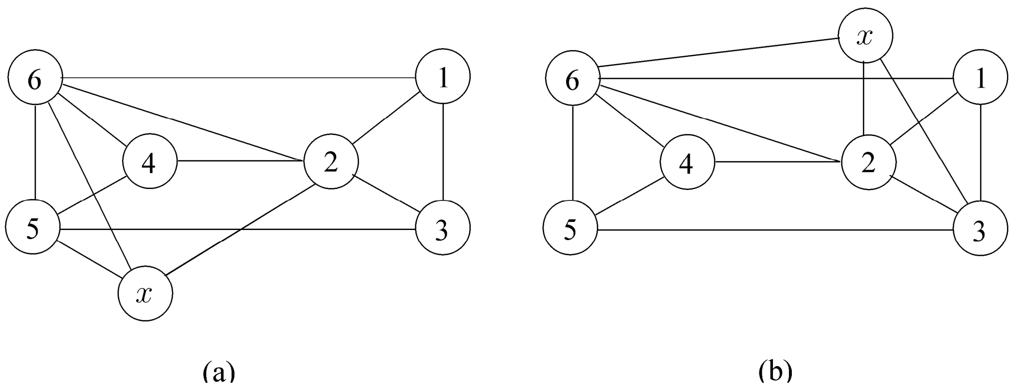


Figure 3:

Proof: If x is universal for $\{1, 2, 3\}$ or for $\{4, 5, 6\}$ then 1) holds. If x is adjacent to none of $\{1, 2, 3\}$ or to none of $\{4, 5, 6\}$ then 1) or 4) holds. We consider the remaining cases as follows.

Case 1 x is adjacent to exactly one of 1, 2, 3 and one of 4, 5, 6. There are 9 cases.

If $N(x) \cap \Sigma = \{1, 6\}$, $\{2, 4\}$, $\{2, 6\}$, or $\{3, 5\}$, then $N(x) \cap \Sigma$ induces a clique. The remaining five cases are as follows. If $N(x) \cap \Sigma = \{1, 4\}$ then G contains a 5-hole $(5, 3, 1, x, 4, 5)$. If $N(x) \cap \Sigma = \{1, 5\}$ then G contains a 5-hole $(5, 4, 2, 1, x, 5)$. If $N(x) \cap \Sigma = \{2, 5\}$ then we have a T-parachute $TP(1, 2, 6, 3, 5, x)$. If $N(x) \cap \Sigma = \{3, 4\}$ then G contains a 5-hole $(4, x, 3, 1, 6, 4)$. If $N(x) \cap \Sigma = \{3, 6\}$ then we have a T-parachute $TP(4, 6, 5, 2, 3, x)$.

Case 2 x is adjacent to exactly one of 1, 2, 3 and two of 4, 5, 6. There are also 9 cases.

Case 2.1 x is adjacent to 1 but not 2 or 3.

If $N(x) \cap \Sigma = \{1, 4, 5\}$ then we have a T-parachute $TP(4, 5, 6, x, 1, 3)$. If $N(x) \cap \Sigma = \{1, 4, 6\}$ then G contains a 5-hole $(5, 3, 1, x, 4, 5)$. If $N(x) \cap \Sigma = \{1, 5, 6\}$ then G contains a 5-hole $(5, 4, 2, 1, x, 5)$.

Case 2.2 x is adjacent to 2 but not 1 or 3.

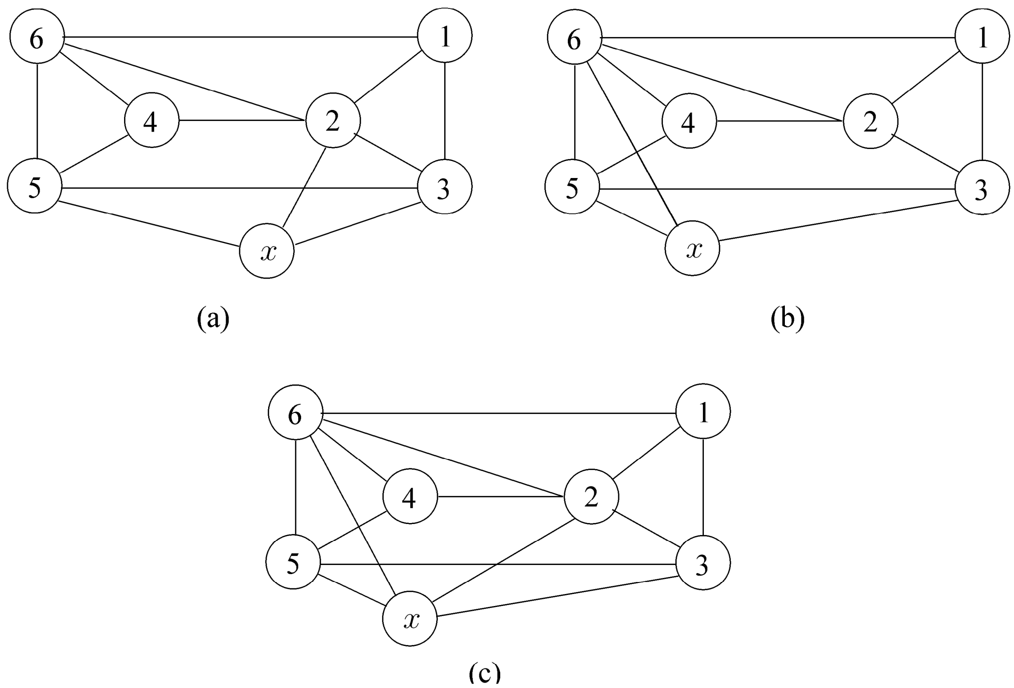


Figure 4:

If $N(x)\cap\Sigma = \{2,4,5\}$ then we have a T-parachute $TP(1,2,6,3,5,x)$. If $N(x)\cap\Sigma = \{2,4,6\}$ then $N(x)\cap\Sigma$ induces a clique. If $N(x)\cap\Sigma = \{2,5,6\}$ then $G(\Sigma\cup\{x\})$ is the graph in Fig.3(a).

Case 2.3 x is adjacent to 3 but not 1 or 2.

If $N(x)\cap\Sigma = \{3,4,5\}$ then G contains a 5-hole $(4,x,3,1,6,4)$. If $N(x)\cap\Sigma = \{3,4,6\}$ then we have a T-parachute $TP(4,6,5,x,3,1)$. If $N(x)\cap\Sigma = \{3,5,6\}$ then $G(\Sigma\cup\{x\})$ is the graph in Fig.4(b).

Case 3 x is adjacent to exactly two of 1, 2, 3 and one of 4, 5, 6. There are also 9 cases.

By symmetry, we get either contradictions or the graph in Fig.3(b) or Fig.4(a).

Case 4 x is adjacent to exactly two of 1, 2, 3 and two of 4, 5, 6. There are also 9 cases.

Case 4.1 x is adjacent to 1 and 2 but not 3.

If $N(x)\cap\Sigma = \{1,2,4,5\}$ then we have a T-parachute $TP(4,5,6,x,1,3)$. If $N(x)\cap\Sigma = \{1,2,4,6\}$ then G contains a 5-hole $(5,3,1,x,4,5)$. If $N(x)\cap\Sigma = \{1,2,5,6\}$ then we have a T-parachute $TP(1,2,x,3,5,4)$.

Case 4.2 x is adjacent to 1 and 3 but not 2.

If $N(x)\cap\Sigma = \{1,3,4,5\}$ then the graph induced by $\{1,2,3,4,5,6,x\}$ is the complement of a 7-hole. If $N(x)\cap\Sigma = \{1,3,4,6\}$ then we have a T-parachute $TP(1,3,2,x,4,5)$. If $N(x)\cap\Sigma = \{1,3,5,6\}$ then G contains a 5-hole $(5,4,2,1,x,5)$.

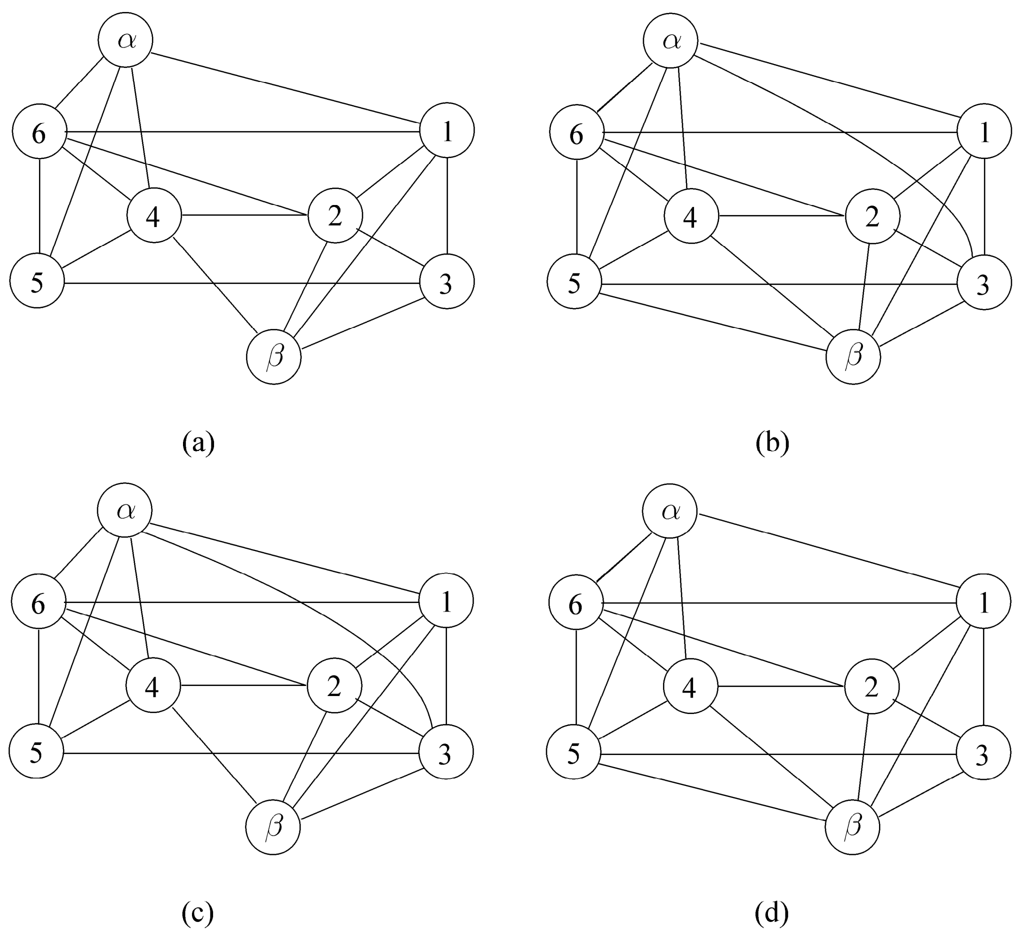
Case 4.3 x is adjacent 2 and 3 but not 1.

If $N(x)\cap\Sigma = \{2,3,4,5\}$ then G contains a 5-hole $(4,x,3,1,6,4)$. If $N(x)\cap\Sigma = \{2,3,4,6\}$

then we have a T-parachute $TP(4,6,5,x,3,1)$. If $N(x)\cap\Sigma=\{2,3,5,6\}$ then $G(\Sigma\cup\{x\})$ is the graph in Fig.4(c).
This completes the proof. $\square$

It is worth looking at the complements of the graphs in Fig.3 and 4. Their complements contain a $P_6(6,3,4,1,5,2)$ and x is adjacent to 1, 4 and at most one of 2, 3, 5 and 6.

Lemma 5 *If G has neither the star cutset $N(2)\cup\{2\}\setminus\{1,4\}$ (center 2) nor the star cutset $N(6)\cup\{6\}\setminus\{1,4\}$ (center 6), then G contains one of the graphs in Fig.5.*



α may be adjacent to β.

Figure 5:

Proof: We prove this lemma by contradiction.
Let $S=N(2)\cup\{2\}\setminus\{1,4\}$. Suppose that S is not a star cutset (center 2). Let $P(x_1,...,x_n)$ be any path in $G\setminus(S\cup\{1,4,5\})$ that minimally connects node 1 and $\{4,5\}$. x_1 is adjacent

to node 1. x_n is adjacent to $\{4,5\}$. Nodes of P may be adjacent to node 3 or 6 but not 2. Suppose that P contains only a single node x_1. Then by Lemma 4, x_1 is adjacent to 1, 4, 5, 6 (and possibly 3) but not 2. Thus we get node α of Fig.5. We assume now that P contains at least one edge. We consider the following cases.

Case 1 x_n is adjacent to node 4 but not 5.

If no node of P is adjacent to 3 then G contains an odd wheel $(5,3,1,P,4,5;2)$. Therefore, $V(P)$ must be adjacent to 3. By Lemma 4, x_n cannot be adjacent to 3. Let x_i be the node of largest index adjacent to 3. Let Q be the subpath of P from x_i to x_n. If no node of $V(Q)$ is adjacent to 6 then G contains an odd wheel $(4,Q,3,1,6,4;2)$. Hence, Q has a node adjacent to 6. Let $x_j (j \geq i)$ be the node of smallest index adjacent to 6. Let Q' be the subpath of P from x_i to x_j. If $j \neq n$ then we have a T-parachute $TP(4,6,5,2,3,Q')$. If $j = n$ then we have a T-parachute $TP(4,6,5,x_n,3,1,Q)$.

Case 2 x_n is adjacent to both node 4 and 5.

$V(P)$ must be adjacent to both 3 and 6. Otherwise, we have a T-parachute $TP(2,1,6,3,5,P)$. The wheel $(4,P,1,2,4;6)$ is either a proper wheel or a universal wheel. The wheel $(4,P,1,2,4;3)$ is either a proper wheel, a T-wheel or a line wheel. If $(4,P,1,2,4;6)$ is a universal wheel, and $(4,P,1,2,4;3)$ is a T-wheel or a line wheel, then a node y in $P \setminus \{x_n\}$ is adjacent to both 3 and 6 (and possibly 1), which contradicts Lemma 4.

Case 3 x_n is adjacent to node 5 but not 4.

The wheel $(5,P,1,2,4,5;6)$ is either a proper wheel or a universal wheel. The wheel $(5,P,1,2,4,5;3)$ is either a proper wheel or a line wheel. If $(5,P,1,2,4,5;6)$ is a universal wheel and $(5,P,1,2,4,5;3)$ is a line wheel, then the wheel $(3,P,1,3;6)$ is a proper wheel unless P contains exactly one edge. In this case, x_1 is adjacent to 1 and 6, and x_2 is adjacent to 3, 5, 6 and x_1, and these are the only adjancies between x_1, x_2 and Σ.

By symmetry, if $N(6) \cup \{6\} \setminus \{1,4\}$ (center 6) is not a star cutset, then either there is a node y_1 adjacent to 1, 2, 3, 4, (and possibly 5) but not 6 (this is node β of Figure 5), or there is an edge (y_1, y_2) with y_1 adjacent to only 2, 4 and y_2 adjacent to only 2, 3, 5 in Σ (symmetrical to Case 3 above). To complete the proof of this lemma, we show that the two following cases lead to a contradiction.

Case i G contains nodes x_1 , x_2 as in Case 3 above and y_1 adjacent to 1, 2, 3, 4 (and possibly 5) but not 6.

If y_1 is not adjacent to x_2 then G contains a 5-hole $(x_2,3,y_1,4,6,x_2)$. Otherwise, if y_1 is adjacent to x_2 then we have a T-parachute $TP(3,y_1,1,x_2,6,4)$.

Case ii G contains nodes x_1 , x_2 as in Case 3 above and an edge (y_1,y_2) with y_1 adjacent to only 2, 4 and y_2 adjacent to only 2, 3, 5 in Σ.

If x_1 is adjacent to y_2 then we have a T-parachute $TP(1,6,x_1,2,y_2,5)$. Therefore, x_1 cannot be adjacent to y_2. Then G contains a proper wheel $(x_2,5,y_2,2,1,x_1,x_2;6)$ if x_2 is not adjacent to y_2, or a 5-hole $(x_2,y_2,2,1,x_1,x_2)$ if x_2 is adjacent to y_2. □

Lemma 6 *If G contains any of the graphs in Fig.3 or Fig.4 then G has a star cutset.*

Proof: This lemma holds by the following claims.

Claim 1 If G contains any graph in Fig.3 then G has a star cutset.

Proof of Claim 1 The graphs in Fig.3(a) and (b) are isomorphic. We prove that if G contains the graph in Fig.3(b) then G has a star cutset. It follows from Lemma 5 that there exists a node β that has the same neighbors in Σ as in Fig.5.

If β is not adjacent to x then G contains a 5-hole $(6, 4, \beta, 3, x, 6)$. Otherwise, if β is adjacent to x then we have a T-parachute $TP(3, \beta, 1, x, 6, 4)$. $\diamond$

Claim 2 If G contains any graph in Fig.4 then G has a star cutset.

Proof of Claim 2 Notice that the graphs in Fig.4(a) and (b) are isomorphic. We prove that if G contains the graph in Fig.4(a) or (c) then G has a star cutset. By Lemma 5, there exists a node α that has the same neighbors in Σ as in Fig.5.

If α is not adjacent to x then G contains a 5-hole $(x, 2, 1, \alpha, 5, x)$. Otherwise, if α is adjacent to x then we have a T-parachute $TP(5, \alpha, x, 4, 2, 1)$. $\diamond$

$\square$

Lemma 7 *If a node $x \in G \setminus \Sigma$ is adjacent to Σ and $N(x) \cap \Sigma$ induces a clique distinct from $\{1, 2, 3\}$ or $\{4, 5, 6\}$ then G has a star cutset.*

Proof: We prove this by contradiction. Let $K = N(x) \cap \Sigma$. By symmetry we can assume that $K \cap \{4, 5, 6\} \neq \emptyset$. We break the proof into the following steps.

Claim 1 If K contains node 5 then G has a star cutset.

Proof of Claim 1 This claim covers $K = \{5\}$, $\{5, 3\}$, $\{5, 4\}$ or $\{5, 6\}$. Let $S = N(5) \cup \{5\} \setminus \{x\}$. Suppose that S is not a star cutset (center 5). Let $P(x_1, ..., x_n)$ be any path in $G \setminus (S \cup \{x, 1, 2\})$ that minimally connects node x and $\{1, 2\}$. x_1 is adjacent to node x. x_n is adjacent to $\{1, 2\}$. $V(P)$ may be adjacent to node 3, 4 or 6 but not 5. Let P' be the path induced by $V(P) \cup \{x\}$. We also use x_0 to denote x for convenience. We consider the following cases.

Case 1 x_n is adjacent to 1 but not 2.

$V(P')$ is adjacent to 4 or x is adjacent to 3. Otherwise, G contains a proper wheel $(x, P, 1, 2, 4, 5, x; 3)$. Suppose that $V(P')$ is not adjacent to 4. This implies that x is adjacent to 3. Then node 6 must be universal for $V(P')$. Otherwise, G contains a proper wheel $(x, P, 1, 2, 4, 5, x; 6)$. This contradicts the fact that x is not adjacent to both 3 and 6. Therefore, $V(P')$ must be adjacent to 4. If $V(P')$ is not adjacent to both 3 and 6 then we have a T-parachute $TP(2, 1, 6, 3, 5, P, x)$ or a T-parachute $TP(2, 1, 3, 6, 5, P, x)$. Hence, $V(P')$ is adjacent to 3, 4 and 6. Let $Q(y_1, ..., y_m)$ be a minimal subpath of P' such that $V(Q)$ is adjacent to 3 and 4. We assume w.l.o.g. that y_1 is adjacent to 4 and y_m is adjacent to 3. Suppose that $V(Q)$ contains x. Notice that x cannot be adjacent to both 3 and 4 by our assumption. If y_1 is x then G contains an odd wheel $(4, x, Q, 3, 2, 4; 5)$. Otherwise, if y_m is x then G contains an odd wheel $(3, 2, 4, Q, x, 3; 5)$. Hence, $V(Q)$ cannot contain x. Now we consider the following cases.

Case 1.1 Q does not contain x_n.

If $V(Q)$ is not adjacent to 6 then G contains an odd wheel $(4, Q, 3, 1, 6, 4; 2)$. Therefore, $V(Q)$ must be adjacent to 6. Let y_i be the node of largest index adjacent to 6. Let Q' denote the subpath of Q from y_i to y_m. If $i \neq 1$ then we have a T-parachute $TP(4, 6, 5, 2, 3, Q')$. Otherwise, if $i = 1$ then we have a T-parachute $TP(4, 6, 5, y_1, 3, 1, Q)$.

Case 1.2 Q contains x_n.

By Lemma 4, y_1 cannot be x_n. Then we have a T-parachute $TP(1, 3, 2, x_n, 4, 5, Q)$.

Case 2 x_n is adjacent to 2 but not 1.

If $V(P')$ is not adjacent to both 3 and 6 then we have a T-parachute $TP(1, 2, 3, 6, 5, P, x)$ or a T-parachute $TP(1, 2, 6, 3, 5, P, x)$. Therefore, $V(P')$ must be adjacent to both 3 and 6. Let $Q(y_1, ..., y_m)$ be a minimal subpath of P' such that $V(Q)$ is adjacent to 3 and 6. We assume w.l.o.g. that y_1 is adjacent to 6 and y_m is adjacent to 3.

Suppose that $V(Q)$ contains x. Notice that x cannot be adjacent to both 3 and 6 by our assumption. If y_1 is x then G contains an odd wheel $(6, x, Q, 3, 1, 6; 5)$. Otherwise, if y_m is x then G contains an odd wheel $(6, Q, x, 3, 1, 6; 5)$. Hence, $V(Q)$ does not contain x.

Suppose that Q does not contain x_n. If $V(Q)$ is not adjacent to 4 then we have a T-parachute $TP(4, 6, 5, 2, 3, Q)$. Hence, $V(Q)$ must be adjacent to 4. Let y_i be the node of largest index adjacent to 4. Let Q' denote the subpath of Q from y_i to y_m. If $i \neq 1$ then G contains an odd wheel $(4, Q', 3, 1, 6, 4; 2)$. Otherwise, if $i = 1$ then we have a T-parachute $TP(4, 6, y_1, 5, 3, 1, Q)$. Therefore, any minimal subpath Q of P that is adjacent to 3 and 6 must contain x_n.

If Q contains only node x_n, then G has a star cutset by Lemmas 4 and 6. So we can assume that Q contains at least one edge. If y_1 is x_n then G contains an odd wheel $(5, 3, Q, 6, 5; 2)$. If y_m is x_n then G contains an odd wheel $(6, Q, 3, 1, 6; 2)$.

Case 3 x_n is adjacent to both 1 and 2.

Suppose that $V(P')$ is not adjacent to 4. If $V(P')$ is not adjacent to 6, either, then we have a T-parachute $TP(1, 2, 6, x_n, 5, 4, P, x)$. Hence, $V(P')$ is adjacent to 6. Then node 6 is universal for $V(P')$ since otherwise, G contains a proper wheel $(5, x, P, 2, 4, 5; 6)$. Furthermore, $V(P')$ is adjacent to 3 since otherwise, G contains a proper wheel $(5, x, P, 2, 3, 5; 6)$. Notice that x cannot be adjacent to both 3 and 6 by our assumption. Then the wheel $(5, x, P, 2, 4, 5; 3)$ is either a proper wheel or a Δ-free wheel. If it is a Δ-free wheel then there exists a node $y \in V(P) \setminus \{x_n\}$ adjacent to both 3 and 6. This implies that we have a T-parachute $TP(4, 6, 5, 2, 3, y)$. Therefore, $V(P')$ must be adjacent to 4.

Let x_i be the node of largest index adjacent to 4. Let Q denote the subpath of P' from x_i to x_n. Suppose $i = 0$ (x is adjacent to 4). Notice that x cannot be adjacent to both 3 and 4 by our assumption. If $V(P)$ is not adjacent to 3 then G contains an odd wheel $(5, x, P, 2, 3, 5; 4)$. Hence, $V(P)$ is adjacent to 3. Let x_j be the node of smallest index adjacent to 3. Let W be the subpath of P' from x to x_j. Suppose that $j = n$. If $V(P')$ is not adjacent to 6 then G contains an odd wheel $(6, 4, x, P, 1, 6; 2)$. Hence, $V(P')$ is adjacent to 6. If x_n is not adjacent to 6 (this implies $|V(P)| \geq 2$) then the wheel $(4, x, P, 2, 4; 6)$ or the wheel $(5, x, P, 3, 5; 6)$ is a proper wheel since x cannot be adjacent to both 4 and 6 by our assumption. However, if x_n is adjacent to 6 then we have a T-parachute $TP(1, x_n, 6, 3, 5, P, x)$. Hence, $j \neq n$. Then G contains an odd wheel $(4, x, W, 3, 2, 4; 5)$. Therefore, $i \neq 0$. In the rest of the proof we assume that $i \geq 1$.

If $V(Q)$ is not adjacent to 3 then G contains a proper wheel $(5, 3, 1, x_n, Q, 4, 5; 2)$. Hence, $V(Q)$ must be adjacent to 3. Let $j(j \geq i)$ be the smallest index such that x_j is adjacent to 3. We use Q' to denote the subpath of P from x_i to x_j.

Case 3.1 $j \neq n$. This also implies that $i \neq n$.

If $V(Q')$ is not adjacent to 6 then G contains an odd wheel $(4, Q', 3, 1, 6, 4; 2)$. Hence, $V(Q')$ is adjacent to 6. Let $k(k \leq j)$ be the largest index such that x_k is adjacent to 6. We use Q'' to denote the subpath of P from x_k to x_j. If $k \neq i$ then we have a T-parachute $TP(4, 6, 5, 2, 3, Q'')$. Otherwise, if $k = i$ then we have a T-parachute $TP(4, 6, 5, x_i, 3, 1, Q')$.

Case 3.2 $j = n$.

If $i \neq n$ then G contains an odd wheel $(5, 3, x_n, Q, 4, 5; 2)$. Hence, $i = n$. Let $P'' = P' \setminus \{x_n\}$. Suppose that $V(P'')$ is not adjacent to 3. Then $n = 1$ and x is adjacent to 4 since, otherwise we have a T-parachute $TP(2, x_n, 4, 3, 5, P, x)$. By our assumption, x cannot be adjacent to 6. Then x_1 must be adjacent to 6 since otherwise, G contains a 5-hole $(5, x, x_1, 2, 6, 5)$. But now we have a T-parachute $TP(2, x_1, 6, 3, 5, x)$. Therefore $V(P'')$ is adjacent to 3. Suppose that $V(P'')$ is not adjacent to 4. Then $n = 1$ and x is adjacent to 3 since, otherwise we have a T-parachute $TP(2, x_n, 3, 4, 5, P, x)$. Then x_1 must be adjacent to 6 since otherwise, G contains a 5-hole $(5, x, x_1, 2, 6, 5)$. In this case, the graph induced by $\{1, 3, 4, 5, 6, x_1, x\}$ is isomorphic to the graph in Fig.4(a) (notice that $\{1, 3, 4, 5, 6, x_1\}$ induces a $\bar{P}_6$). By Lemma 6, G has a star cutset.

Therefore, $V(P'')$ must be adjacent to both 3 and 4. Let $W(z_1, ..., z_m)$ be a minimal subpath of P'' such that $V(W)$ is adjacent to both 3 and 4. We assume w.l.o.g. that z_1 is adjacent to 4 and z_m is adjacent to 3. If $V(W)$ is not adjacent to 6 then G contains an odd wheel $(4, W, 3, 1, 6, 4; 2)$. Hence, $V(W)$ is adjacent to 6. Let l be the largest index such that z_l is adjacent to 6. We use W' to denote the subpath of W from z_l to z_m. Notice that x cannot be adjacent to both 4 and 6. If $l \neq 1$ then we have a T-parachute $TP(4, 6, 5, 2, 3, W')$. Otherwise, if $l = 1$ then we have a T-parachute $TP(4, 6, 5, z_1, 3, 1, W)$. This completes the proof. ◇

Claim 2 If K contains node 6 but not 5 then G has a star cutset.

Proof of Claim 2 By symmetry between $\{1, 2, 6\}$ and $\{2, 4, 6\}$, we can assume that $K \neq \{1, 2, 6\}$. Therefore, this case covers the following $K = \{6\}$, $\{6, 1\}$, $\{6, 2\}$, $\{6, 4\}$ or $\{6, 2, 4\}$. Let $S = N(6) \cup \{6\} \setminus \{x\}$. Suppose that S is not a star cutset (center 6). Let $P(x_1, ..., x_n)$ be any path in $G \setminus (S \cup \{x, 3\})$ that minimally connects node x and 3. x_1 is adjacent to node x. x_n is adjacent to 3. $V(P)$ may be adjacent to node 1, 2, 4 or 5 but not 6. By Claim 1, we can assume that $N(x_n) \cap \Sigma \neq \{3\}$, $\{3, 1\}$, $\{3, 2\}$ and $\{3, 5\}$, since node 3 is symmetric to node 5. It follows from Lemmas 4 and 6 that x_n is also adjacent to 1 and 2 (and possibly 4, 5). We consider the following cases.

Case 1 P contains only a single node x_1.

Notice that x cannot be adjacent to 5 by our assumption. If x_1 is not adjacent to 5 then G contains an odd wheel $(6, 5, 3, x_1, x, 6; 2)$. Therefore, x_1 must be adjacent to 5. Notice also that x cannot be adjacent to both 1 and 2 by our assumption. Then we have a T-parachute

$TP(3, x_1, 1, 5, 6, x)$ or a T-parachute $TP(3, x_1, 2, 5, 6, x)$.

Case 2 P contains at least one edge.

Suppose that $V(P)$ is not adjacent to 5. Then the wheel $(x, P, 3, 5, 6, x; 2)$ is either a proper wheel or a line wheel. The wheel $(x, P, 3, 5, 6, x; 1)$ is either a proper wheel or a line wheel. If both the wheel $(x, P, 3, 5, 6, x; 1)$ and the wheel $(x, P, 3, 5, 6, x; 2)$ are line wheels then x is adjacent to both 1 and 2. This contradicts our assumption. Therefore, $V(P)$ is adjacent to 5. Let P' denote the path induced by $V(P) \cup \{x\}$. Let $Q(y_1, ..., y_m)$ be a minimal subpath of P' such that $V(Q)$ is adjacent to both 5 and 1. We assume w.l.o.g. that y_1 is adjacent to 5 and y_m is adjacent to 1.

y_1 cannot be x by our assumption. Suppose that y_m is x. That is, x is adjacent 1. Then Q contains at least one edge by our assumption. If y_1 is not x_n then G contains an odd wheel $(5, 3, 1, x, Q, 5; 6)$. Otherwise, if y_1 is x_n then we have a T-parachute $TP(3, x_n, 1, 5, 6, Q, x)$. Therefore, y_m is not x, either.

Case 2.1 Q does not contain x_n.

Case 2.1.1 $V(Q)$ is not adjacent to 4.

If $V(Q)$ is not adjacent to 2 then G contains an odd wheel $(5, 4, 2, 1, Q, 5; 3)$. Therefore, $V(Q)$ is adjacent to 2. Let y_i be the node of smallest index adjacent to 2. If $i = m$ then we have a T-parachute $TP(1, 2, y_m, 3, 5, 4, Q)$. Otherwise, if $i \neq m$ then we have a T-parachute $TP(1, 2, 6, 3, 5, Q')$, where Q' is the subpath of Q from y_1 to y_i.

Case 2.1.2 $V(Q)$ is adjacent to 4.

Let y_j be the node of largest index adjacent to 4. Let Q'' be the subpath of Q from y_j to y_m. If $j \neq 1$ then the wheel $(5, 3, 1, Q'', 4, 5; 2)$ is either a proper wheel or a line wheel. If it is a line wheel then the wheel $(6, 4, Q'', 1, 6; 2)$ is a proper wheel. If $j = 1$ then we have a T-parachute $TP(4, 5, 6, y_1, 1, 3, Q)$.

Case 2.2 Q contains x_n.

Case 2.2.1 Q contains at least one edge.

This implies that y_m is x_n since x_n is adjacent to 1. Then G contains an odd wheel $(6, 5, Q, x_n, 1, 6; 3)$.

Case 2.2.2 Q contains only node x_n.

Let $P'' = P' \setminus \{x_n\}$. $V(P'')$ must be adjacent to both 5 and 1. Otherwise, we have a T-parachute $TP(3, x_n, 1, 5, 6, P, x)$ or a T-parachute $TP(3, x_n, 5, 1, 6, P, x)$. Let $W(z_1, ..., z_k)$ be a minimal subpath of P'' such that $V(W)$ is adjacent to 5 and 1. We assume w.l.o.g. that z_1 is adjacent to 5 and z_k is adjacent to 1.

z_1 cannot be adjacent to x by our assumption. Suppose that z_k is x. This implies that W contains at least one edge. Then G contains an odd wheel $(5, 3, 1, x, W, 5; 6)$. Hence, z_k cannot be x, either.

If $V(W)$ is not adjacent to 2 then we have a T-parachute $TP(2, 1, 6, 3, 5, W)$. Therefore, $V(W)$ is adjacent to 2. Let z_l be the node of smallest index adjacent to 2. Let W' be the subpath of W from z_1 to z_l. If $l \neq k$ then we have a T-parachute $TP(1, 2, 6, 3, 5, W')$. Hence, $l = k$. If $V(W)$ is not adjacent to 4 then we have a T-parachute $TP(1, 2, z_k, 3, 5, 4, W)$. Therefore, $V(W)$ is adjacent to 4. Notice that W contains at least one edge by Lemma 4. The wheel $(5, W, 2, 6, 5; 4)$ is either a universal wheel or a proper wheel. If it is a universal wheel then the wheel $(5, W, 2, 3, 5; 4)$ is a proper wheel. ◇

Claim 3 If $N(x) \cap \Sigma = \{4\}$ then G has a star cutset.

Proof of Claim 3 Let $S = N(4) \cup \{4\} \setminus \{x\}$. Suppose that S is not a star cutset. Let $P(x_1, ..., x_n)$ be any path in $G \setminus (S \cup \{x, 1, 3\})$ that minimally connects node x and $\{1, 3\}$. x_1 is adjacent to node x. x_n is adjacent to $\{1, 3\}$. By Claims 1 and 2 (and Lemmas 4 and 6), we can assume that $N(x_n) \cap \Sigma = \{1\}$ or $N(x_n) \cap \Sigma \supseteq \{1, 2, 3\}$ since 3 is symmetric to 5 and 2 is symmetric to 6. $V(P)$ may be adjacent to node 2, 5 or 6 but not 4.

Case 1 $N(x_n) \cap \Sigma = \{1\}$

$V(P)$ must be adjacent to 5. Otherwise, G contains a proper wheel $(5, 3, 1, P, x, 4, 5; 2)$. Let x_i be the node of largest index adjacent to 5. Let Q be the subpath of P from x_i to x_n. If $V(Q)$ is not adjacent to 2 then G contains an odd wheel $(5, 4, 2, 1, Q, 5; 3)$. Hence, $V(Q)$ must be adjacent to 2. Let $j (j \geq i)$ be the smallest index such that x_j is adjacent to 2. Let Q' be the subpath of P from x_i to x_j. If Q' contains at least one edge then we have a T-parachute $TP(1, 2, 6, 3, 5, Q')$. Therefore, Q' contains only node x_i. If x_i is not adjacent to 6 then we still have a T-parachute $TP(1, 2, 6, 3, 5, Q')$. Hence, x_i is adjacent to 6. Then G contains the graph in Fig.3(a). By Lemma 6, G has a star cutset.

Case 2 $N(x_n) \cap \Sigma \supseteq \{1, 2, 3\}$

If $V(P)$ is not adjacent to 5 then G contains a proper wheel $(5, 3, x_n, P, x, 4, 5; 2)$. Therefore, $V(P)$ must be adjacent to 5.

Case 2.1 x_n is adjacent to 5.

Let $P' = P \setminus \{x_n\}$. Then $n > 1$ and $V(P')$ must be adjacent to both 2 and 5. Otherwise, we have a T-parachute $TP(3, x_n, 2, 5, 4, P, x)$ or a T-parachute $TP(3, x_n, 5, 2, 4, P, x)$. Let Q be a minimal subpath of P' such that $V(Q)$ is adjacent to both 2 and 5. If Q contains at least one edge, or $V(Q)$ is not adjacent to 6 then we have a T-parachute $TP(1, 2, 6, 3, 5, Q)$. Otherwise, if Q contains only a single node, and it is adjacent to 6 then G contains the graph in Fig.3(a). By Lemma 6, G has a star cutset.

Case 2.2 x_n is not adjacent to 5.

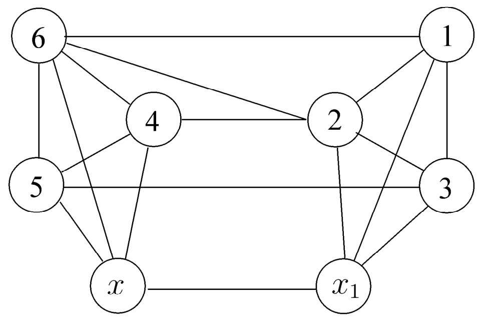
Let x_i be the node of largest index adjacent to 5. Let Q be the subpath of P from x_i to x_{n-1}. If $V(Q)$ is not adjacent to 2 then G contains an odd wheel $(5, 4, 2, x_n, Q, x_i, 5; 3)$. Therefore, $V(Q)$ must be adjacent to 2. Let $j (j \geq i)$ be the smallest index such that x_j is adjacent to 2. Let Q' denote the subpath of Q from x_i to x_j. If Q' contains at least one edge, or $V(Q')$ is not adjacent to 6 then we have a T-parachute $TP(1, 2, 6, 3, 5, Q')$. Otherwise, if Q' contains only a single node, and it is adjacent to 6 then G contains the graph in Fig.3(a). By Lemma 6, G has a star cutset. This completes the proof. ◇

□

These lemmas imply Theorem 2.

3 Proof of Theorem 3

In this section, we prove our main result as follows. Recall the notation introduced in Section 1.4: B is a maximally P_6-connected subgraph of $\bar{G}$. Lemma 10 shows that every node outside $V(B)$ is universal for B or has no neighbor in B. Lemma 11 shows that B is bipartite. Before proving these lemmas, we prove two technical lemmas (Lemmas 8 and 9).



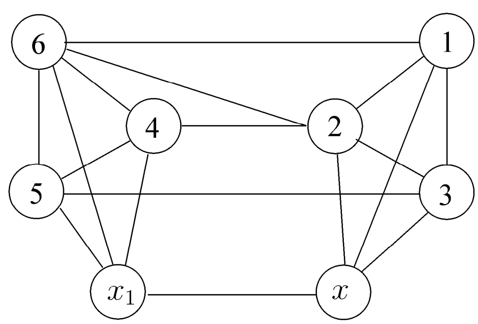
x_1 may be also adjacent to 4 or 5 or both but not 6.

Figure 6:

Lemma 8 *Let G be a WP′-free perfect graph that contains $\bar{P}_6$. Let Σ be the node set of a $\bar{P}_6$ of G as labeled in Fig.2. A node x in $G \setminus \Sigma$ adjacent to Σ satisfies the following properties.*

1) *If $N(x) \cap \Sigma = \{4, 5, 6\}$ then G has a star cutset or G contains the graph in Fig. 6 or both.*

2) *If $N(x) \cap \Sigma = \{1, 2, 3\}$ then G has a star cutset or G contains the graph in Fig. 7 or both.*



x_1 may be also adjacent to 1 or 3 or both but not 2.

Figure 7:

Proof: By symmetry, the proof of 2) is similar to the proof of 1). We prove 1) as follows. Let $S = N(6) \cup \{6\} \setminus \{x, 1, 2\}$. Suppose that S is not a star cutset (center 6). Let $P(x_1, ..., x_n)$ be any path in $G \setminus (S \cup \{1, 2, 3, x\})$ that minimally connects node x and $\{1, 2, 3\}$. x_1 is adjacent to node x. x_n is adjacent to $\{1, 2, 3\}$. $V(P)$ may be adjacent to 4 or 5 but not 6. By Theorem 2, we can assume that x_n is adjacent to 1, 2, 3 (and possibly 4, 5).

If P contains only node x_1 then G contains the graph in Fig. 6. In the rest of this proof we assume that P contains at least one edge.

If $V(P)$ is not adjacent to 4 then we have a T-parachute $TP(1, 2, 6, x_n, x, 4, P)$. If $V(P)$ is adjacent to 4 then the wheel $(x, P, 2, 6, x; 4)$ is either a proper wheel or a universal wheel. If it is a universal wheel then the wheel $(x, P, 1, 6, x; 4)$ is a proper wheel. This completes the proof. □

Let G be a WP′-free perfect graph that contains a $\bar{P}_6$ and let B be a maximally P_6-connected induced subgraph of $\bar{G}$. In the rest of this section, we work on $\bar{G}$ unless specified otherwise. Recall that the complements of the graphs in Fig.5 are formed by a $P_6(6, 3, 4, 1, 5, 2)$, a node α adjacent to 2 (and possibly 3) but not 1, 4, 5 or 6, and a node β adjacent to 6 (and possibly 5) but not 1, 2, 3 or 4.

Lemma 9 *Suppose that G has no star cutset. Let Σ be the node set of a P_6 of B. If a node $y \notin \Sigma$ has a neighbor x in Σ but y is not universal for Σ, then y belongs to B. Furthermore, if edge (x, y) does not belong to any P_6 then B contains one of the graphs in Fig.8.*

Proof: Let $P_6(6, 3, 4, 1, 5, 2)$ denote the path induced by Σ. $\Sigma \cup \{y\}$ induces a bipartite graph by Theorem 2. It is easy to check that, in the graph induced by $\Sigma \cup \{y\}$, the edge (x, y) belongs to some P_6 that shares an edge with $P_6(6, 3, 4, 1, 5, 2)$ except in the case where y is only adjacent to 4 in the $P_6(6, 3, 4, 1, 5, 2)$ and in the case where y is only adjacent to 4, 5, and 6 in the $P_6(6, 3, 4, 1, 5, 2)$ (x may be 4, 5 or 6) and their symmetric cases. We consider these two cases as follows.

Case 1 y is only adjacent to 4 in the $P_6(6, 3, 4, 1, 5, 2)$

Recall that by Lemma 5, G contains one of the graphs in Fig.5 since G has no star cutset. We further consider the following cases.

Case 1.1 $N(\alpha) \cap \Sigma = \{2\}$.

$(4, y)$ belongs to $P_6(y, 4, 1, 5, 2, \alpha)$ if α is not adjacent to y, or $P_6(6, 3, 4, y, \alpha, 2)$ if α is adjacent to y. Both of these P_6's share an edge with $P_6(6, 3, 4, 1, 5, 2)$.

Case 1.2 $N(\alpha) \cap \Sigma = \{2, 3\}$.

If y is not adjacent to α then $(4, y)$ belongs to $P_6(y, 4, 1, 5, 2, \alpha)$, which shares an edge with $P_6(6, 3, 4, 1, 5, 2)$. We assume now that y is adjacent to α. Notice that β cannot be adjacent to y since G is perfect. If $N(\beta) \cap \Sigma = \{5, 6\}$ then $(4, y)$ belongs to $P_6(y, 4, 3, 6, \beta, 5)$, which shares an edge with $P_6(6, 3, 4, 1, 5, 2)$. Notice that (α, y) belongs to $P_6(\alpha, y, 4, 1, 5, \beta)$ if α is not adjacent to β, and (α, y), (α, β) belong to $P_6(1, 4, y, \alpha, \beta, 6)$ if α is adjacent to β. Thus, we can assume that $N(\beta) \cap \Sigma = \{6\}$.

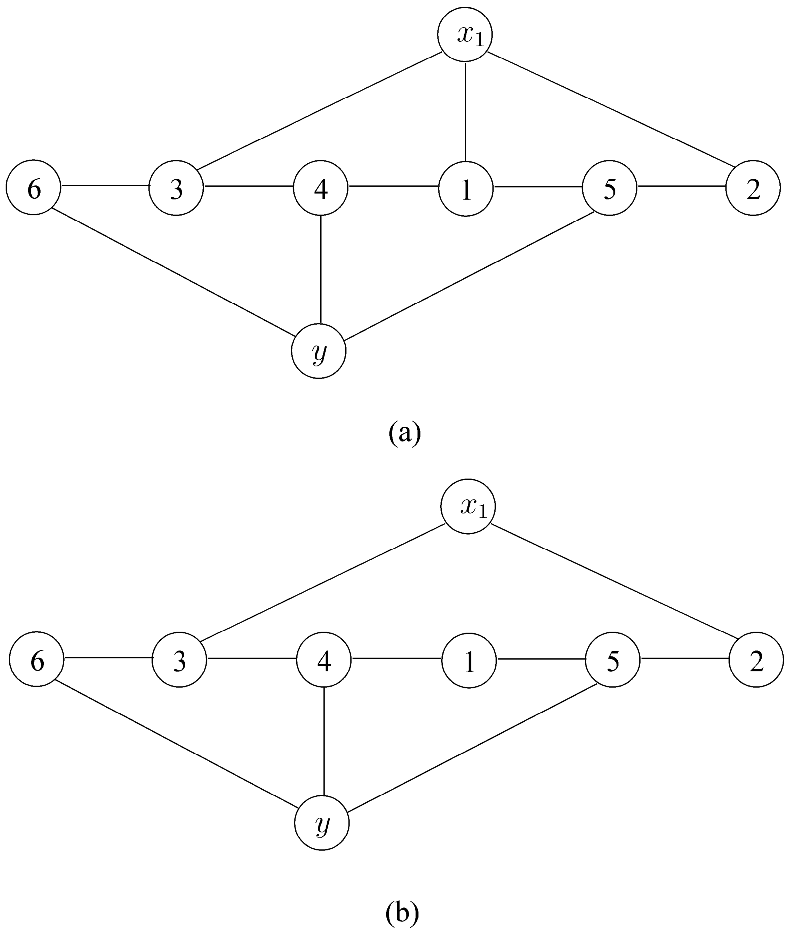


Figure 8: Neither $(3, x_1)$ nor $(5, y)$ belongs to any P_6 in (a). $(4, y)$ does not belong to any P_6 in (b).

If β is adjacent to α then $(4, y)$ belongs to $P_6(5,1,4,y,\alpha,\beta)$, which shares an edge with $P_6(6,3,4,1,5,2)$. If β is not adjacent to α then B contains the $P_6(\beta,6,3,4,1,5)$. Note that β is adjacent to none of α, y and 2. By Lemma 5, there exists one more node α' such that $N(\alpha') \cap V(P_6(\beta,6,3,4,1,5)) = \{\beta\}$ or $\{1,\beta\}$ in $\bar{G}$. α' is not adjacent to α in $\bar{G}$ since G is perfect. If α' is not adjacent to y then $(4, y)$ belongs to $P_6(y,4,3,6,\beta,\alpha')$, which shares an edge with $P_6(6,3,4,1,5,2)$. Notice that (α, y) belongs to $P_6(\alpha',\beta,6,3,\alpha,y)$ in this case. So we can assume that α' is adjacent to y. In the case where $N(\alpha') \cap V(P_6(\beta,6,3,4,1,5)) = \{\beta\}$, the edge $(4, y)$ belongs to $P_6(1,4,y,\alpha',\beta,6)$, which shares an edge with $P_6(6,3,4,1,5,2)$. In the case where $N(\alpha') \cap P_6(\beta,6,3,4,1,5) = \{1,\beta\}$, the subgraph induced by $\{\alpha,3,4,1,\alpha',\beta,y,6\}$ is isomorphic to the graph in Fig.8(b). Furthermore, y belongs to $P_6(1,\alpha',y,\alpha,3,6)$, which shares an edge with $P_6(6,3,4,1,5,2)$. Therefore, $y \in B$. Notice that (α, y) belongs to $P_6(5,2,\alpha,y,\alpha',\beta)$ in both cases.

Case 2 y is only adjacent to 4, 5, and 6 in the $P_6(6,3,4,1,5,2)$.

By Lemma 8, there exists a node x_1 adjacent to 2 (and possibly 1, 3) but not y, 4, 5 or 6 in $\bar{G}$.

If x_1 is adjacent to 2 (and possibly 1) but not 3 in the $P_6(6,3,4,1,5,2)$ then $(4, y)$ and $(5, y)$ belong to the $P_6(3,4,y,5,2,x_1)$, and $(6, y)$ belongs to the $P_6(3,6,y,5,2,x_1)$. Both of these P_6's share an edge with $P_6(6,3,4,1,5,2)$.

If x_1 is only adjacent to 2 and 3 in the $P_6(6,3,4,1,5,2)$ then the graph induced by $\{6,3,4,1,5,2,y,x_1\}$ is the graph in Fig.8(b). As noted above, $y \in B$ in this case.

If x_1 is only adjacent to 2, 1 and 3 in the $P_6(6,3,4,1,5,2)$ then the graph induced by $\{6,3,4,1,5,2,y,x_1\}$ is the graph in Fig.8(a), and $(4, y)$, $(6, y)$, $(1, x_1)$ and $(2, x_1)$ belong to the $P_6(6,y,4,1,x_1,2)$. Notice that neither $(5, y)$ nor $(3, x_1)$ belongs to any P_6 in this subgraph. Finally, note that x_1 and y belongs to the $P_6(6,y,4,1,x_1,2)$, which shares an edge with $P_6(6,3,4,1,5,2)$. Therefore, $x_1, y \in B$. □

Lemma 10 *Suppose that G has no star cutset. A node $y \notin V(B)$ adjacent to $V(B)$ is universal for $V(B)$.*

Proof: By Lemma 9, node y is universal for the node set Σ of some P_6 in B. It follows from Theorem 2 that y is also universal for the node set of the P_6's that share an edge with Σ. Since B is P_6-connected graph, this implies that y is universal for $V(B)$. □

Lemma 11 *Suppose that G has no star cutset. Then B is bipartite.*

Proof: Suppose that B is not bipartite. Since B is perfect, B contains a triangle (x, y, z). Obviously, these three nodes cannot belong to the same P_6. Suppose that two nodes of these three nodes, say x and y, belong to some P_6. Let Σ denote the node set of this P_6. Then by Theorem 2, the third node z should be universal for Σ. Now we prove that it is also universal for $V(B) \setminus \{z\}$. Suppose that z is universal for $S \subset V(B)$, where $\Sigma \subset S$ and $S \cup \{z\} \neq V(B)$. Since B is P_6-connected, in B there exists another P_6, denote by Σ' the node set of this P_6, not entirely contained in S which shares an edge e with some P_6 in S. Since, z is adjacent to both ends of e, $z \notin \Sigma'$. Furthermore, by Theorem 2, z is universal for Σ'. Therefore, z is universal for $S \cup \Sigma'$. By induction, z is universal for $V(B) \setminus \{z\}$. This contradicts the fact that B is P_6-connected. Hence, no two of the three nodes x, y, z belong to the same P_6.

Therefore, we only need to consider the following two cases by Lemma 9.

Case 1 B contains the graph in Fig.8(a) plus z adjacent to 5 and y. That is, $\{5, y, z\}$ induces a triangle.

By Theorem 2 applied to the $P_6(6, 3, 4, 1, 5, 2)$, z cannot be adjacent to 1, 2 or 3. By Theorem 2 applied to the $P_6(6, y, 4, 1, x_1, 2)$, z cannot be adjacent to 4, 6 or x_1. But now the $P_6(6, 3, x_1, 2, 5, z)$ plus node y contradicts Lemma 4.

Case 2 B contains the graph in Fig.8(b) plus z adjacent to 4 and y. That is, $\{4, y, z\}$ induces a triangle.

By Theorem 2 applied to the $P_6(6, 3, 4, 1, 5, 2)$, z cannot be adjacent to 1, 2 or 3. By Theorem 2 applied to the $P_6(1, 5, y, 6, 3, x_1)$, z cannot be adjacent to 5, 6 or x_1. But now the $P_6(z, y, 6, 3, x_1, 2)$ plus node 4 contradicts Lemma 4. □

Proof of Theorem 3: By Lemma 11, B is bipartite. Suppose B is not a connected component of $\bar{G}$. By Lemma 10, any node y in $\bar{G} \setminus B$ that has a neighbor in B is universal for $V(B)$. Therefore, $N(x) \cup \{x\}$ is a star cutset of G for any node x in $V(B)$, since $S = V(B) \setminus (N(x) \cup \{x\})$ is nonempty, and y and S are in distinct connected components of $G \setminus (N(x) \cup \{x\})$. This completes the proof. □

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