A new infinite class of ideal minimally non-packing clutters

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Abstract

The $\tau = 2$ *Conjecture* predicts that every ideal minimally non-packing clutter has covering number two. In the original paper where the conjecture was proposed, in addition to an infinite class of such clutters, thirteen small instances were provided. The construction of the small instances followed an ad-hoc procedure and why it worked has remained a mystery, until now. In this paper, using the theory of *clean tangled clutters*, we identify key structural features about these small instances, in turn leading us to a second infinite class of ideal minimally non-packing clutters with covering number two. Unlike the previous infinite class consisting of *cuboids* with unbounded *rank*, our class is made up of non-cuboids, all with rank three.

Keywords. Clutters, ideal minimally non-packing clutters, the Q_6 property, the $\tau = 2$ Conjecture, clean tangled clutters.

1 Introduction

Let $\mathcal{C}$ be a clutter over ground set V.[1] The *packing number*, denoted $\nu(\mathcal{C})$, is the maximum number of pairwise disjoint members. The *covering number*, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover, i.e. the minimum number of elements needed to intersect every member. We have $\nu(\mathcal{C}) \leq \tau(\mathcal{C})$; this motivates the following standard definitions:

- $\mathcal{C}$ *packs* if $\nu(\mathcal{C}) = \tau(\mathcal{C})$.
- $\mathcal{C}$ has the *packing property* if every minor of $\mathcal{C}$, including $\mathcal{C}$ itself, packs [9].
- $\mathcal{C}$ is *minimally non-packing* if $\mathcal{C}$ does not pack, but every proper minor of $\mathcal{C}$ packs [9].

Observe that a clutter has the packing property if, and only if, it has no minimally non-packing minor. It was proved in [9] that if $\mathcal{C}$ has the packing property, then the set covering polyhedron

$$\left\{ x \in \mathbb{R}^V_+ : \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C} \right\}$$

[1]To pick up the pace of the introduction, we have assumed familiarity with standard notions such as clutters, minors, etc. and have postponed their definition to §1.1.

is integral, that is, $\mathcal{C}$ is *ideal* [10]. This implication is a consequence of a powerful theorem of Lehman [16] on the structure of a *minimally non-ideal* clutter, i.e. a non-ideal clutter whose proper minors are ideal. Since the packing property implies idealness, every minimally non-packing clutter is either ideal or minimally non-ideal.

An important conjecture in the area is the *Replication Conjecture* [7], stating that a minimally non-packing clutter cannot have replicated elements. Lehman's theorem verifies the conjecture for the minimally non-ideal clutters (see [9]). It therefore remains to prove that an ideal minimally non-packing clutter cannot admit replicated elements. To solve the remaining case, Cornuéjols, Guenin and Margot made the following stronger conjecture:

Conjecture 1 ($\tau = 2$ Conjecture [9])**.** *Every ideal minimally non-packing clutter has covering number two.*

All the examples of ideal minimally non-packing clutters known at the time had covering number two, making the authors of [9] believe the conjecture above. The reader may think that the reason for believing the conjecture is somewhat superficial but, recently, geometric evidence supporting the conjecture was provided in [1].

Given the $\tau = 2$ Conjecture, a natural research direction is to study, give examples of, and characterize ideal minimally non-packing clutters with covering number two. In [9], the authors provided an infinite class of such clutters along with thirteen small examples. The infinite class of ideal minimally non-packing clutters consists of *cuboids*, and in papers [4, 1], by using the theory of cuboids, more than 700 new *small* cuboid examples were generated via a computer program. Twelve of the thirteen small examples [9], however, are not cuboids. Other than the fact that they belong to chains of ideal minimally non-packing clutters starting with Q_6 [4], not much else has been known about them.

In this paper, we use the theory of *clean tangled clutters* to identify key structural properties about the thirteen small examples of ideal minimally non-packing clutters with covering number two, and in general about those with *rank* three. This investigation leads us to a new infinite class of ideal minimally non-packing clutters with covering number two, clutters which share the identified structural properties with the thirteen small examples.

Since the new infinite class of ideal minimally non-packing clutters is easy to describe, and the proof of correctness does not need much overhead knowledge, we present these two things first. Given a clutter $\mathcal{C}$ over ground set V, we define $G(\mathcal{C})$ as the graph with vertices V and edges corresponding to the two-element minimal covers of $\mathcal{C}$.

Theorem 2. *Let $\mathcal{C}$ be a clutter, and let $G = G(\mathcal{C})$. Assume that*

- *G is bipartite and has exactly 3 connected components,*
- *the first connected component of G has two vertices $1, 2$ and an edge between them,*
- *the second connected component of G has two vertices $3, 4$ and an edge between them,*
- *the third connected component of G is a path on at least four edges, where the first edge is $\{5, 6\}$, the last edge is $\{5', 6'\}$, $5, 5'$ belong to the same part of the bipartition, and $6, 6'$ belong to the other part of the bipartition, and*

- *the minimal covers of $\mathcal{C}$ of cardinality different from two are precisely*

$$\{2,4,6\},\{2,3,5\},\{1,4,5'\},\{1,3,6'\} \quad \textit{and} \quad \{3,5,6'\},\{4,5',6\}.$$

Then $\mathcal{C}$ is an ideal minimally non-packing clutter. (See Figure 1 for an illustration of the graph G.)

After introducing some definitions and a preliminary in §1.1, we prove Theorem 2 in §2. Then we prove in §3 that all ideal minimally non-packing clutters with covering number two and *rank three* share certain structural properties – properties that our new examples enjoy. We conclude the paper in §4 by describing the thirteen examples of ideal minimally non-packing clutters of [9] that we alluded to, and noting that these examples also enjoy the structural properties discussed in §3.

1.1 Definitions and a preliminary

Given a finite set V, a *clutter* $\mathcal{C}$ over ground set V is a family of subsets of V, such that $C \not\subseteq C'$ for all $C, C' \in \mathcal{C}$. We refer to elements of V simply as *elements*, and sets in $\mathcal{C}$ as *members*. A *cover* of $\mathcal{C}$ is a subset $B \subseteq V$ with $B \cap C \neq \emptyset$ for all $C \in \mathcal{C}$. A *transversal* of $\mathcal{C}$ is a cover $B \subseteq V$ with $|B \cap C| = 1$ for all $C \in \mathcal{C}$. A cover is *minimal* if it does not contain another cover. The *blocker* $b(\mathcal{C})$ of $\mathcal{C}$ is the clutter over ground set V consisting of the minimal covers of $\mathcal{C}$ [11]. For all clutters $\mathcal{C}$, we have $b(b(\mathcal{C})) = \mathcal{C}$ [11, 14].

For disjoint $I, J \subseteq V$, the *minor* $\mathcal{C} \setminus I/J$ of $\mathcal{C}$ obtained by *deleting* I and *contracting* J is the clutter over ground set $V \setminus (I \cup J)$ consisting of the inclusion-wise minimal sets in $\{C \setminus J : C \in \mathcal{C},\ C \cap I = \emptyset\}$. We say that $\mathcal{C} \setminus I/J$ is a *proper* minor of $\mathcal{C}$ if $I \cup J \neq \emptyset$. Deletions in $\mathcal{C}$ correspond to contractions in $b(\mathcal{C})$, and contractions in $\mathcal{C}$ correspond to deletions in $b(\mathcal{C})$, so that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ [20].

Recall that a clutter $\mathcal{C}$ is ideal if its set covering polyhedron

$$\left\{x \in \mathbb{R}^V_+ : \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C}\right\}$$

is integral. Idealness is closed under taking the blocker and minors. That is, $\mathcal{C}$ is ideal if and only if $b(\mathcal{C})$ is ideal [13, 15], and if $\mathcal{C}$ is ideal, then all minors of $\mathcal{C}$ are ideal [21].

We now give some examples of non-ideal clutters. For any integer $n \geq 3$, the clutter Δ_n over ground set $[n] = \{1, \ldots, n\}$ is given by

$$\Delta_n = \{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\}$$

More generally, a *delta* is any clutter whose elements can be relabeled to obtain Δ_n for some n. An *extended odd hole* is any clutter whose elements can be relabeled as $[n]$, for odd $n \geq 5$, to obtain a clutter $\mathcal{C}$ of the form

$$\mathcal{C} = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\} \cup \mathcal{C}'$$

where $\mathcal{C}'$ consists of members with cardinality three or more. Note that the set covering polyhedron of Δ_n has a fractional vertex $\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$, and the set covering polyhedron of any extended odd hole has a fractional vertex $\left(\frac{1}{2},\ldots,\frac{1}{2}\right)$. Since a clutter $\mathcal{C}$ is ideal if and only if $b(\mathcal{C})$ is ideal, the blocker of any extended odd hole is also non-ideal. We might also remark that $b(\Delta_n)$ is non-ideal, except that $b(\Delta_n) = \Delta_n$. We need the following result for the proof of Theorem 2:

Theorem 3 ([16], see also [19])**.** *Every minimally non-packing clutter with covering number two is ideal, a delta, or the blocker of an extended odd hole.*

2 Validity of our construction

We begin with a few definitions and notions. Let $\mathcal{C}$ be a clutter over ground set V. Recall that $G(\mathcal{C})$ is the graph with vertices V and edges corresponding to the two-element minimal covers of $\mathcal{C}$.

Definition 4 ([2, 3])**.** *A clutter $\mathcal{C}$ is* tangled *if $\tau(\mathcal{C}) = 2$ and every element of $\mathcal{C}$ is in a cardinality-two cover. That is, $\mathcal{C}$ is tangled if $G(\mathcal{C})$ has no isolated vertex.*

Note that for a tangled clutter $\mathcal{C}$, every member of $\mathcal{C}$ is a vertex cover of $G(\mathcal{C})$. (The converse, however, is not true.)

Definition 5. *The* rank *of a tangled clutter $\mathcal{C}$, denoted $\operatorname{rank}(\mathcal{C})$, is the number of connected components of $G(\mathcal{C})$.*

Note that the clutters $\mathcal{C}$ described in Theorem 2 are tangled, have rank three, and $G(\mathcal{C})$ is bipartite. We will use these properties, along with the following lemma, to show that the clutters described in Theorem 2 do not pack:

Lemma 6. *Let $\mathcal{C}$ be a tangled clutter with rank three, and suppose $G(\mathcal{C})$ is bipartite. Denote by $\{U_1, V_1\}$, $\{U_2, V_2\}$, $\{U_3, V_3\}$ the bipartitions of the connected components of $G(\mathcal{C})$. If $V_1 \cup V_2 \cup V_3$, $U_1 \cup U_2 \cup V_3$, $U_1 \cup V_2 \cup U_3$, $V_1 \cup U_2 \cup U_3$ are covers of $\mathcal{C}$, then $\mathcal{C}$ does not pack.*

Proof. Suppose for contradiction that $\mathcal{C}$ packs, so that $\mathcal{C}$ has disjoint members C_1, C_2. Then each edge of $G(\mathcal{C})$ must have exactly one end in each of C_1, C_2, so each C_i respects the bipartition of each connected component of $G(\mathcal{C})$; that is, $C_i \cap (U_j \cup V_j) \in \{U_j, V_j\}$ for $i \in [2]$ and $j \in [3]$. We may assume that $C_1 \cap (U_1 \cup V_1) = U_1$ and so $C_2 \cap (U_1 \cup V_1) = V_1$. We have two cases:

Case 1: $C_1 \cap (U_2 \cup V_2) = U_2$ and so $C_2 \cap (U_2 \cup V_2) = V_2$. As $V_1 \cup V_2 \cup V_3$ is a cover, it follows that $C_1 \cap (U_3 \cup V_3) = V_3$ and so $C_2 \cap (U_3 \cup V_3) = U_3$. But then C_2 is disjoint from the cover $U_1 \cup U_2 \cup V_3$.

Case 2: $C_1 \cap (U_2 \cup V_2) = V_2$ and so $C_2 \cap (U_2 \cup V_2) = U_2$. As $V_1 \cup U_2 \cup U_3$ is a cover, it follows that $C_1 \cap (U_3 \cup V_3) = U_3$ and so $C_2 \cap (U_3 \cup V_3) = V_3$. But then C_2 is disjoint from the cover $U_1 \cup V_2 \cup U_3$.

In both cases we have a contradiction, so $\mathcal{C}$ does not pack. $\square$

We will use the following lemma to show that the clutters described in Theorem 2 are minimally non-packing. Note that this lemma applies even to clutters $\mathcal{C}$ where $G(\mathcal{C})$ has isolated vertices:

Lemma 7. *Let $\mathcal{C}$ be a clutter over ground set V, and let $G = G(\mathcal{C})$. Assume that:*

(i) G *is a bipartite graph with bipartition* $\{U_0, V_0\}$*,*

(ii) $|\{B \in b(\mathcal{C}) : |B| > 2\}| \leq 1$*, and*

(iii) if $B \in b(\mathcal{C})$ *satisfies* $|B| > 2$*, then* $B = \{u, v, w\}$ *where*

- $u \in U_0$ *and* $\{v, w\} \subseteq V_0$*, and*
- *in* G*, either* v, w *belong to different connected components, or some neighbor of* u *is a cut-vertex of* G *separating* v *and* w*.*

Then $\mathcal{C}$ *has the packing property.*

Proof. We proceed by induction on $|V| \geq 2$. The base case $|V| = 2$ holds trivially, so we may assume $|V| \geq 3$. Also, we may assume $\tau(\mathcal{C}) \geq 2$.

Claim 1. $\mathcal{C}$ *packs.*

Proof of Claim. If $\tau(\mathcal{C}) \geq 3$, then every minimal cover of $\mathcal{C}$ has cardinality at least 3, so $b(\mathcal{C}) = \{\{u, v, w\}\}$, so $\mathcal{C} = \{\{u\}, \{v\}, \{w\}\}$, which clearly packs. Otherwise, $\tau(\mathcal{C}) = 2$. Notice that both U_0, V_0 are covers of $b(\mathcal{C})$, so they each contain members of $\mathcal{C}$, implying in turn that $\mathcal{C}$ has two disjoint members, so $\mathcal{C}$ packs. $\Diamond$

Claim 2. *For each* $x \in V$*,* $\mathcal{C}/x$ *has the packing property.*

Proof of Claim. Notice that $\mathcal{C}/x$ satisfies (i)-(iii), so the claim follows from the induction hypothesis. $\Diamond$

Claim 3. *For each* $x \in V$ *and* $N = \{y \in V : \{x, y\} \in b(\mathcal{C})\}$*,* $\mathcal{C} \setminus x/N$ *has the packing property.*

Proof of Claim. Let $\mathcal{C}' = \mathcal{C} \setminus x/N$. If $N \neq \emptyset$, then $\mathcal{C}'$ has the packing property by Claim 2. We may therefore assume that $N = \emptyset$. By the induction hypothesis, it suffices to prove that $\mathcal{C}'$ satisfies (i)-(iii). If $x \notin \{u, v, w\}$, then $\mathcal{C}'$ clearly satisfies (i)-(iii). If $x \in \{v, w\}$, then since $u \in U_0$ and $\{v, w\} \subseteq V_0$, it follows that $\mathcal{C}'$ satisfies (i)-(iii). Otherwise $x = u$. As $N = \emptyset$, we see that v, w belong to different connected components of G, so $\mathcal{C}'$ satisfies (i)-(iii), as required. $\Diamond$

These three claims imply that $\mathcal{C}$ has the packing property. To see this, consider an arbitrary minor $\mathcal{C} \setminus I/J$ of $\mathcal{C}$. If $I = J = \emptyset$, then the minor packs by Claim 1. If $J \neq \emptyset$, then the minor packs by Claim 2. Otherwise, $J = \emptyset$ and $I \neq \emptyset$. If $\tau(\mathcal{C} \setminus I/J) < 2$, then the minor obviously packs. Otherwise, $\tau(\mathcal{C} \setminus I/J) \geq 2$, so by Claim 3, the minor packs. We have completed the induction step, thereby finishing the proof of the lemma. $\square$

We are now ready to prove Theorem 2, stating that if $\mathcal{C}$ is tangled where $G(\mathcal{C})$ is as illustrated in Figure 1, and $\{2, 4, 6\}, \{2, 3, 5\}, \{1, 4, 5'\}, \{1, 3, 6'\}$ and $\{3, 5, 6'\}, \{4, 5', 6\}$ are the minimal covers of cardinality different from two, then $\mathcal{C}$ is ideal minimally non-packing:

Proof of Theorem 2. Let $\mathcal{C}$ be a clutter satisfying the hypotheses of Theorem 2. Note that $\mathcal{C}$ is tangled. Let $\{U_3, V_3\}$ be the bipartition of the third connected component of G, where $\{5, 5'\} \subseteq U_3$ and $\{6, 6'\} \subseteq V_3$. (See Figure 1 for an illustration of G.)

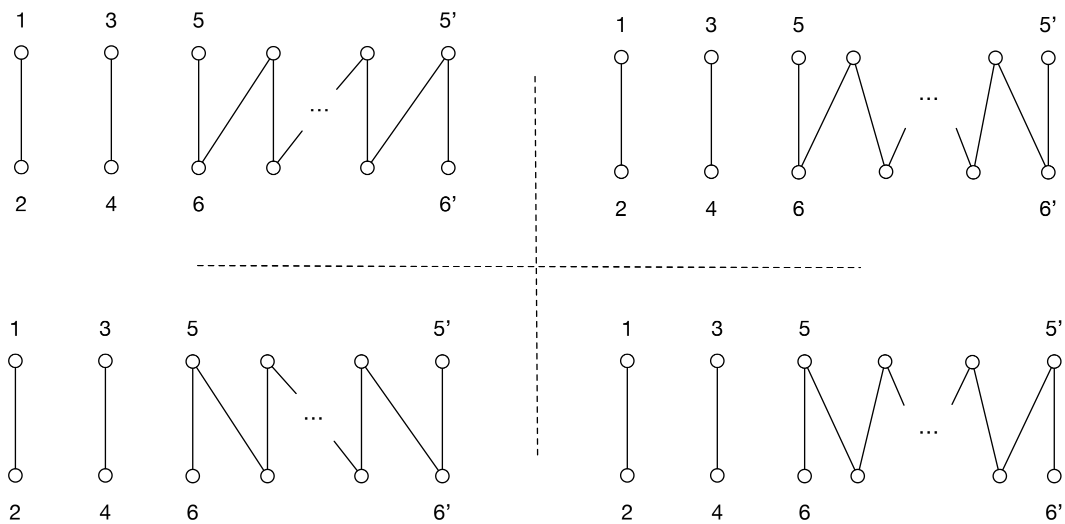

Figure 1: The four possibilities for G from Theorem 2.

Claim 1. *$\mathcal{C}$ does not pack.*

Proof of Claim. As $\{2,4,6\},\{2,3,5\},\{1,4,5'\},\{1,3,6'\}$ are minimal covers, $\{2,4\}\cup V_3$, $\{2,3\}\cup U_3$, $\{1,4\}\cup U_3$ and $\{1,3\}\cup V_3$ are covers, so $\mathcal{C}$ does not pack by Lemma 6. $\Diamond$

In what follows, notice that in our setup, there is symmetry between $1,2$, between $3,4$, between $5,6$, between $5',6'$, and between $\{5,6\},\{5',6'\}$.

Claim 2. *Every proper contraction minor of $\mathcal{C}$ packs.*

Proof of Claim. Choose $I\subseteq V$ such that $\tau(\mathcal{C}/I)\geq 2$. Let $\mathcal{C}'=\mathcal{C}/I$ and notice that $b(\mathcal{C}')=b(\mathcal{C})\setminus I$. As a result, $\tau(\mathcal{C}')\in\{2,3\}$. If $\tau(\mathcal{C}')=3$, then it can be readily checked that $b(\mathcal{C}')$ has at most two members, each of cardinality 3, implying in turn that $\mathcal{C}'$ packs. Otherwise, $\tau(\mathcal{C}')=2$, in which case we need to look for disjoint members in $\mathcal{C}'$. As disjoint members remain disjoint in contraction minors, we may assume that $I=\{x\}$. In what follows, we find disjoint covers in $b(\mathcal{C}')$. Assume in the first case that $x\in\{1,2,3,4,5,6,5',6'\}$. By symmetry, we may assume that $x\notin\{2,4,6,5',6'\}$.

- If $x=1$, then $\{2,3\}\cup V_3$, $\{4\}\cup U_3$ are disjoint covers of $b(\mathcal{C}')$.
- If $x=3$, then $\{1,4\}\cup V_3$, $\{2\}\cup U_3$ are disjoint covers of $b(\mathcal{C}')$.
- Otherwise, $x=5$. Then $\{2,3\}\cup(U_3\setminus\{5\})$, $\{1,4\}\cup V_3$ are disjoint covers of $b(\mathcal{C}')$.

Assume in the remaining case that $x\in V\setminus\{1,2,3,4,5,6,5',6'\}=(U_3\cup V_3)\setminus\{5,6,5',6'\}$. Notice that deleting x from G disconnects the third connected component, that is, $G\setminus x$ has four connected components with bipartitions $\{\{1\},\{2\}\}$, $\{\{3\},\{4\}\}$, $\{U_3',V_3'\}$, $\{U_3'',V_3''\}$, where $U_3'\cup U_3''=U_3\setminus\{x\}$, $V_3'\cup V_3''=V_3\setminus\{x\}$,

$5 \in U_3', 6 \in V_3', 5' \in U_3''$ and $6' \in V_3''$. Observe now that $\{1,3\} \cup (V_3' \cup U_3''), \{2,4\} \cup (U_3' \cup V_3'')$ are disjoint covers of $b(\mathcal{C}')$.

In each case, we proved that $b(\mathcal{C}')$ has disjoint covers, giving disjoint members of $\mathcal{C}'$ in turn, as desired. $\diamondsuit$

Claim 3. *Let $I \subseteq V$ be nonempty, such that I is disjoint from $\{1,2,3,4,5,6,5',6'\}$ and not a cover of $\mathcal{C}$. Let*

$$N = \big\{y \in V \setminus I : \{x,y\} \in b(\mathcal{C}) \text{ for some } x \in I\big\}.$$

Then $\mathcal{C} \setminus I/N$ packs.

Proof of Claim. Let $\mathcal{C}' = \mathcal{C} \setminus I/N$, and note that $b(\mathcal{C}') = b(\mathcal{C})/I \setminus N$. As $I \cap \{1,2,3,4,5,6,5',6'\} = \emptyset$, it follows that $b(\mathcal{C})/I \setminus N = b(\mathcal{C}) \setminus (I \cup N)$, implying that $\mathcal{C}' = \mathcal{C}/(I \cup N)$, so $\mathcal{C}'$ packs by Claim 2. $\diamondsuit$

Claim 4. *Let $x \in \{1,2,3,4,5,6,5',6'\}$ and $N = \{y \in V : \{x,y\} \in b(\mathcal{C})\}$. Then $\mathcal{C} \setminus x/N$ has the packing property.*

Proof of Claim. By symmetry, we may assume that $x \notin \{2,4,6,5',6'\}$. To prove the claim, it suffices to show that $\mathcal{C} \setminus 1/2, \mathcal{C} \setminus 3/4$ and $\mathcal{C} \setminus 5/6$ have the packing property.

Every minimal cover of $\mathcal{C} \setminus 1/2$ has cardinality two, and the graph over vertex set $V \setminus \{1,2\}$ of the minimal covers of the minor is bipartite with bipartition $\{3\} \cup U_3, \{4\} \cup V_3$, so $\mathcal{C} \setminus 1/2$ has the packing property by Lemma 7.

Every minimal cover of $\mathcal{C} \setminus 3/4$ also has cardinality two, and the graph over vertex set $V \setminus \{3,4\}$ of the minimal covers of the minor is bipartite with bipartition $\{1\} \cup U_3, \{2\} \cup V_3$, so once again $\mathcal{C} \setminus 3/4$ has the packing property by Lemma 7.

Finally, let $\mathcal{C}' = \mathcal{C} \setminus 5/6$, and let G' be the graph over vertex set $V \setminus \{5,6\}$ whose edges correspond to the cardinality two minimal covers of $\mathcal{C}'$. Notice that G' is a bipartite graph with bipartition $\{1,3\} \cup (U_3 \setminus \{5\}), \{2,4\} \cup (V_3 \setminus \{6\})$. Moreover, there is only one minimal cover with cardinality greater than two, namely $\{1,4,5'\}$. Furthermore, the neighbor 3 of 4 in G' is a cut-vertex separating $1, 5'$. Thus $\mathcal{C}'$ has the packing property by Lemma 7. $\diamondsuit$

These four claims imply that $\mathcal{C}$ is a minimally non-packing clutter. To see this, note first that the clutter does not pack by Claim 1. Let $\mathcal{C} \setminus I/J$ be a proper minor. If $I = \emptyset$, then the minor packs by Claim 2. Otherwise, $I \neq \emptyset$. If $\tau(\mathcal{C} \setminus I/J) < 2$, then the minor clearly packs. Otherwise, $\tau(\mathcal{C} \setminus I/J) \geq 2$. If $I \cap \{1,2,3,4,5,6,5',6'\} \neq \emptyset$, then the minor packs by Claim 4. Otherwise, I is disjoint from $\{1,2,3,4,5,6,5',6'\}$. In this case, as $\tau(\mathcal{C} \setminus I/J) \geq 2$, it must be that $J = \big\{y \in V \setminus I : \{x,y\} \in b(\mathcal{C}) \text{ for some } x \in I\big\}$, so the minor packs by Claim 3. We have exhausted all cases, so every proper of $\mathcal{C}$ packs, so $\mathcal{C}$ is a minimally non-packing clutter.

By Theorem 3, $\mathcal{C}$ is ideal, a delta, or the blocker of an extended odd hole. However, as G is a bipartite graph with at least two connected components, $\mathcal{C}$ must be an ideal clutter, thereby finishing the proof. $\square$

Take an integer $r \geq 1$. Given a set $S \subseteq \{0,1\}^r$, the *cuboid* of S is the clutter over ground set $[2r]$ whose members have incidence vectors $\{(p_1, 1-p_1, \ldots, p_r, 1-p_r) : p \in S\}$. We call a clutter a *cuboid* if it can be obtained from the cuboid of some set S by relabeling elements of the ground set $[2r]$. In other words, a cuboid is a clutter $\mathcal{C}$ whose ground set can be relabeled as $[2r]$ for some integer $r \geq 1$, such that $\{2i-1, 2i\}$ for each $i \in [r]$ is a transversal:

$$|\{1,2\} \cap C| = |\{3,4\} \cap C| = \cdots = |\{2r-1, 2r\} \cap C| = 1 \quad \forall C \in \mathcal{C}.$$

Notice that every member of $\mathcal{C}$ has cardinality r, and that $\{2i-1, 2i\}$ for each $i \in [r]$ is a cover. Thus, if $\mathcal{C}$ has no cover of cardinality one, then $\mathcal{C}$ is a tangled clutter where $G(\mathcal{C})$ has at least the edges $\{1,2\}, \{3,4\}, \ldots, \{2r-1, 2r\}$. Our construction has the promised properties:

Remark 8. *The clutters $\mathcal{C}$ described in Theorem 2 have $\tau(\mathcal{C}) = 2$, $\operatorname{rank}(\mathcal{C}) = 3$, and are not cuboids.*

Proof. The first two properties are immediate. To see that $\mathcal{C}$ is not a cuboid, note first that $|V| \geq 9$. If $|V|$ is odd, then we are clearly done. Otherwise, $|V|$ is even. We prove that $\mathcal{C}$ has a member of cardinality greater than $|V|/2$, thereby finishing the proof. To this end, let $\{U_3, V_3\}$ be the bipartition of the third connected component of $G(\mathcal{C})$, where $\{5, 5'\} \subseteq U_3$ and $\{6, 6'\} \subseteq V_3$. The third connected component is either a $56'$- or $5'6$-path (see Figure 1). In the first case, $\{1,4,5\} \cup V_3$ is a member of $\mathcal{C}$ of cardinality $(|V|+2)/2$, while in the remaining case, $\{1,3,6\} \cup U_3$ is a member of $\mathcal{C}$ of cardinality $(|V|+2)/2$, as claimed. $\square$

That our infinite class of ideal minimally non-packing clutters consists of non-cuboids all with rank three is interesting because the only other known infinite class of such examples, due to [9], is a family of cuboids of unbounded rank (see [4]).

3 Structure of ideal minimally non-packing clutters

Our goal in this section is to prove Theorem 21, that all ideal minimally non-packing clutters with covering number two and rank three share certain structural properties. To describe these properties, we need a few concepts first.

3.1 Clean tangled clutters, the core, and the setcore

Let us begin with the following definition:

Definition 9. *A clutter $\mathcal{C}$ is* clean *if no minor of $\mathcal{C}$ is a delta or the blocker of an extended odd hole.*

In particular, ideal clutters are clean, so the clutters obtained by our construction are clean. We are particularly interested in clean tangled clutters. To see why the tangled assumption is reasonable, note that in general we have $\tau(\mathcal{C} \setminus \{v\}) \leq \tau(\mathcal{C})$ and $\nu(\mathcal{C} \setminus \{v\}) \leq \nu(\mathcal{C})$. But if v is in no cover of cardinality $\tau(\mathcal{C})$, then $\tau(\mathcal{C} \setminus \{v\}) = \tau(\mathcal{C})$, and in particular $\mathcal{C}$ cannot be minimally non-packing. To summarize:

Remark 10. *Every ideal minimally non-packing clutter with covering number two is a clean tangled clutter.*

For clean tangled clutters, the graph $G(\mathcal{C})$ of cardinality-two covers takes a nice form:

Theorem 11 ([6], Remark 6 and Theorem 7)**.** *If $\mathcal{C}$ is a clean tangled clutter, then:*

(i) $G(\mathcal{C})$ is a bipartite graph without isolated vertices ([6], Remark 6).

If, in addition, $G(\mathcal{C})$ is connected and $\{U_1, V_1\}$ is the bipartition of $G(\mathcal{C})$, then:

(ii) Neither of U_1, V_1 is a cover of $\mathcal{C}$ ([6], Theorem 7).

(iii) U_1, V_1 are members of $\mathcal{C}$.

Proof. **(iii)** It follows from (ii) that the complement of U_1, namely V_1, contains a member of $\mathcal{C}$. Since every member of $\mathcal{C}$ is a vertex cover of $G(\mathcal{C})$ and no proper subset of V_1 is a vertex cover, it follows that V_1 itself is a member of $\mathcal{C}$. Similarly, U_1 is a member of $\mathcal{C}$, as required. □

For any clutter $\mathcal{C}$, a *fractional packing* of $\mathcal{C}$ is a feasible point of the following linear program:

$$\begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum (y_C : v \in C \in \mathcal{C}) \leq 1 \quad v \in V \\ & y \geq \mathbf{0}. \end{array}$$

The *value* of a fractional packing y is the value $\mathbf{1}^\top y$ of the linear program. It can be readily verified that by weak duality, the value of any fractional packing is bounded above by $\tau(\mathcal{C})$. The following result was proved recently:

Theorem 12 ([6], Theorem 3 and [5], Lemma 1.6)**.** *If $\mathcal{C}$ is a clean tangled clutter, then $\mathcal{C}$ has a fractional packing of value two.*

As a result, we may study the structure of the members of $\mathcal{C}$ used in some such fractional packing:

Definition 13. *Let $\mathcal{C}$ be a clean tangled clutter. Then the* core *of $\mathcal{C}$ is the clutter*

$$\textit{core}(\mathcal{C}) = \{C \in \mathcal{C} : y_C > 0 \textit{ for some fractional packing } y \textit{ of value two}\}.$$

We use a complementary slackness argument to prove the following:

Lemma 14. *Let $\mathcal{C}$ be a clean tangled clutter over ground set V. Then every member of $\mathrm{core}(\mathcal{C})$ is a transversal of the cardinality-two covers of $\mathcal{C}$. Moreover, for every fractional packing y of value two and for every element $v \in V$, we have $\sum (y_C : v \in C \in \mathcal{C}) = 1$.*

Proof. Let $\{u, v\}$ be a cover of $\mathcal{C}$, and let y be a fractional packing of $\mathcal{C}$. Then we have

$$\sum(y_C : C \in \mathcal{C}) \leq \sum(y_C : u \in C \in \mathcal{C}) + \sum(y_C : v \in C \in \mathcal{C}) \leq 2$$

The first inequality follows from the fact that each $C \in \mathcal{C}$ contains either u or v, and the second follows from adding the congestion inequalities for each of u, v. If y has value two, then $\sum(y_C : C \in \mathcal{C}) = 2$, so we have equality above. The first equality implies that if $y_C > 0$, then C contains exactly one of u, v. Therefore, every member of $\mathrm{core}(\mathcal{C})$ is a transversal of the cardinality-two covers $\{u, v\}$.

The second equality implies that $\sum(y_C : u \in C \in \mathcal{C}) = \sum(u_C : v \in C \in \mathcal{C}) = 1$. Since $\mathcal{C}$ is tangled, every element $v \in V$ appears in some cardinality-two cover of $\mathcal{C}$, so $\sum(y_C : v \in C \in \mathcal{C}) = 1$ for all $v \in V$. □

As a result, since $G(\mathcal{C})$ is bipartite, members of the core must respect the bipartition of each connected component:

Remark 15. *Let $\mathcal{C}$ be a clean tangled clutter, and let $r = \mathrm{rank}(\mathcal{C})$. For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of $G(\mathcal{C})$. If $C \in \mathrm{core}(\mathcal{C})$, then $C \cap (U_i \cup V_i) \in \{U_i, V_i\}$ for $i \in [r]$.*

In other words, each $C \in \mathrm{core}(\mathcal{C})$ is determined by r binary choices; in each connected component, C must contain exactly one of the two parts of the bipartition. This allows a more concise representation of the core:

Definition 16. *Let $\mathcal{C}$ be a clean tangled clutter, and let $r = \mathrm{rank}(\mathcal{C})$. For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of $G(\mathcal{C})$. For each $C \in \mathrm{core}(\mathcal{C})$, define $p_C \in \{0,1\}^r$ such that*

$$(p_C)_i = \begin{cases} 0 & \text{if } C \cap (U_i \cup V_i) = V_i \\ 1 & \text{if } C \cap (U_i \cup V_i) = U_i \end{cases}$$

The setcore *of $\mathcal{C}$ is the subset of $\{0,1\}^r$ given by $\mathrm{setcore}(\mathcal{C}) = \{p_C : C \in \mathrm{core}(\mathcal{C})\}$. (The setcore is defined up to relabeling and twisting coordinates, since our definition would change if $U_1, \dots, U_r$ were relabeled, or if the roles of U_i, V_i were swapped.)*

Given a clutter $\mathcal{C}$ over ground set V, we may obtain a new clutter by *duplicating* a chosen element $v \in V$; specifically, we obtain the clutter $\mathcal{C}'$ over ground set $V \cup \{v'\}$ where $v' \notin V$, given by

$$\mathcal{C}' = \{C \cup \{v'\} : C \in \mathcal{C}, v \in C\} \cup \{C : C \in \mathcal{C}, v \notin C\}.$$

In general, if $\mathcal{C}'$ can be obtained from $\mathcal{C}$ by a finite number of duplications, we say that $\mathcal{C}'$ is a *duplication* of $\mathcal{C}$. Note that:

Remark 17. *Given a clean tangled clutter $\mathcal{C}$, $\mathrm{core}(\mathcal{C})$ is a duplication of the cuboid of $\mathrm{setcore}(\mathcal{C})$.*

Now consider the task of finding a fractional packing of $\mathcal{C}$ of value two. By definition, our fractional packing may only use members in $\mathrm{core}(\mathcal{C})$, and by Lemma 14, our fractional packing must assign a total weight of 1 to members C with $(p_C)_i = 0$, and a total weight of 1 to members C with $(p_C)_i = 1$, for each $i \in [r]$. Therefore, after a $\frac{1}{2}$-scaling, finding a fractional packing of value two becomes equivalent to expressing $\frac{1}{2} \cdot \mathbf{1} \in [0,1]^r$ as a convex combination of $\mathrm{setcore}(\mathcal{C})$. Then Theorem 12 implies:

Remark 18. *If $\mathcal{C}$ is a clean tangled clutter, then the convex hull of $\mathrm{setcore}(\mathcal{C})$ contains $\frac{1}{2} \cdot \mathbf{1}$. Moreover, for each $x \in \mathrm{setcore}(\mathcal{C})$, we can write $\frac{1}{2} \cdot \mathbf{1}$ as a convex combination of $\mathrm{setcore}(\mathcal{C})$ which assigns a nonzero weight to x.*

3.2 Ideal minimally non-packing clutters of rank three

We just saw in Remark 18 that if $\mathcal{C}$ is a clean tangled clutter, then the setcore contains the center of the unit hypercube in its convex hull. If we additionally assume $\mathcal{C}$ does not pack and has rank three, i.e. $G(\mathcal{C})$ has exactly three connected components, then the setcore is determined up to twisting:

Lemma 19. *Let $\mathcal{C}$ be a clean tangled clutter with rank three, and assume that $(0, 0, 0) \in \operatorname{setcore}(\mathcal{C})$ and $\mathcal{C}$ does not pack. Then $\operatorname{setcore}(\mathcal{C}) = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$.*

For the proof of the lemma, note that finding an integral packing of value two in $\mathcal{C}$ is equivalent to finding two points $p, q \in \operatorname{setcore}(\mathcal{C})$ such that $p + q = \mathbf{1}$. That is, finding two disjoint members in $\mathcal{C}$ amounts to finding a pair of antipodal points in $\operatorname{setcore}(\mathcal{C})$.

Proof. Since $\mathcal{C}$ is tangled, we have $\tau(\mathcal{C}) = 2$. Since $\mathcal{C}$ does not pack, $\operatorname{setcore}(\mathcal{C})$ does not have antipodal points, so $(1, 1, 1) \notin \operatorname{setcore}(\mathcal{C})$. By Remark 18, we can write $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ as a convex combination of $\operatorname{setcore}(\mathcal{C})$ which assigns a nonzero weight to $(0, 0, 0)$. Since $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ lies on the plane $x_1 + x_2 = 1$, and $(0, 0, 0)$ lies on one side of the plane, $\operatorname{setcore}(\mathcal{C})$ must contain a point on the other side, which must be $(1, 1, 0)$. Similarly, $\operatorname{setcore}(\mathcal{C})$ contains $(1, 0, 1)$ and $(0, 1, 1)$, and cannot contain any other points by the antipodality restriction. $\square$

The hypothesis that $\operatorname{setcore}(C) = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ guarantees the presence of four specific members of $\mathcal{C}$, which in turn imposes restrictions on the minimal covers of $\mathcal{C}$:

Lemma 20. *Let $\mathcal{C}$ be a clean tangled clutter with rank three, and denote by $\{U_1, V_1\}, \{U_2, V_2\}, \{U_3, V_3\}$ the bipartitions of the connected components of $G(\mathcal{C})$. Assume that*

$$C_1 = V_1 \cup V_2 \cup V_3 \qquad C_2 = V_1 \cup U_2 \cup U_3 \qquad C_3 = U_1 \cup V_2 \cup U_3 \qquad C_4 = U_1 \cup U_2 \cup V_3$$

are members of $\mathcal{C}$. Then:

(i) If C_i is a cover for some $i \in [4]$, then C_i contains a minimal cover of cardinality three, consisting of one element from each connected component of $G(\mathcal{C})$.

(ii) Assume that $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, v_3'\}$ are minimal covers of $\mathcal{C}$ for some $u_1 \in U_1$, $u_2 \in U_2$, $v_1 \in V_1$, $v_2 \in V_2$, $v_3, v_3' \in V_3$ (see Figure 2). Then there exists a minimal cover B of $\mathcal{C}$ with $B \cap (U_1 \cup V_1) = \{u_1, v_1\}$ and $|B| \leq 3$.

Proof. **(i)** By symmetry, it suffices to prove the statement for C_1. Choose a cover $B \subseteq V_1 \cup V_2 \cup V_3$ of minimum cardinality. Then B intersects each V_i, else B would be disjoint from one of C_2, C_3, C_4. Suppose for contradiction $|B \cap V_1| \geq 2$, and consider $\mathcal{C}' = \mathcal{C} \setminus I/J$ for $I = B \setminus V_1$, $J = (U_2 \cup V_2 \cup U_3 \cup V_3) \setminus B$, so that $\mathcal{C}'$ has ground set $U_1 \cup V_1$. We have two cases:

Case 1: $\tau(\mathcal{C}') \geq 2$. Since $\mathcal{C}'$ is a minor of $\mathcal{C}$, it follows that $\mathcal{C}'$ is clean. Since the induced subgraph $G(\mathcal{C})[U_1 \cup V_1]$ is a subgraph of $G(\mathcal{C}')$, it follows that $\mathcal{C}'$ is tangled. Then $G(\mathcal{C}')$ is bipartite by Theorem 11 (i). The subgraph relation implies that $G(\mathcal{C}')$ is connected and has bipartition $\{U_1, V_1\}$, so U_1 is a member of $\mathcal{C}'$ by Theorem 11 (iii). But $B \setminus I = B \cap V_1$ is a cover of $\mathcal{C}'$ disjoint from U_1, a contradiction.

Case 2: $\tau(\mathcal{C}') \leq 1$; that is, there exists $D \in b(\mathcal{C})$ with $D \subseteq U_1 \cup V_1 \cup I$ and $|D \cap (U_1 \cup V_1)| \leq 1$. Therefore, if D contains an element of U_1, then $D \cap V_1 = \emptyset$, so D is disjoint from $C_4 = V_1 \cup U_2 \cup U_3$, a contradiction. Therefore, D is disjoint from U_1, so $D \subseteq V_1 \cup V_2 \cup V_3$, and

$$|D| \leq |D \setminus I| + |I| \leq 1 + |B \setminus V_1| < |B \cap V_1| + |B \setminus V_1| = |B|,$$

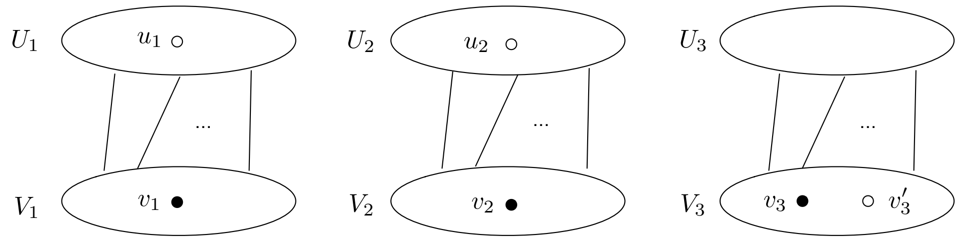

Figure 2: The two minimal covers $\{v_1, v_2, v_3\}, \{u_1, u_2, v_3'\}$ provided in Lemma 20 (ii).

a contradiction to the minimality of B. Hence $|B \cap V_1| = 1$, and similarly $|B \cap V_2| = 1$ and $|B \cap V_3| = 1$.

(ii) Suppose otherwise. Let $J = (U_1 \cup V_1) \setminus \{u_1, v_1\}$, and let $\mathcal{C}' = \mathcal{C} \setminus \{u_1, v_1\}/J$; then our contrary assumption implies $\tau(\mathcal{C}') \geq 2$. Since $\mathcal{C}'$ is a minor of $\mathcal{C}$, it follows that $\mathcal{C}'$ is clean. Since the induced subgraph $G(\mathcal{C})[U_2 \cup V_2 \cup U_3 \cup V_3]$ is a subgraph of $G(\mathcal{C}')$, it follows that $\mathcal{C}'$ is tangled. Then $G(\mathcal{C}')$ is bipartite by Theorem 11 (i).

Note that u_2, v_2 are connected by an odd length path in $G(\mathcal{C})[U_2 \cup V_2]$ and hence also in $G(\mathcal{C}')[U_2 \cup V_2]$. Similarly, v_3, v_3' are connected by an even length path in $G(\mathcal{C})[U_3 \cup V_3]$ and hence also in $G(\mathcal{C}')[U_3 \cup V_3]$. But since $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, v_3'\}$ are minimal covers of $\mathcal{C}$, it follows that $\{u_2, v_3'\}, \{v_2, v_3\}$ are covers of $\mathcal{C}'$, hence edges of $G(\mathcal{C}')$, which gives an odd cycle in $G(\mathcal{C}')$, a contradiction. $\square$

The following theorem summarizes the discussion in this section:

Theorem 21. *Let $\mathcal{C}$ be an ideal minimally non-packing clutter with covering number two and rank three. Then:*

(i) *$\mathcal{C}$ is a clean tangled clutter.*

(ii) *$G(\mathcal{C})$ is bipartite, and the bipartitions of the connected components may be labeled as $\{U_1, V_1\}, \{U_2, V_2\}$, $\{U_3, V_3\}$, in such a way that* $\mathrm{setcore}(\mathcal{C}) = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$*, so that the following are members of $\mathcal{C}$:*

$$C_1 = V_1 \cup V_2 \cup V_3, \qquad C_2 = U_1 \cup U_2 \cup V_3, \qquad C_3 = U_1 \cup V_2 \cup U_3, \qquad C_4 = V_1 \cup U_2 \cup U_3$$

(iii) *Each of C_1, C_2, C_3, C_4 contains a minimal cover of cardinality three, consisting of one element from each connected component of $G(\mathcal{C})$.*

(iv) *If $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, v_3'\}$ are minimal covers of $\mathcal{C}$ for some $u_1 \in U_1$, $u_2 \in U_2$, $v_1 \in V_1$, $v_2 \in V_2$, $v_3, v_3' \in V_3$, then there exists a minimal cover B of $\mathcal{C}$ with $B \cap (U_1 \cup V_1) = \{u_1, v_1\}$ and $|B| \leq 3$, and similarly, there exists a minimal cover B' of $\mathcal{C}$ with $B' \cap (U_2 \cup V_2) = \{u_2, v_2\}$ and $|B'| \leq 3$. (The analogous statements obtained by using any of U_1, U_2, U_3, V_1, V_2 in the place of V_3 also hold.)*

Proof. Statement (i) follows from Remark 10. By Theorem 11 (i), $G(\mathcal{C})$ is bipartite, and by Theorem 12, $\mathrm{core}(\mathcal{C})$ is nonempty, so we may label the bipartitions of the connected components as $\{U_1, V_1\}, \{U_2, V_2\}$,

$\{U_3, V_3\}$ in such a way that $(0,0,0) \in \mathrm{setcore}(\mathcal{C})$. Then by Lemma 19, we have $\mathrm{setcore}(\mathcal{C}) = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$, so that C_1, C_2, C_3, C_4 are members of $\mathcal{C}$.

Now we observe that C_1 is also a cover of $\mathcal{C}$, since otherwise, $\mathcal{C}$ would have a member disjoint from C_1, resulting in a pair of disjoint members, contradicting the assumption that $\mathcal{C}$ does not pack. Similarly, C_2, C_3, C_4 are covers of $\mathcal{C}$. Hence (iii) and (iv) follow from Lemma 20. □

Consider the new ideal minimally non-packing clutters of Theorem 2. We leave it to the reader to verify that the four minimal covers $\{2,4,6\}, \{2,3,5\}, \{1,4,5'\}, \{1,3,6'\}$ "come from" Theorem 21 (iii), while the two minimal covers $\{3,5,6'\}, \{4,5',6\}$ come from Theorem 21 (iv).

The well-known clutter Q_6 over ground set [6] is defined by

$$Q_6 = \{\{2,4,6\}, \{1,3,6\}, \{1,4,5\}, \{2,3,5\}\}.$$

Theorem 21 (ii) and Remark 17 immediately imply:

Corollary 22. *Let $\mathcal{C}$ be an ideal minimally non-packing clutter with covering number two and rank three. Then* $\mathrm{core}(\mathcal{C})$ *is a duplication of* Q_6.

It is known that all ideal minimally non-packing clutters with covering number two share a weaker property:

Theorem 23 ([9]). *Let $\mathcal{C}$ be an ideal minimally non-packing clutter with covering number two. Then some subset of $\mathcal{C}$ is a duplication of Q_6.*

In general, this Q_6-like subset need not be $\mathrm{core}(\mathcal{C})$ itself, since $\mathrm{rank}(\mathcal{C})$ may be greater than three. It is a feature of our construction that the clutters obtained, having rank three, exhibit Q_6-like structure in their core.

4 Previously known constructions

We conclude by describing the small instances of ideal minimally non-packing clutters provided by Cornuéjols, Guenin and Margot [9]. These constructions give thirteen ideal minimally non-packing clutters with covering number two and rank three. In Figures 3 and 4, these clutters are depicted via their blockers; cardinality-two minimal covers are shown as edges of a bipartite graph, and cardinality-three minimal covers are listed below the graph (and there are no other minimal covers).

The first twelve clutters (see Figure 3) are obtained by a common construction and are denoted $Q_6 \otimes X$, where $X \subseteq [6]$, subject to some restrictions on X. The construction produces clutters with exactly four minimal covers of cardinality three and no minimal covers of higher cardinality. The clutter $Q_6 \otimes \emptyset$ is just Q_6. The thirteenth clutter (see Figure 4) is a one-off example not conforming to the construction, and has the following incidence matrix:

$$\begin{array}{c} \begin{array}{cccccccc} 1 & 1' & 2 & 2' & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

All of these clutters share the properties required by Theorem 21. For instance, for the thirteenth clutter with the incidence matrix shown above, we have $U_1 = \{1, 1'\}, V_1 = \{2, 2'\}$, $U_2 = \{3\}, V_2 = \{4\}$ and $U_3 = \{6\}, V_3 = \{5\}$. The first four rows form the core, reaffirming (ii). The four members contain the minimal covers $\{1', 3, 5\}, \{1, 4, 6\}, \{2', 3, 6\}, \{2, 4, 5\}$, respectively, reaffirming (iii). Lastly, the five cardinality-two minimal covers, along with the minimal cover $\{1, 2', 6\}$, reaffirm (iv).

Late note on dijoins. After writing this paper, we noticed an intimate connection between our findings and some results on *dijoins* from the early 2000s. Let us explain this connection.

Ideal minimally non-packing clutters arise naturally from directed graphs. Let $D = (V, A)$ be a digraph. A *dicut* is a cut of the form $\delta^+(U) \subseteq A$ where $\delta^-(U) = \emptyset$, for some nonempty and proper subset U of V. A *dijoin* is any cover of the clutter of minimal dicuts. Equivalently, $J \subseteq A$ is a dijoin if D/J is strongly connected.

Denote by $\mathcal{C}(D)$ be the clutter of minimal dijoins of D. It follows from a well-known result of Lucchesi and Younger that $\mathcal{C}(D)$ is ideal [17]. *Woodall's Conjecture* states that the clutter $\mathcal{C}(D)$ must pack, a problem that has remained open to this date despite its simple statement [23]. The history of the problem is further muddled by the fact that $\mathcal{C}$ does *not* have the packing property. (Deletion in the clutter does *not* correspond to deletion in the digraph.)

Consider the three digraphs D_1, D_2, D_3 depicted in Figures 5, 6 and 7. The first digraph is due to Schrijver [18], while the other two are due to Cornuéjols and Guenin [8]. These digraphs were found as counterexamples to a conjecture of Edmonds and Giles on dijoins [12]. Even though $\mathcal{C}(D_i), i \in [3]$ pack, they do not have the packing property. More specifically, for $i \in [3]$, denote by I_i the arc subset corresponding to the dashed arcs of D_i. Then the minor $\mathcal{C}(D_i) \setminus I_i, i \in [3]$ is an ideal minimally non-packing clutter with covering number two. In fact, $\mathcal{C}(D_1) \setminus I_1 = Q_6 \otimes \{2, 4, 5\}$. The blocker of $\mathcal{C}(D_i) \setminus I_i$ is depicted next to D_i, where cardinality-two minimal covers are shown as edges of a bipartite graph, and all the other minimal covers are listed below the graph. The reader will notice that each $\mathcal{C}(D_i) \setminus I_i, i \in [3]$ has rank three. This can already be observed by looking at the digraph, as the solid arcs form three connected components, each of which is an alternating path of sources and sinks.

For more information, we refer the reader to Aaron Williams's very interesting Master's thesis [22]. He has shown that, up to a novel reduction called *folding*, the three clutters $\mathcal{C}(D_i) \setminus I_i, i \in [3]$ are the only ideal minimally non-packing clutters of covering number two and rank three coming from dijoins (see Chapter 6). He has also shown that $\mathcal{C}(D) \setminus I$ cannot be ideal minimally non-packing with covering number two and rank four (see Chapter 7).

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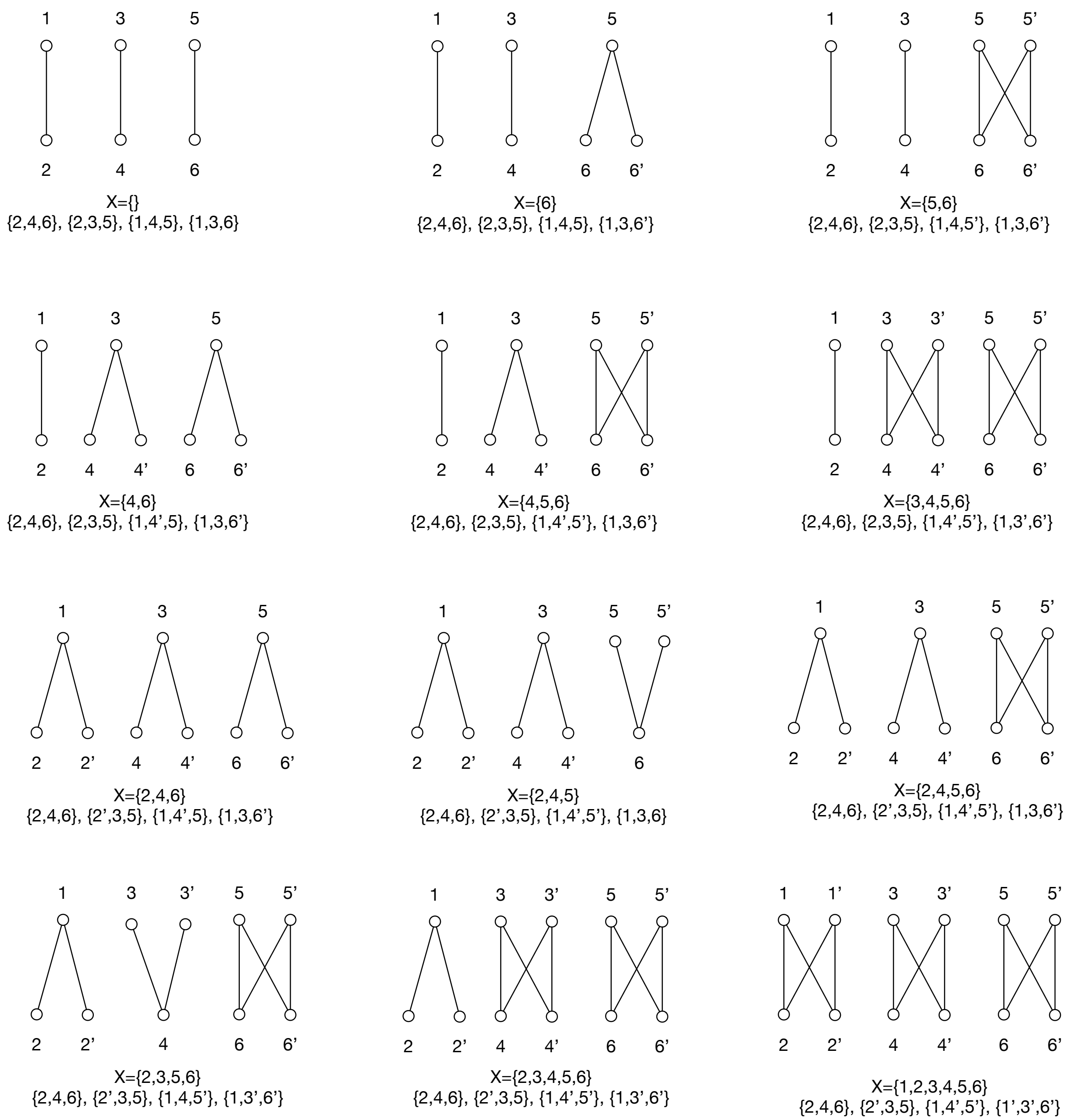

Figure 3: The blockers of the twelve clutters $Q_6 \otimes X$ from [9].

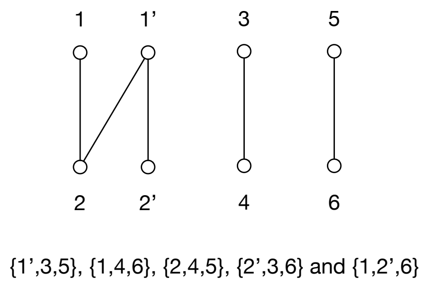

Figure 4: The blocker of the thirteenth ideal minimally non-packing clutter of rank three from [9].

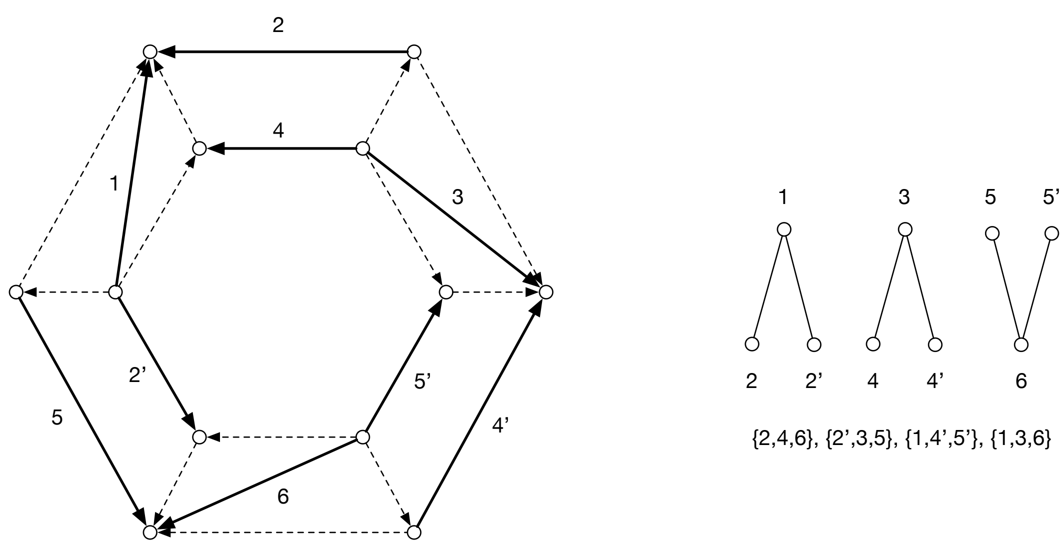

Figure 5: Left: The digraph D_1 from [18]. The unlabeled dashed arcs form the set I_1. Right: An illustration of the blocker of $\mathcal{C}(D_1) \setminus I_1$.

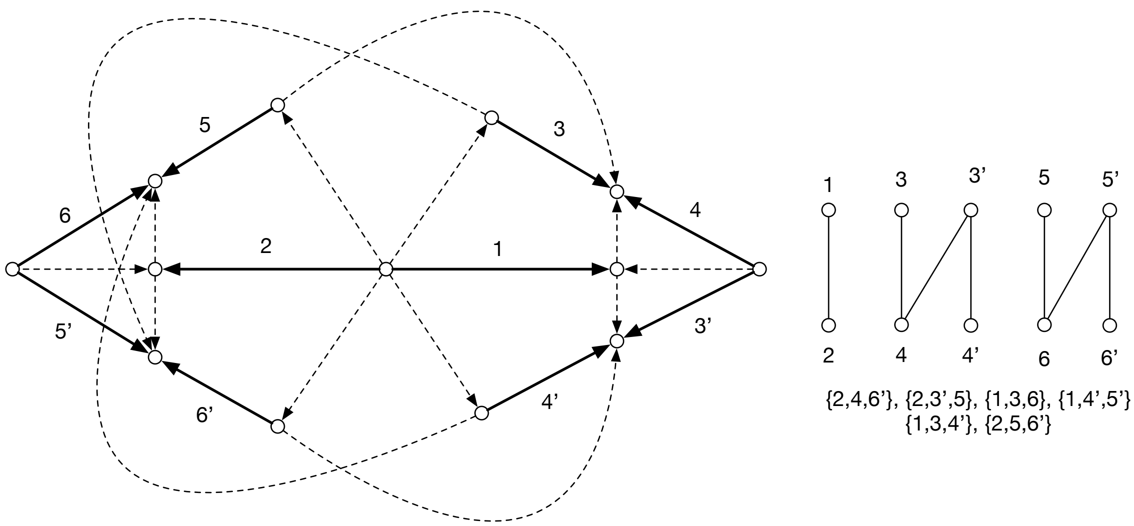

Figure 6: Left: The digraph D_2 from [8]. The unlabeled dashed arcs form the set I_2. Right: An illustration of the blocker of $\mathcal{C}(D_2) \setminus I_2$.

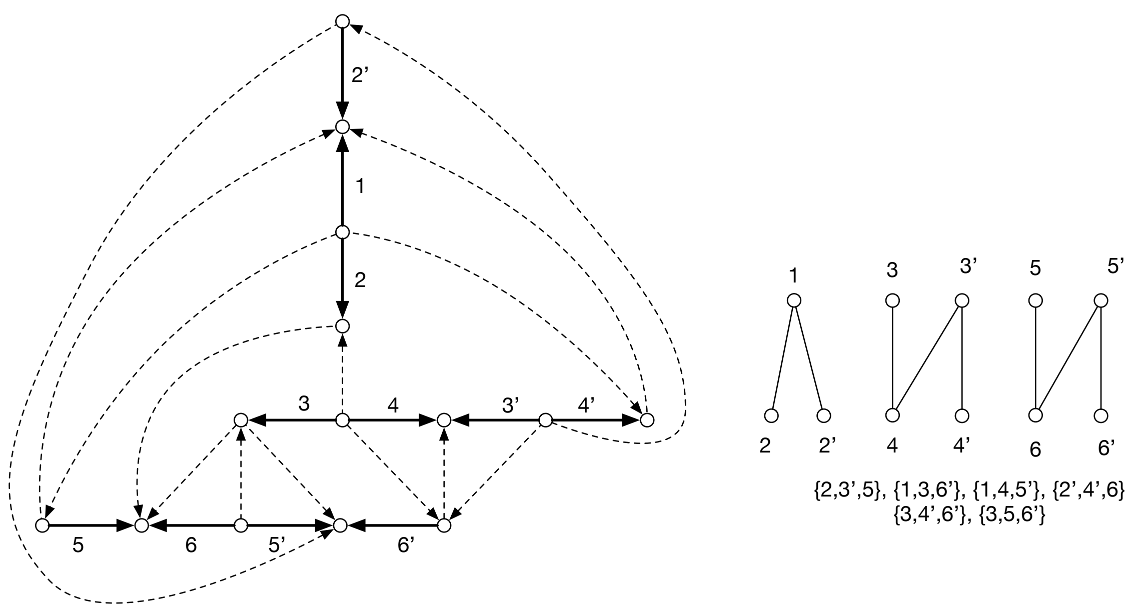

Figure 7: Left: The digraph D_3 from [8]. The unlabeled dashed arcs form the set I_3. Right: An illustration of the blocker of $\mathcal{C}(D_3) \setminus I_3$.

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