A min-max theorem for clean tangled clutters

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Abstract

A clutter is *clean* if it has no delta or the blocker of an extended odd hole minor. There are combinatorial and geometric classes of clean clutters, namely ideal clutters, clutters without an intersecting minor, and binary clutters. A clutter is *tangled* if it has covering number two and every element is in a minimum cover. Clean tangled clutters arise, for example, when studying the packing property in Clutter Theory. We define two parameters on clean tangled clutters, one is the optimum of a combinatorial minimization problem called the *monochromatic covering number*, extending notions such as matroid *girth* and clutter *covering number*, while the other is the optimum of a geometric maximization problem called the *depth*, intimately linked to another parameter defined recently called the *notch* of a set system. We prove a min-max relation between the two parameters, exposing an intriguing interplay between the combinatorics and the geometry of such clutters. As a consequence of our results, we also prove that the core of an ideal tangled clutter is an ideal clutter.

Keywords. Clutters, clean tangled clutters, min-max relation, notch, covering number, girth.

1 Introduction

Take an integer $n \geq 3$. Denote by Δ_n the clutter over ground set $[n] := \{1, \ldots, n\}$ whose members are $\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}, \{2, 3, \ldots, n\}$. Any clutter isomorphic to Δ_n is called a *delta of dimension* n. A delta is equal to its blocker. Given an odd integer $n \geq 5$, an *extended odd hole of dimension* n is any clutter whose ground set can be relabeled as $[n]$ so that its minimum cardinality members are precisely $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}$. The deltas, extended odd holes, and their blockers form a basic class of non-ideal clutters.

It was proved in [13] that testing idealness of a clutter is co-NP-complete, a result that is surprising given that testing perfection of a clutter belongs to P [10]. The bottleneck is that given a clutter, finding a minor that is a delta or an extended odd hole, is an NP-hard task. In another surprising turn of events, it was recently shown that the same problem would belong to P if the input were the blocking clutter rather than the clutter itself. More precisely,

Definition 1.1. *A clutter is* clean *if it has no minor that is a delta or the blocker of an extended odd hole.*

Unlike idealness, testing cleanness of a clutter belongs to P [5]. The class of clean clutters contains ideal clutters, binary clutters, as well as clutters without an intersecting minor. There are several conjectures in Combinatorial Optimization, Graph Theory, and Matroid Theory that relate to these classes of clutters. The Flowing Conjecture relating to ideal binary clutters [21], and the $\tau = 2$ Conjecture relating to ideal clutters without an intersecting minor [11] are two highlights. In fact, it suffices to prove these two conjectures, as well as a few others, for clean *tangled* clutters.

Definition 1.2. *A clutter $\mathcal{C}$ is* tangled *if it has covering number two, and every element appears in a minimum cover. Given a tangled clutter $\mathcal{C}$ over ground set V, denote by $G(\mathcal{C})$ the graph over vertex set V whose edges correspond to the minimum covers of $\mathcal{C}$.*

Clean tangled clutters have been a recent subject of study by the authors. In [1], we proved the Dyadic Conjecture for clean tangled clutters. In [6], the structure of these clutters was used to obtain a new infinite class of ideal minimally non-packing clutters. In [7], an interplay between the geometry and the combinatorics of these clutters were exposed, where a connection between simplices and projective geometries was shown. We continue the study of clean tangled clutters, further cementing the interplay between the geometry and the combinatorics of such clutters. To this end, let $\mathcal{C}$ be a clean tangled clutter.

Rank It can be readily seen that $G(\mathcal{C})$ is a bipartite graph. The *rank of* $\mathcal{C}$, denoted $\mathrm{rank}(\mathcal{C})$, is the number of connected components of $G(\mathcal{C})$.

Monochromatic covering number A cover of $\mathcal{C}$ is *monochromatic* if it is monochromatic in some proper 2-vertex-coloring of $G(\mathcal{C})$. The *monochromatic covering number of* $\mathcal{C}$ is defined as

$$\mu(\mathcal{C}) := \min\{|B| : B \text{ is a monochromatic cover of } \mathcal{C}\};$$

if there is no monochromatic cover, then $\mu(\mathcal{C}) := \infty$. Observe that $\mu(\mathcal{C}) \geq 3$.

Depth The *core of* $\mathcal{C}$, denoted $\mathrm{core}(\mathcal{C})$, is the set of all members $C \in \mathcal{C}$ that intersect every minimum cover of $\mathcal{C}$ exactly once. Alternatively, the core collects the members that can be used with a nonzero coefficient in a fractional packing of value two. It is known that $\mathrm{core}(\mathcal{C})$ is a duplication of the cuboid of a set $S \subseteq \{0,1\}^{\mathrm{rank}(\mathcal{C})}$. Moreover, it is known that S is unique up to relabeling and twisting the coordinates, and its convex hull is a full-dimensional polytope containing $\frac{1}{2} \cdot \mathbf{1}$ in its interior. The set S is called the *setcore of* $\mathcal{C}$ [7]. Let us call the points in S *feasible*, and the points in $\{0,1\}^{\mathrm{rank}(\mathcal{C})} - S$ *infeasible*. The *depth of* $\mathcal{C}$ is defined as

$$\mathrm{depth}(\mathcal{C}) := \max\{d : \mathrm{setcore}(\mathcal{C}) \text{ has a } d\text{-dimensional infeasible hypercube}\};$$

if $\mathrm{setcore}(\mathcal{C})$ has no infeasible hypercube, then $\mathrm{depth}(\mathcal{C}) := -\infty$.

The min-max relation In §3 we prove that for every clean tangled clutter $\mathcal{C}$, we have

$$\mathrm{rank}(\mathcal{C}) - \mu(\mathcal{C}) = \mathrm{depth}(\mathcal{C})\,.$$

Observe that $\mu(\mathcal{C})$ is a parameter defined on all of $\mathcal{C}$, while $\mathrm{depth}(\mathcal{C})$ is a parameter defined only on the core of $\mathcal{C}$, so the min-max relation above is non-trivial. Observe that the monochromatic covering number is the optimum of a *combinatorial* minimization problem, while the depth is the optimum of a *geometric* maximization problem. Thus, the min-max relation manifest an intriguing interplay between the combinatorics and the geometry of clean tangled clutters. On the combinatorial side, we see in §4 that our notion of monochromatic covering number extends the conventional notion of girth for simple graphs and more generally simple binary matroids, as well as the notion of covering number for clean clutters with covering number at least three. On the geometric side, our notion of depth is intimately linked to the notion of *notch* defined recently in [9]. More precisely, when the depth of $\mathcal{C}$ is finite, then it is equal to the notch of $\mathrm{setcore}(\mathcal{C})$ minus 1. In §4, we also introduce *irreducible* monochromatic covers, and prove as a consequence that the core of an ideal tangled clutter is also ideal.

2 Definitions and preliminaries

Clutters Let V be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of V called *members*. $\mathcal{C}$ is a *clutter* over *ground set* V if no member contains another [14]. Two clutters are *isomorphic* if one is obtained from the other by relabeling the ground set. Two distinct elements u, v are *duplicates in* $\mathcal{C}$ if for each $C \in \mathcal{C}$, $u \in C$ if and only if $v \in C$. To *duplicate an element* u *of* $\mathcal{C}$ is to introduce a new element v and replace $\mathcal{C}$ by the clutter over ground set $V \cup \{v\}$ whose members are $\{C : u \notin C \in \mathcal{C}\} \cup \{C \cup \{v\} : u \in C \in \mathcal{C}\}$. A *duplication of* $\mathcal{C}$ is any clutter obtained from $\mathcal{C}$ after repeatedly duplicating some elements. A *transversal* is a subset $B \subseteq V$ such that $|B \cap C| = 1$ for all $C \in \mathcal{C}$. A *cover* is a subset $B \subseteq V$ such that $B \cap C \neq \emptyset$ for all $C \in \mathcal{C}$. The *covering number* of $\mathcal{C}$, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. A cover is *minimal* if it does not contain another cover. The *blocker* of $\mathcal{C}$, denoted $b(\mathcal{C})$, is the clutter over ground set V whose members are the minimal covers of $\mathcal{C}$ [14]. It is well-known that $b(b(\mathcal{C})) = \mathcal{C}$ [16, 14]. Take disjoint $I, J \subseteq V$. The *minor* of $\mathcal{C}$ obtained after *deleting* I and *contracting* J, denoted $\mathcal{C} \setminus I/J$, is the clutter over ground set $V - (I \cup J)$ whose members consist of the inclusion-wise minimal sets of $\{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$. The minor is *proper* if $I \cup J \neq \emptyset$. It is well-known that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ [20].

Cuboids Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$. We refer to the points in S as *feasible* and the points in $\{0,1\}^r - S$ as *infeasible*. The *cuboid of* S, denoted $\mathrm{cuboid}(S)$, is the clutter over ground set $[2r]$ whose members have incidence vectors $\{(p_1, 1-p_1, \ldots, p_r, 1-p_r) : p \in S\}$ [8, 2]. Observe that for each $i \in [r]$, $|C \cap \{2i-1, 2i\}| = 1$ for all $C \in \mathrm{cuboid}(S)$, so $\{2i-1, 2i\}$ is a cover of the cuboid. Observe that if the points in S do not agree on a coordinate, then $\mathrm{cuboid}(S)$ is a tangled clutter; that is, any cuboid without a cover of size one is tangled. Take a point $q \in \{0,1\}^r$. To *twist* S *by* q is to replace S by $S \triangle q := \{p \triangle q : p \in S\}$, where the second $\triangle$ denotes coordinate-wise addition modulo 2. Take a coordinate $i \in [r]$. Denote by e_i the i^{th} unit vector of appropriate dimension. To *twist coordinate* i *of* S is to replace S by $S \triangle e_i$. Two sets S_1, S_2 are *isomorphic*, written as $S_1 \cong S_2$, if one is obtained from the other after relabeling and twisting some coordinates. Two distinct coordinates $i, j \in [r]$ are *duplicates in* S if $S \subseteq \{x : x_i = x_j\}$ or $S \subseteq \{x : x_i + x_j = 1\}$. Observe that if two coordinates are duplicates in a set, then they are duplicates in any isomorphic set. Observe further that S has duplicated coordinates if, and only if, $\mathrm{cuboid}(S)$ has duplicated elements.

2.1 Core and setcore

Let $\mathcal{C}$ be a clean tangled clutter over ground set V. Recall that $G(\mathcal{C})$ is the graph over vertex set V whose edges correspond to the minimum covers of $\mathcal{C}$. Recall that if $\mathcal{C}$ is a clean tangled clutter, then the rank of $\mathcal{C}$, denoted $\mathrm{rank}(\mathcal{C})$, is the number of connected components of $G(\mathcal{C})$. It can be readily checked that $G(\mathcal{C})$ is a bipartite graph. Let $G := G(\mathcal{C})$ and $r := \mathrm{rank}(\mathcal{C})$. For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G. Recall that the core of $\mathcal{C}$ is defined as

$$\mathrm{core}(\mathcal{C}) := \{C \in \mathcal{C} : |C \cap B| = 1 \text{ for every minimum cover } B \text{ of } \mathcal{C}\}.$$

Theorem 2.2 below shows that the core is nonempty, and has covering number two.

Theorem 2.1 ([7], Theorem 2.9)**.** $\mathrm{core}(\mathcal{C}) = \{C \in \mathcal{C} : C \cap (U_i \cup V_i) \in \{U_i, V_i\}$ *for each* $i \in [r]\}$.

In particular, $\mathrm{core}(\mathcal{C})$ is a duplication of a cuboid – let us elaborate. The *setcore of* $\mathcal{C}$ *with respect to* $(U_1, V_1; U_2, V_2; \ldots; U_r, V_r)$ is the set $S \subseteq \{0,1\}^r$ defined as follows: start with $S = \emptyset$, and for each $C \in \mathrm{core}(\mathcal{C})$, add a point p to S such that

$$p_i = 0 \quad \Leftrightarrow \quad C \cap (U_i \cup V_i) = U_i \quad \forall i \in [r].$$

By Theorem 2.1, the set S is well-defined and $\mathrm{core}(\mathcal{C})$ is a duplication of $\mathrm{cuboid}(S)$. We denote S by the notation $\mathrm{setcore}(\mathcal{C} : U_1, V_1; U_2, V_2; \ldots; U_r, V_r)$. As the reader can imagine, we will not use this notation often, and use $\mathrm{setcore}(\mathcal{C})$ as short-hand notation. Note however that $\mathrm{setcore}(\mathcal{C})$ is defined only up to isomorphism.

Theorem 2.2 ([7], Theorem 1.5)**.** $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$ *is a full-dimensional polytope contained in* $[0,1]^r$ *and containing* $\frac{1}{2} \cdot \mathbf{1}$ *in its interior. In particular,* $\mathrm{setcore}(\mathcal{C})$ *does not have duplicated coordinates,* $\mathrm{core}(\mathcal{C})$ *is nonempty, and has covering number two.*

We also need the following earlier results.

Theorem 2.3 ([7], Theorem 2.10)**.** *The following statements hold:*

(i) If $r = 1$*, then* $\mathrm{core}(\mathcal{C}) = \{U_1, V_1\}$.

(ii) If $r = 2$*, then* $\mathrm{core}(\mathcal{C}) = \{U_1 \cup U_2, U_1 \cup V_2, V_1 \cup U_2, V_1 \cup V_2\}$.

Theorem 2.4 ([1], Theorem 2.5, and [7], Lemma 2.6)**.** *Suppose* G *is not connected. Let* $\{U, U'\}$ *be the bipartition of a connected component of* G*. Then* $\mathcal{C} \setminus U/U'$ *is a clean tangled clutter such that* $\mathrm{core}(\mathcal{C} \setminus U/U') \subseteq \mathrm{core}(\mathcal{C}) \setminus U/U'$.

Finally, we will need the following lemma.

Lemma 2.5. *Suppose for some $u, v \in V$, every member of $\mathrm{core}(\mathcal{C})$ containing u also contains v. Then u, v belong to the same part of the bipartition of a connected component of G.*

Proof. By Theorem 2.1, it suffices to show that u, v belong to the same connected component of G. Suppose otherwise. In particular, G is not connected. Let $\{U, U'\}$ be the bipartition of the connected component containing u where $u \in U'$. Then $\mathcal{C} \setminus U/U'$ is a clean tangled clutter such that $\mathrm{core}(\mathcal{C} \setminus U/U') \subseteq \mathrm{core}(\mathcal{C}) \setminus U/U'$ by Theorem 2.4. Let w be a neighbor of u in G; so $w \in U$. Then $\{w, u\}$ is a cover of $\mathcal{C}$. As every member of $\mathrm{core}(\mathcal{C})$ containing u also contains v, it follows that $\{w, v\}$ is a cover of $\mathrm{core}(\mathcal{C})$, implying in turn that $\mathrm{core}(\mathcal{C}) \setminus U/U'$ has $\{v\}$ as a cover. However, $\mathrm{core}(\mathcal{C} \setminus U/U') \subseteq \mathrm{core}(\mathcal{C}) \setminus U/U'$, so $\mathrm{core}(\mathcal{C} \setminus U/U')$ has a cover of cardinality 1, a contradiction to Theorem 2.2. □

3 Proof of the min-max relation

Let $\mathcal{C}$ be a clean tangled clutter, $G := G(\mathcal{C})$, and $r := \mathrm{rank}(\mathcal{C})$. For each $i \in [r]$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G. Let B be a cover of $\mathcal{C}$. Recall that B is a monochromatic cover if it is monochromatic in some proper 2-vertex-coloring of G. Equivalently, B is a monochromatic cover if $B \cap U_i = \emptyset$ or $B \cap V_i = \emptyset$ for each $i \in [r]$. Let $\mu := \mu(\mathcal{C})$ and $\delta := \mathrm{depth}(\mathcal{C})$.

Proposition 3.1. $r - \mu \leq \delta$.

Proof. If $\mu = \infty$, then we are done. Otherwise, let B be a monochromatic minimal cover of cardinality μ. After relabeling and swapping $U_i, V_i, i \in [r]$, if necessary, we may assume that $B \subseteq U_1 \cup U_2 \cup \cdots \cup U_\mu$. In particular, $U_1 \cup \cdots \cup U_\mu$ is a cover of $\mathcal{C}$. Thus for each $C \in \mathrm{core}(\mathcal{C})$,

$$C \cap (U_i \cup V_i) = U_i \quad \text{for some } i \in [\mu].$$

That is, if $S \subseteq \{0, 1\}^r$ is the setcore of $\mathcal{C}$ with respect to $(U_1, V_1; \ldots; U_r, V_r)$, then

$$S \cap \{x : x_1 = \cdots = x_\mu = 1\} = \emptyset.$$

Thus $\{x : x_1 = \cdots = x_\mu = 1\}$ is an infeasible hypercube of S of dimension $r - \mu$, implying in turn that $\delta \geq r - \mu$, as required. □

In §3.1, we introduce irreducible monochromatic covers, and also prove that every monochromatic cover intersects at least $\mu(\mathcal{C})$ many connected components of G. In §3.2, we prove that in fact every monochromatic cover of $\mathrm{core}(\mathcal{C})$ intersects $\mu(\mathcal{C})$ many connected components of G. Finally, in §3.3, we prove the min-max relation $r - \mu = \delta$.

3.1 Monochromatic covers

Lemma 3.2. *If $V_1 \cup \cdots \cup V_k$ is a monochromatic cover for some integer $k \in [r]$, then there exists a monochromatic minimal cover B such that $B \subseteq \bigcup_{i=1}^{k} (U_i \cup V_i)$ and $|B \cap (U_i \cup V_i)| \leq 1$ for each $i \in [k]$.*

Proof. Denote by V the ground set of $\mathcal{C}$. Out of all the monochromatic minimal covers of $\mathcal{C}$ contained in $\bigcup_{i=1}^k (U_i \cup V_i)$, pick one of minimum cardinality, call it B. Pick U, U' such that $\{U, U'\} = \{U_i, V_i\}$ for some $i \in [k]$ and $\emptyset \neq B \cap (U \cup U') \subseteq U'$. We claim that $|B \cap (U \cup U')| = 1$, thereby proving the lemma. Suppose for a contradiction that $|B \cap (U \cup U')| \geq 2$. Let $I := B - (U \cup U')$, $J := V - (U \cup U' \cup I)$, and $\mathcal{C}' := \mathcal{C} \setminus I/J$, a minor over ground set $U \cup U'$. Assume in the first case that $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is clean and tangled, and $G[U \cup U'] \subseteq G(\mathcal{C}')$. Thus $G(\mathcal{C}')$ is a connected, bipartite graph whose bipartition is inevitably $\{U, U'\}$. It therefore follows from Theorem 2.3 (i) that $U, U' \in \mathcal{C}'$. However, $B \cap U' = B \cap (U \cup U') = B - I$ is a minimal cover of $\mathcal{C}'$ disjoint from U, a contradiction. Assume in the remaining case that $\tau(\mathcal{C}') \leq 1$. That is, there is a $D \in b(\mathcal{C})$ such that $D \subseteq U \cup U' \cup I$ and $|D - I| \leq 1$. But then D is a monochromatic minimal cover of $\mathcal{C}$ contained in $\bigcup_{i=1}^k (U_i \cup V_i)$ and

$$|D| = |D - I| + |D \cap I| \leq 1 + |B - (U \cup U')| < |B \cap (U \cup U')| + |B - (U \cup U')| = |B|,$$

a contradiction to our minimal choice of B. As a result, $|B \cap (U \cup U')| = 1$, as desired. □

As a consequence, we get the following:

Theorem 3.3. *Suppose a monochromatic cover exists. Then every monochromatic cover of $\mathcal{C}$ intersects at least μ connected components of G. In particular,*

- *every monochromatic cover intersects at least* 3 *connected components of G, and*
- *$\mu \leq r$.*

Proof. Let B be a monochromatic cover, and $\mathcal{K}$ the set of the connected components intersected by B. Then by Lemma 3.2 there is a monochromatic minimal cover B' that intersects only a subset of $\mathcal{K}$ and intersects every connected component at most once. As a result, $|\mathcal{K}| \geq |B'| \geq \mu$, thereby finishing the proof. □

3.2 The effect of deletion-contraction on r and μ

We saw in Theorem 3.3 that every monochromatic cover of $\mathcal{C}$ intersects at least μ connected components of G. Here we strengthen this result by proving that every monochromatic cover of $\mathrm{core}(\mathcal{C})$ intersects at least μ connected components of G. We need two lemmas about how the rank and the monochromatic covering number change when the two parts of a connected component of G are deleted and contracted.

Lemma 3.4. *Suppose G is not connected, and let $\{U, U'\}$ be the bipartition of a connected component of G. Then the following statements hold:*

(i) $\mathrm{rank}(\mathcal{C} \setminus U/U') \leq r - 1$*, and if equality holds, then the vertex sets of the connected components of $G(\mathcal{C} \setminus U/U')$ are precisely the vertex sets of the connected components of G different from $G[U \cup U']$,*

(ii) equality does not *hold in (i) if and only if there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly* 3 *connected components of G, and*

(iii) if $\mu > 3$, then $\mathrm{rank}(\mathcal{C} \setminus U/U') = r - 1$.

Proof. We may assume that $U \cup U' = U_r \cup V_r$. Let $\mathcal{C}' := \mathcal{C} \setminus U/U'$, which is clean and tangled by Theorem 2.4, and let $G' := G(\mathcal{C}')$. As $G[U_i \cup V_i] \subseteq G'$ for all $i \in [r-1]$, G' has at most $r-1$ connected components, so $\mathrm{rank}(\mathcal{C}') \leq r-1$, and if equality holds, then the connected components of G' are precisely $G'[U_i \cup V_i], i \in [r-1]$. Thus **(i)** holds.

Claim 1. *Assume that B is a monochromatic cover of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly 3 connected components of G. Then* $\mathrm{rank}(\mathcal{C}') < r - 1$.

Proof of Claim. We may assume that $B \subseteq U_{r-2} \cup U_{r-1} \cup U$. As $B - U$ is a cover of $\mathcal{C}'$, it follows that $U_{r-2} \cup U_{r-1}$ is a cover of $\mathcal{C}'$. Let us now prove that $\mathrm{rank}(\mathcal{C}') < r - 1$. Suppose otherwise. Then the connected components of G' are precisely $G'[U_i \cup V_i], i \in [r-1]$ by (i). But then $U_{r-2} \cup U_{r-1}$ is a monochromatic cover of $\mathcal{C}'$, one that intersects only 2 connected components of G', a contradiction to Theorem 3.3 applied to $\mathcal{C}'$. $\lozenge$

Claim 2. *Assume that* $\mathrm{rank}(\mathcal{C}') < r - 1$. *Then there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly 3 connected components of G.*

Proof of Claim. Observe that $r \geq 3$. As G' has fewer than $r-1$ connected components, and as $G[U_i \cup V_i] \subseteq G'$ for $i \in [r-1]$, we may assume that G' has an edge between U_{r-2} and U_{r-1}. In particular, $U_{r-2} \cup U_{r-1}$ is a cover of $\mathcal{C}'$, implying in turn that $B := U_{r-2} \cup U_{r-1} \cup U$ is the desired cover of $\mathcal{C}$. $\lozenge$

Claims 1 and 2 prove **(ii)**.

Claim 3. **(iii)** *holds.*

Proof of Claim. We prove the contrapositive. Assume that $\mathrm{rank}(\mathcal{C}') < r - 1$. Then by (ii) there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly 3 connected components of G. Theorem 3.3 now applies to $\mathcal{C}$ and tells us that $3 \leq \mu \leq 3$, so $\mu = 3$, as required. $\lozenge$

This finishes the proof of Lemma 3.4. $\square$

Lemma 3.5. *Suppose G is not connected, and let $\{U, U'\}$ be the bipartition of a connected component of G. Then*

$$\mu - 1 \leq \mu(\mathcal{C} \setminus U/U').$$

Moreover, assuming μ is finite and greater than 3, equality holds above if and only if there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly μ connected components of G.

Proof. Let $\mathcal{C}' := \mathcal{C} \setminus U/U'$, a clean tangled clutter by Theorem 2.4, and let $G' := G(\mathcal{C}')$. Let $\mu' := \mu(\mathcal{C}')$. Note that $\mu' \geq 3$ because $\mathcal{C}'$ is a clean tangled clutter. If $\mu = 3$, then $\mu - 1 = 2 < \mu'$, so we are done. We may therefore assume that $\mu > 3$. We may assume that $U \cup U' = U_r \cup V_r$. It follows from Lemma 3.4 (iii) that $\mathrm{rank}(\mathcal{C}') = r - 1$, so by Lemma 3.4 (i), the connected components of G' are precisely $G'[U_i \cup V_i], i \in [r-1]$. This immediately implies Claim 1 below:

Claim 1. *The bipartitions of the connected components of G' are $\{U_i, V_i\}, i \in [r-1]$.*

Claim 2. *$\mu - 1 \le \mu'$, and if μ is finite and equality holds, then there is a monochromatic cover B of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly μ connected components of G.*

Proof of Claim. If $\mu' = \infty$, we are done. Otherwise, let B' be a monochromatic cover of $\mathcal{C}'$ of cardinality μ'. Let $B := B' \cup U$, a monochromatic cover of $\mathcal{C}$. In particular, μ is finite. On one hand, B intersects at most $\mu' + 1$ connected components of G, as $|B'| = \mu'$. On the other hand, B intersects at least μ connected components of G by Theorem 3.3 applied to $\mathcal{C}$. Thus $\mu \le \mu' + 1$, and if equality holds, then B is the desired set. $\Diamond$

Claim 3. *Suppose B is a monochromatic cover of $\mathcal{C}$ such that $B \cap U \neq \emptyset$ and B intersects exactly μ connected components of G. Then $\mu' = \mu - 1$.*

Proof of Claim. As B is monochromatic, $B \cap U' = \emptyset$, so $B - U$ is a cover of $\mathcal{C}'$. In fact, $B - U$ is a monochromatic cover of $\mathcal{C}'$ by Claim 1. As $B - U$ intersects only $\mu - 1$ connected components of G', $\mu' \le \mu - 1$ by Theorem 3.3 applied to $\mathcal{C}'$, so $\mu' = \mu - 1$ by Claim 2, as claimed. $\Diamond$

Claims 2 and 3 finish the proof. $\square$

3.3 The min-max relation $r - \mu = \delta$

Lemma 3.6. *The following statements are equivalent:*

- *$\mu = \infty$, i.e. $\mathcal{C}$ does not have a monochromatic cover,*
- *$\delta = -\infty$, i.e. $\mathrm{setcore}(\mathcal{C}) = \{0,1\}^r$, i.e. $\bigcup_{i=1}^r W_i \in \mathrm{core}(\mathcal{C})$ whenever $W_i \in \{U_i, V_i\}, i \in [r]$.*

Proof. ($\Rightarrow$) Let us first prove that $\bigcup_{i=1}^r U_i \in \mathcal{C}$. For if not, then $\bigcup_{i=1}^r V_i$ is a monochromatic cover, which is not the case. Thus $\bigcup_{i=1}^r U_i \in \mathcal{C}$ and so $\bigcup_{i=1}^r U_i \in \mathrm{core}(\mathcal{C})$ by Theorem 2.1. Similarly, $\bigcup_{i=1}^r W_i \in \mathrm{core}(\mathcal{C})$ whenever $W_i \in \{U_i, V_i\}, i \in [r]$. ($\Leftarrow$) is immediate. $\square$

Theorem 3.7. *Assume that $U_1 \cup \cdots \cup U_k$ is a cover of $\mathrm{core}(\mathcal{C})$ for some $k \in [r]$. Then $k \ge 3$. In fact, $k \ge \mu$.*

Proof. It follows from Theorem 2.3 (i)-(ii) that $r \ge 3$, and from Lemma 3.6 that μ is finite.

Let us first prove that $k \ge 3$. Suppose otherwise. Then $U_1 \cup U_2$ is a cover of $\mathrm{core}(\mathcal{C})$. Pick $u_1 \in U_1, u_2 \in U_2$ and $v_2 \in V_2$. It follows from Theorem 2.1 that $\{u_1, u_2\}$ is a cover of $\mathrm{core}(\mathcal{C})$, and that every member of $\mathrm{core}(\mathcal{C})$ containing v_2 also contains u_1. But then Lemma 2.5 implies that u_1, v_2 belong to the same connected component of G, a contradiction.

Let us next prove that $k \ge \mu$. We proceed by induction on $r \ge 3$. For the base case $r = 3$, as the monochromatic covering number is finite, we may apply Theorem 3.3 and conclude that $k \ge 3 = r \ge \mu$. For the induction step, assume that $r \ge 4$. If $\mu = 3$, then we are done. Otherwise, $\mu > 3$. Let $\mathcal{C}' := \mathcal{C} \setminus U_k/V_k$, which is a clean tangled clutter such that $\mathrm{core}(\mathcal{C}') \subseteq \mathrm{core}(\mathcal{C}) \setminus U_k/V_k$ by Theorem 2.4. Let $G' := G(\mathcal{C}')$. Then $U_1 \cup \cdots \cup U_{k-1}$ is a cover of $\mathrm{core}(\mathcal{C}')$. As $\mu > 3$, it follows from Lemma 3.4 (iii) that $\mathrm{rank}(\mathcal{C}') = r - 1$, so by Lemma 3.4 (i), the connected components of G' are precisely $G'[U_i \cup V_i], i \in [r] - \{k\}$. We may therefore apply the induction hypothesis to conclude that $k - 1 \geq \mu(\mathcal{C}')$. By Lemma 3.5, $\mu(\mathcal{C}') \geq \mu - 1$. Putting the last two inequalities together gives us that $k \geq \mu$, thereby completing the induction step. $\square$

Notice that Theorem 3.7 non-trivially relates a core dependent parameter to a parameter defined on the whole clutter.

Theorem 3.8. $r - \mu = \delta$.

Proof. If $\delta = -\infty$, then $\mu = \infty$ by Lemma 3.6, so we are done. Otherwise, pick a subset $I \subseteq [r]$ of cardinality $r - \delta$ and decisions $a_1, \ldots, a_{r-d} \in \{0, 1\}$ such that $\mathrm{setcore}(\mathcal{C}) \cap \{x : x_i = a_i, i \in I\} = \emptyset$. That is, for some $W_i \in \{U_i, V_i\}, i \in I$, the union $\bigcup_{i \in I} W_i$ is a cover of $\mathrm{core}(\mathcal{C})$. It therefore follows from Theorem 3.7 that $r - \delta \geq \mu$. By Proposition 3.1, $r - \delta \leq \mu$, so $r - \delta = \mu$, thereby finishing the proof. $\square$

4 Applications

In this section we discuss four applications of our results. In §4.1, we discuss *irreducible* monochromatic covers. In §4.2, we discuss *ideal* tangled clutters, and prove that the core of such clutters is also ideal. In §4.3, we show how the monochromatic covering number extends the notion of girth for simple binary matroids and simple graphs. In §4.4, we show how the monochromatic covering number extends the notion of covering number for clean clutters with covering number at least three.

Moving forward we need the following remark.

Remark 4.1. *Take an integer $r \geq 1$, a set $S \subseteq \{0, 1\}^r$ without an infeasible hypercube of dimension at least $r - 2$, and let $\mathcal{C} := \mathrm{cuboid}(S)$. Suppose $\mathcal{C}$ is a clean tangled clutter. Then $\mathrm{rank}(\mathcal{C}) = r$ and $\mathrm{setcore}(\mathcal{C}) \cong S$.*

4.1 Irreducible monochromatic covers

Let $\mathcal{C}$ be a clean tangled clutter, where $G := G(\mathcal{C})$, $r := \mathrm{rank}(\mathcal{C})$, and denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G for $i \in [r]$. Take a subset $I \subseteq [r]$. We say that $\bigcup_{i \in I} V_i$ is an *irreducible monochromatic cover of* $\mathcal{C}$ if $\bigcup_{i \in I} V_i$ is a cover of $\mathcal{C}$, and for each $j \in I$, $\left(\bigcup_{i \in I, i \neq j} V_i\right) \cup U_j$ is not a cover. In particular, if $\bigcup_{i \in I} V_i$ an irreducible monochromatic cover, then $\bigcup_{i \in I, i \neq j} V_i$ is not a cover for any $j \in I$. Let $\mu := \mu(\mathcal{C})$.

Theorem 4.2. *Suppose μ is finite and $U_1 \cup \cdots \cup U_\mu$ is a cover of $\mathcal{C}$. Then $U_1 \cup \cdots \cup U_\mu$ is an irreducible monochromatic cover of $\mathcal{C}$.*

Proof. By symmetry, it suffices to show that $U_1 \cup \cdots \cup U_{\mu-1} \cup V_\mu$ is not a cover. Suppose otherwise. Let $S \subseteq \{0, 1\}^r$ be the setcore of $\mathcal{C}$ with respect to $(U_1, V_1; \ldots; U_r, V_r)$. Then the two covers $U_1 \cup \cdots \cup U_{\mu-1} \cup U_\mu, U_1 \cup \cdots \cup U_{\mu-1} \cup V_\mu$ yield the following infeasible hypercubes in S:

$$\{x : x_1 = \cdots = x_{\mu-1} = x_\mu = 1\} \quad \text{and} \quad \{x : x_1 = \cdots = x_{\mu-1} = 1, x_\mu = 0\},$$

implying in turn that $\{x : x_1 = \cdots = x_{\mu-1} = 1\}$ is an infeasible hypercube of S. This implies that $\mathrm{depth}(\mathcal{C}) \geq r - (\mu - 1) = r - \mu + 1$. However, $\mathrm{depth}(\mathcal{C}) = r - \mu$ by Theorem 3.8, a contradiction. $\square$

Lemma 4.3. *If $V_1 \cup \cdots \cup V_k$ is an irreducible monochromatic cover for some integer $k \in [r]$, then there exists a monochromatic minimal cover B such that $B \subseteq \bigcup_{i=1}^k V_i$ and $|B \cap V_i| = 1$ for each $i \in [k]$.*

Proof. Denote by V the ground set of $\mathcal{C}$. The proof of this lemma is very similar to Lemma 3.2, except for our minimal choice. Out of all the monochromatic minimal covers of $\mathcal{C}$ contained in $\bigcup_{i=1}^k V_i$, pick one of minimum cardinality, call it B. As $\bigcup_{i=1}^k V_i$ is an irreducible monochromatic cover, it follows that $B \cap V_i \neq \emptyset, i \in [k]$. To finish the proof of the lemma, it suffices to show that $|B \cap V_1| = 1$. Suppose for a contradiction that $|B \cap V_1| \geq 2$. Let $I := B - V_1$, $J := V - (U_1 \cup V_1 \cup I)$, and $\mathcal{C}' := \mathcal{C} \setminus I/J$, a minor over ground set $U_1 \cup V_1$. Assume in the first case that $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is clean and tangled, and $G[U_1 \cup V_1] \subseteq G(\mathcal{C}')$. Thus $G(\mathcal{C}')$ is a connected, bipartite graph whose bipartition is inevitably $\{U_1, V_1\}$. It therefore follows from Theorem 2.3 (i) that $U_1, V_1 \in \mathcal{C}'$. However, $B \cap V_1 = B - I$ is a minimal cover of $\mathcal{C}'$ disjoint from U_1, a contradiction. Assume in the remaining case that $\tau(\mathcal{C}') \leq 1$. That is, there is a $D \in b(\mathcal{C})$ such that $D \subseteq U_1 \cup V_1 \cup I$ and $|D - I| \leq 1$. As $D \subseteq (V_1 \cup \cdots \cup V_k) \cup U_1$, and $V_1 \cup \cdots \cup V_k$ is an irreducible monochromatic cover, it follows that $D \subseteq \bigcup_{i=1}^k V_i$. But then D is a monochromatic minimal cover of $\mathcal{C}$ contained in $\bigcup_{i=1}^k V_i$ and

$$|D| = |D - I| + |D \cap I| \leq 1 + |B - (U_1 \cup V_1)| < |B \cap (U_1 \cup V_1)| + |B - (U_1 \cup V_1)| = |B|,$$

a contradiction to our minimal choice of B. As a result, $|B \cap V_1| = 1$, as desired. $\square$

Theorem 4.4. *Suppose for some integer $k \in [r]$ that $V_1 \cup \cdots \cup V_k$ is a cover of $\mathcal{C}$. Then $k \geq \mu$. Moreover, if $k = \mu$, then $V_1 \cup \cdots \cup V_k$ contains a minimal cover of cardinality k picking exactly one element from each $V_i, i \in [k]$.*

Proof. That $k \geq \mu$ follows from Theorem 3.3. If $k = \mu$, then by Theorem 4.2, $V_1 \cup \cdots \cup V_k$ is an irreducible monochromatic cover, so by Lemma 4.3, $V_1 \cup \cdots \cup V_k$ contains a minimal cover of cardinality three picking exactly one element from each $V_i, i \in [k]$, as claimed. $\square$

This theorem for the case of $\mu = 3$ was proved in [7], Theorem 5.2, and served as an important tool for finding the clutter of *the lines of the Fano plane* as a minor in clean tangled clutters.

4.2 Ideal tangled clutters

$\mathcal{C}$ is an *ideal* clutter if *the associated set covering polyhedron*

$$Q(\mathcal{C}) := \left\{ x \in \mathbb{R}_+^V : \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C} \right\}$$

is integral [12]. The deltas and extended odd holes are non-ideal clutters. It is well-known that a clutter is ideal if and only if its blocker is ideal [15, 18]. In particular, the blocker of an extended odd hole is also non-ideal. Moreover, if a clutter is ideal, so is every minor of it [21]. Thus ideal clutters are clean.

Let $\mathcal{C}$ be an ideal tangled clutter, where $G := G(\mathcal{C})$, $r := \operatorname{rank}(\mathcal{C})$, and denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G for $i \in [r]$. Let $S := \operatorname{setcore}(\mathcal{C} : U_1, V_1; \cdots ; U_r, V_r)$.

Theorem 4.5 ([3], Theorem 22a). $\operatorname{conv}(S)$ *is described by* $\mathbf{0} \le x \le \mathbf{1}$*, and*

$$\sum_{B\cap V_i\neq\emptyset} |B\cap V_i| x_i + \sum_{B\cap U_j\neq\emptyset} |B\cap U_j|(1-x_j) \ge 1$$

for every monochromatic cover B *of* $\mathcal{C}$.

For the sake of completion, we have provided a proof of this theorem in the appendix.

Lemma 4.6. *Consider a facet-defining inequality for* $\operatorname{conv}(S)$ *different from any of* $\mathbf{0} \le x \le \mathbf{1}$*. Then the inequality is of the form*

$$\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \ge 1$$

for some disjoint $I, J \subseteq [r]$ *where* $\left(\cup_{i\in I} U_i\right) \cup \left(\cup_{j\in J} V_j\right)$ *is an irreducible monochromatic cover of* $\mathcal{C}$.

Proof. By Theorem 4.5, the inequality is of the form

$$\sum_{B\cap V_i\neq\emptyset} |B\cap V_i| x_i + \sum_{B\cap U_j\neq\emptyset} |B\cap U_j|(1-x_j) \ge 1 \tag{$\star$}$$

for some monochromatic cover B of $\mathcal{C}$. We may assume that $B \subseteq V_1 \cup \cdots \cup V_k$, and $B \cap V_i \neq \emptyset$ for each $i \in [k]$. By Theorem 3.3, $k \ge \mu(\mathcal{C}) \ge 3$.

We claim that the monochromatic cover $V_1 \cup \cdots \cup V_{k-1} \cup V_k$ is irreducible. Suppose otherwise; say $V_1 \cup \cdots \cup V_{k-1} \cup U_k$ is a monochromatic cover, too. Then these covers yield the following infeasible hypercubes in S:

$$\{x : x_1 = \cdots = x_{k-1} = x_k = 0\} \quad \text{and} \quad \{x : x_1 = \cdots = x_{k-1} = 0, x_k = 1\},$$

implying in turn that $\{x : x_1 = \cdots = x_{k-1} = 0\}$ is an infeasible hypercube of S, implying in turn that $\sum_{i=1}^{k-1} x_i \ge 1$ is valid for $\operatorname{conv}(S)$. However,

$$\sum_{B\cap V_i\neq\emptyset} |B\cap V_i| x_i + \sum_{B\cap U_j\neq\emptyset} |B\cap U_j|(1-x_j) = \sum_{i=1}^{k} |B\cap V_i| x_i \ge \sum_{i=1}^{k} x_i, \tag{$\diamond$}$$

so the valid inequality $\sum_{i=1}^{k-1} x_i \ge 1$ dominates the facet-defining inequality ($\star$), a contradiction.

Since $V_1 \cup \cdots \cup V_k$ is an irreducible monochromatic cover, we may apply Theorem 4.4 to conclude that there exists a minimal cover $B' \subseteq V_1 \cup \cdots \cup V_k$ of $\mathcal{C}$ intersecting each $V_i, i \in [k]$ exactly once. Thus, by Theorem 4.5,

$$\sum_{i=1}^{k} x_i = \sum_{B'\cap V_i\neq\emptyset} |B'\cap V_i| x_i \ge 1$$

is valid for $\operatorname{conv}(S)$. Since ($\diamond$) holds, and ($\star$) is facet-defining, ($\star$) must be of the form $\sum_{i=1}^{k} x_i \ge 1$, thereby finishing the proof. $\square$

A subset $S \subseteq \{0,1\}^r$ is *cube-ideal* if its convex hull is described by $\mathbf{0} \leq x \leq \mathbf{1}$, and some *generalized set covering* inequalities, which are of the form $\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \geq 1$ for disjoint $I, J \subseteq [r]$. It is known that S is cube-ideal if, and only if, $\mathrm{cuboid}(S)$ is an ideal clutter [2]. Putting Theorem 4.5 and Lemma 4.6, we get the following.

Theorem 4.7. *Let $\mathcal{C}$ be an ideal tangled clutter. Then* $\mathrm{setcore}(\mathcal{C})$ *is a cube-ideal set, and* $\mathrm{core}(\mathcal{C})$ *is an ideal clutter.* □

This theorem was first claimed in [4], Theorem 22, but the authors found a flaw in the proof [3].

4.3 The girth of a binary matroid

$\mathcal{C}$ is a *binary* clutter if the symmetric difference of any odd number of members contains a member; equivalently, $\mathcal{C}$ is binary if $|C \cap B| \equiv 1 \pmod 2$ for all $C \in \mathcal{C}, B \in b(\mathcal{C})$ [17]. In particular, a clutter is binary if and only if its blocker is binary. Observe that the deltas, extended odd holes and their blockers are not binary. If a clutter is binary, so is every minor of it [20]. Thus binary clutters are clean.

Remark 4.8. *Take an integer $r \geq 1$ and let $S \subseteq \{0,1\}^r$ be a vector space over $GF(2)$ whose points do not agree on a coordinate. Then* $\mathrm{cuboid}(S)$ *is a clean tangled clutter.*

Here we relate the monochromatic covering number of $\mathrm{cuboid}(S)$ to a natural matroidal parameter. To this end, take an integer $r \geq 1$ and let $S \subseteq \{0,1\}^r$ be a vector space over $GF(2)$. Basic Linear Algebra tells us that there is a $0-1$ matrix A with r columns such that $S = \{x : Ax \equiv \mathbf{0} \pmod 2\}$. Let M be the binary matroid over ground set $EM := [r]$ that is represented by A. The *cycle space of* M is the set $\mathrm{cycle}(M) := S$ and the *cocycle space of* M, denoted $\mathrm{cocycle}(M) \subseteq \{0,1\}^r$, is the row space of A over $GF(2)$. Notice that $\mathrm{cycle}(M)\,, \mathrm{cocycle}(M)$ are binary spaces that are orthogonal complements over $GF(2)$. Observe that the binary matroid M can be fully determined by either A, its cycle space or its cocycle space.

A *cycle of* M is a subset $C \subseteq EM$ such that $\chi_C \in \mathrm{cycle}(M)$, and a *cocycle of* M is a subset $D \subseteq EM$ such that $\chi_D \in \mathrm{cocycle}(M)$. In particular, $\emptyset$ is both a cycle and a cocycle. Notice that every cycle and every cocycle have an even number of elements in common. A *circuit of* M is a nonempty cycle that does not contain another nonempty cycle, and a *cocircuit of* M is a nonempty cocycle that does not contain another nonempty cocycle. It is well-known that every cycle is either empty or the disjoint union of some circuits, and that every cocycle is either empty or the disjoint union of some cocircuits [19].

An element $e \in EM$ is a *loop of* M if $\{e\}$ is a circuit, and two distinct elements $e, f \in EM$ are *parallel in* M if $\{e, f\}$ is a circuit. M is a *simple* binary matroid if it has no loop and no parallel elements, i.e. if every circuit has cardinality at least three. If M is simple, then its *girth*, denoted $\mathrm{girth}(M)$, is the minimum cardinality of a circuit.

Theorem 4.9. *Let M be a simple binary matroid, and let* $\mathcal{C} := \mathrm{cuboid}(\mathrm{cocycle}(M))$. *Then* $\mu(\mathcal{C}) = \mathrm{girth}(M)$.

Proof. Write $EM = [r]$, and let $S^\perp := \mathrm{cocycle}(M)$, viewed as a subset of $\{0,1\}^r$. As $S^\perp$ is a vector space over $GF(2)$, there is a $0-1$ matrix B with r columns such that $S^\perp = \{x \in \{0,1\}^r : Bx \equiv \mathbf{0} \pmod 2\}$.

Observe that the row space of B over $GF(2)$, which is the orthogonal complement of $S^\perp$ over $GF(2)$, is the cycle space of M.

Claim. $\mathrm{girth}(M) = r - \max\{d : S^\perp$ *has an infeasible hypercube of dimension* $d\}$.

Proof of Claim. Let $g := \mathrm{girth}(M)$ and $d^\star := \max\{d : S^\perp$ has an infeasible hypercube of dimension $d\}$. $(\geq)$ Let $C \subseteq [r]$ be a circuit of M of length g. We may assume that $C = \{1, \ldots, g\}$. As C intersects every cocycle an even number of times, it follows that $S^\perp \cap \{x : x_1 = \cdots = x_{g-1} = 0, x_g = 1\} = \emptyset$, so there is an infeasible hypercube of dimension $r - g$, implying in turn that $g \geq r - d^\star$. $(\leq)$ Pick $I \subseteq [r]$ of cardinality $r - d^\star$ such that $S^\perp \cap \{x : x_i = a_i \ \forall i \in I\} = \emptyset$ for some decisions $a_i \in \{0, 1\}, i \in I$. As a result, the following linear system of equations has no $0 - 1$ solution:

$$Bx \equiv \mathbf{0} \pmod 2 \quad \text{and} \quad x_i \equiv a_i \pmod 2 \quad \forall i \in I.$$

There must be an infeasibility certificate. That is, assuming B has m rows, there exist $c \in \{0, 1\}^m$ and $d_i \in \{0, 1\}, i \in I$ such that

$$B^\top c + \sum_{i \in I} d_i e_i \equiv \mathbf{0} \pmod 2 \quad \text{and} \quad \sum_{i \in I} d_i a_i \equiv 1 \pmod 2.$$

Let C be the cycle of M such that $\chi_C = B^\top c$. Then the first equation above tells us that $C \subseteq I$ while the second equation tells us that $C \neq \emptyset$. As a result, M has a nonempty cycle of length at most $|I| = r - d^\star$, implying in turn that there is a circuit of length at most $r - d^\star$, so $g \leq r - d^\star$, as required. ♢

As M has no loop, the points in $S^\perp$ do not agree on a coordinate, so $\mathcal{C} = \mathrm{cuboid}(S^\perp)$ is a clean tangled clutter by Remark 4.8. Moreover, as M has no parallel elements, $\mathrm{girth}(M) \geq 3$, so $S^\perp$ has no infeasible hypercube of dimension at least $r - 2$, by the claim above. We may therefore apply Remark 4.1 to conclude that $\mathrm{rank}(\mathcal{C}) = r$ and $\mathrm{setcore}(\mathcal{C}) \cong S^\perp$. It therefore follows from the claim above that $\mathrm{girth}(M) = \mathrm{rank}(\mathcal{C}) - \mathrm{depth}(\mathcal{C})$, so $\mathrm{girth}(M) = \mu(\mathcal{C})$ by Theorem 3.8. □

Given distinct elements $e, f, g \in EM$, if e, f are parallel and f, g are parallel, then so are e, g. A *parallel class of* M is a maximal subset of EM of parallel elements. The *simplification* of M, denoted $\mathrm{si}(M)$, is the binary matroid obtained from M after deleting all loops, and for every parallel class, keeping one representative and deleting all the other elements. Observe that $\mathrm{si}(M)$ is a simple binary matroid.

Theorem 4.10. *Let M be a binary matroid without a loop, and let $\mathcal{C} := \mathrm{cuboid}(\mathrm{cocycle}(M))$. Then $\mathrm{rank}(\mathcal{C})$ is equal to the number of parallel classes of M, and $\mu(\mathcal{C}) = \mathrm{girth}(\mathrm{si}(M))$.*

Proof. This is a rather immediate consequence of Theorem 4.9. We leave this as an exercise for the reader. □

Take an integer $k \geq 1$. An important example of a binary matroid is the *projective geometry* $M = PG(k-1, 2)$ represented by the matrix A whose columns are $\{a \in \{0, 1\}^k : a \neq \mathbf{0}\}$. The second projective geometry $PG(1, 1)$ is the graphic matroid of a triangle, while the third one $PG(2, 1)$ is the Fano matroid. It can be readily checked that $PG(k-1,1)$ is a simple matroid where every element appears in a triangle, so in particular, $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-1,2)))$ has monochromatic covering number 3. These clutters played a starring role in [7].

4.4 The covering number of a clean clutter

Let us start with the following easy remark:

Remark 4.11. *Neither a delta nor the blocker of an extended odd hole has a transversal of cardinality two.*

Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$. For a point $x \in \{0,1\}^r$, the *induced clutter of S with respect to x*, denoted $\mathrm{ind}(S \triangle x)$, is the clutter over ground set $[r]$ whose members are the inclusion-wise minimal sets of $\{C \subseteq [r] : \chi_C \in S \triangle x\}$. Observe that S has 2^r induced clutters, and that these clutters are in a one-to-one correspondence with the 2^r minors of $\mathrm{cuboid}(S)$ obtained after contracting, for each $i \in [r]$, exactly one of $2i-1, 2i$ (see [2]). It follows from Remark 4.11 that,

Remark 4.12. *Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$ whose induced clutters are clean. Then $\mathrm{cuboid}(S)$ is clean.*

S is *up-monotone* if for all $x, y \in \{0,1\}^r$ such that $x \geq y$, if $y \in S$ then $x \in S$. An element of a clutter is *free* if it is not contained in any member.

Remark 4.13 ([2], Remark 4.6)**.** *Take an integer $r \geq 1$, an up-monotone set $S \subseteq \{0,1\}^r$, and a point $x \in \{0,1\}^r$. Then $\mathrm{ind}(S \triangle x)$ is, after deleting free elements, equal to $\mathrm{ind}(S \triangle \mathbf{0}) / \{i \in [r] : x_i = 1\}$.*

Let $\mathcal{A}$ be a clutter over ground set $[r]$. The *up-monotone set associated with $\mathcal{A}$* is the up-monotone set $\{\chi_C : C \subseteq [r] \text{ contains a member of } \mathcal{A}\} \subseteq \{0,1\}^r$. Notice that the induced clutter of this set with respect to $\mathbf{0}$ is $\mathcal{A}$.

Theorem 4.14. *Let $\mathcal{A}$ be a clean clutter such that $\tau(\mathcal{A}) \geq 3$. Let S be the associated up-monotone set, and let $\mathcal{C} := \mathrm{cuboid}(S)$. Then $\mathcal{C}$ is clean and tangled, and $\mu(\mathcal{C}) = \tau(\mathcal{A})$.*

Proof. As $\tau(\mathcal{A}) > 1$, the points in S do not agree on a coordinate, so $\mathcal{C}$ is tangled. As $\mathrm{ind}(S \triangle \mathbf{0}) = \mathcal{A}$ is clean, it follows from Remark 4.13 that every induced clutter of S is clean, so $\mathcal{C}$ is clean by Remark 4.12. It remains to prove that $\mu(\mathcal{C}) = \tau(\mathcal{A})$. To this end, label the ground set of $\mathcal{A}$ as $[r]$ for some integer $r \geq 1$, let $\tau := \tau(\mathcal{A})$ and

$$d^\star := \max\{d : S \text{ has an infeasible hypercube of dimension } d\}.$$

Claim. $\tau = r - d^\star$.

Proof of Claim. ($\geq$) Let $B \subseteq [r]$ be a cover of $\mathcal{A}$ of cardinality τ. Then $S \cap \{x : x_i = 0 \; \forall i \in B\} = \emptyset$, implying in turn that $d^\star \geq r - \tau$. ($\leq$) Pick $I \subseteq [r]$ of cardinality $r - d^\star$ such that $S \cap \{x : x_i = a_i \; \forall i \in I\} = \emptyset$ for some decisions $a_i \in \{0,1\}, i \in I$. As S is up-monotone, and as the infeasible hypercube above is maximum, it follows that $a_i = 0$ for all $i \in I$. As a result, I must be a cover of $\mathrm{ind}(S \triangle \mathbf{0}) = \mathcal{A}$, implying in turn that $r - d^\star \geq \tau$, as required. $\Diamond$

In particular, as $\tau \geq 3$, S has no infeasible hypercube of dimension at least $r-2$. We may therefore apply Remark 4.1 and conclude that $\mathrm{rank}(\mathcal{C}) = r$ and $\mathrm{setcore}(\mathcal{C}) \cong S$. As a result, $\tau = \mathrm{rank}(\mathcal{C}) - \mathrm{depth}(\mathcal{C})$ by the claim above, implying that $\tau = \mu(\mathcal{C})$ by Theorem 3.8, thereby finishing the proof. □

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A Proof of Theorem 4.5

Theorem A.1 ([3], Theorem 22a)**.** *Let $\mathcal{C}$ be an ideal tangled clutter over ground set V, where $G := G(\mathcal{C})$, $r := \mathrm{rank}(\mathcal{C})$, and denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G for $i \in [r]$. Let $S := \mathrm{setcore}(\mathcal{C} : U_1, V_1; \cdots ; U_r, V_r)$. Then $\mathrm{conv}(S)$ is described by $\mathbf{0} \le x \le \mathbf{1}$, and*

$$\sum_{B \cap V_i \neq \emptyset} |B \cap V_i| x_i + \sum_{B \cap U_j \neq \emptyset} |B \cap U_j|(1 - x_j) \ge 1$$

for every monochromatic cover B of $\mathcal{C}$.

Proof. Denote by E the edge set of G. We know that

$$\{\chi_C : C \in \mathrm{core}(\mathcal{C})\} = \{\chi_C : C \in \mathcal{C}\} \cap \big\{x : x_u + x_v = 1,\ \{u, v\} \in E\big\}. \qquad (\star)$$

Claim 1. $\mathrm{conv}\{\chi_C : C \in \mathrm{core}(\mathcal{C})\} = Q\big(b(\mathcal{C})\big) \cap \big\{x : x_u + x_v = 1,\ \{u, v\} \in E\big\}$.

Proof of Claim. ($\subseteq$) follows immediately from ($\star$). ($\supseteq$) Pick a point $x^\star$ in the set on the right-hand side. Clearly, $x^\star \in Q\big(b(\mathcal{C})\big)$. Since $\mathcal{C}$ is ideal, so is $b(\mathcal{C})$, implying that for some $\lambda \in \mathbb{R}^{\mathcal{C}}_+$ with $\sum_{C \in \mathcal{C}} \lambda_C = 1$, we have that

$$x^\star \ge \sum_{C \in \mathcal{C}} \lambda_C \chi_C.$$

Since for all $\{u, v\} \in E$, we have that $x^\star_u + x^\star_v = 1$ and $\{u, v\} \in b(\mathcal{C})$, equality must hold above and by ($\star$), if $\lambda_C > 0$ then $C \in \mathrm{core}(\mathcal{C})$. Hence, $x^\star \in \mathrm{conv}\{\chi_C : C \in \mathrm{core}(\mathcal{C})\}$, as required. $\lozenge$

For each $i \in [r]$, pick $u_i \in U_i$ and $v_i \in V_i$, and let $\mathcal{C}'$ be the clutter over ground set $\{u_1, v_1, \ldots, u_r, v_r\}$ obtained from $\mathrm{core}(\mathcal{C})$ after contracting $V - \{u_1, v_1, \ldots, u_r, v_r\}$.

Claim 2. $\mathrm{conv}(\{\chi_C : C \in \mathcal{C}'\})$ *can be described by $z \ge \mathbf{0}$, $z_{u_i} + z_{v_i} = 1, i \in [r]$, and*

$$\sum_{B \cap V_i \neq \emptyset} |B \cap V_i| z_{v_i} + \sum_{B \cap U_j \neq \emptyset} |B \cap U_j| z_{u_j} \ge 1$$

for every monochromatic minimal cover B of $\mathcal{C}$.

Proof of Claim. Observe that $\mathrm{conv}\{\chi_C : C \in \mathcal{C}'\}$ is the projection of $\mathrm{conv}\{\chi_C : C \in \mathrm{core}(\mathcal{C})\}$ onto the coordinates $\{u_i, v_i : i \in [r]\}$. Thus, to give a description for $\mathrm{conv}\{\chi_C : C \in \mathcal{C}'\}$, we may apply Fourier-Motzkin Elimination to the description of $\mathrm{conv}\{\chi_C : C \in \mathrm{core}(\mathcal{C})\}$ given by Claim 1, thereby giving us the claimed description. $\lozenge$

Claim 3. $\mathrm{conv}(S)$ *is defined by $\mathbf{0} \le x \le \mathbf{1}$, and*

$$\sum_{B \cap V_i \neq \emptyset} |B \cap V_i| x_i + \sum_{B \cap U_j \neq \emptyset} |B \cap U_j|(1 - x_j) \ge 1$$

for every monochromatic minimal cover B of $\mathcal{C}$.

Proof of Claim. Observe that $\mathcal{C}' \cong \mathrm{cuboid}(S)$. The claim therefore follows from Claim 2 by another application of Fourier-Motzkin Elimination. $\Diamond$

Claim 3 finishes the proof. $\square$