IDEAL BINARY CLUTTERS, CONNECTIVITY, AND A CONJECTURE OF SEYMOUR

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ABSTRACT. A binary clutter is the family of odd circuits of a binary matroid, that is, the family of circuits that intersect with odd cardinality a fixed given subset of elements. Let A denote the $0,1$ matrix whose rows are the characteristic vectors of the odd circuits. A binary clutter is ideal if the polyhedron $\{x \geq \mathbf{0} : \ Ax \geq \mathbf{1}\}$ is integral. Examples of ideal binary clutters are st-paths, st-cuts, T-joins or T-cuts in graphs, and odd circuits in weakly bipartite graphs. In 1977, Seymour conjectured that a binary clutter is ideal if and only if it does not contain $\mathcal{L}_{F_7}$, $\mathcal{O}_{K_5}$, or $b(\mathcal{O}_{K_5})$ as a minor. In this paper, we show that a binary clutter is ideal if it does not contain five specified minors, namely the three above minors plus two others. This generalizes Guenin's characterization of weakly bipartite graphs, as well as the theorem of Edmonds and Johnson on T-joins and T-cuts.

1. INTRODUCTION

A *clutter* $\mathcal{H}$ is a finite family of sets, over some finite ground set $E(\mathcal{H})$, with the property that no set of $\mathcal{H}$ contains, or is equal to, another set of $\mathcal{H}$. A clutter is said to be *ideal* if the polyhedron $\{x \in \mathbb{R}_+^{|E(\mathcal{H})|} : \sum_{i \in S} x_i \geq 1, \forall S \in \mathcal{H}\}$ is an integral polyhedron, that is, all its extreme points have $0,1$ coordinates. A clutter $\mathcal{H}$ is *trivial* if $\mathcal{H} = \emptyset$ or $\mathcal{H} = \{\emptyset\}$. Given a nontrivial clutter $\mathcal{H}$, we write $A(\mathcal{H})$ for a 0,1 matrix whose columns are indexed by $E(\mathcal{H})$ and whose rows are the characteristic vectors of the sets $S \in \mathcal{H}$. With this notation, a nontrivial clutter $\mathcal{H}$ is ideal if and only if $\{x \geq \mathbf{0} : A(\mathcal{H})x \geq \mathbf{1}\}$ is an integral polyhedron.

Given a clutter $\mathcal{H}$, a set $T \subseteq E(\mathcal{H})$ is a *transversal* of $\mathcal{H}$ if T intersects all the members of $\mathcal{H}$. The clutter $b(\mathcal{H})$, called the *blocker* of $\mathcal{H}$, is defined as follows: $E\big(b(\mathcal{H})\big) = E(\mathcal{H})$ and $b(\mathcal{H})$ is the set of inclusion-wise minimal transversals of $\mathcal{H}$. It is well known that $b\big(b(\mathcal{H})\big) = \mathcal{H}$ [13]. Hence we say that $\mathcal{H}, b(\mathcal{H})$ form a *blocking pair* of clutters. Lehman [14] showed that, if a clutter is ideal, then so is its blocker. A clutter is said to be *binary* if, for any $S_1, S_2, S_3 \in \mathcal{H}$, their symmetric difference $S_1 \triangle S_2 \triangle S_3$ contains, or is equal to, a set of $\mathcal{H}$.

Given a clutter $\mathcal{H}$ and $i \in E(\mathcal{H})$, the *contraction* $\mathcal{H}/i$ and *deletion* $\mathcal{H} \setminus i$ are clutters defined as follows: $E(\mathcal{H}/i) = E(\mathcal{H} \setminus i) = E(\mathcal{H}) - \{i\}$, the family $\mathcal{H}/i$ is the set of inclusion-wise minimal members of $\{S - \{i\} : S \in \mathcal{H}\}$, and $\mathcal{H} \setminus i = \{S : i \notin S \in \mathcal{H}\}$. Contractions and deletions can be performed sequentially, and the result does not depend on the order. A clutter obtained from $\mathcal{H}$ by a set of deletions J_d and a set of contractions J_c, (where $J_c \cap J_d = \emptyset$) is called a *minor* of $\mathcal{H}$ and is denoted by $\mathcal{H} \setminus J_d / J_c$. It is a *proper* minor if $J_c \cup J_d \neq \emptyset$. A clutter is said to be *minimally nonideal* (mni) if it is not ideal but all its proper minors are ideal.

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The clutter $\mathcal{O}_{K_5}$ is defined as follows: $E(\mathcal{O}_{K_5})$ is the set of 10 edges of the complete graph K_5 and $\mathcal{O}_{K_5}$ is the set of odd circuits of K_5 (the triangles and the circuits of length 5). The 10 constraints corresponding to the triangles define a fractional extreme point $(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$ of the associated polyhedron $\{x \geq \mathbf{0} : A(\mathcal{O}_{K_5})x \geq \mathbf{1}\}$. Thus $\mathcal{O}_{K_5}$ is not ideal and neither is its blocker. The clutter $\mathcal{L}_{F_7}$ is the family of circuits of length three of the Fano matroid (or, equivalently, the family of lines of the Fano plane), i.e. $E(\mathcal{L}_{F_7}) = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$\mathcal{L}_{F_7} = \big\{\{1,3,5\}, \{1,4,6\}, \{2,3,6\}, \{2,4,5\}, \{1,2,7\}, \{3,4,7\}, \{5,6,7\}\big\}.$$

The fractional point $(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$ is an extreme point of the associated polyhedron, hence $\mathcal{L}_{F_7}$ is not ideal. The blocker of $\mathcal{L}_{F_7}$ is $\mathcal{L}_{F_7}$ itself. The following excluded minor characterization is predicted.

Seymour's Conjecture [Seymour [23] p. 200, [26] (9.2), (11.2)]
A binary clutter is ideal if and only if it has no $\mathcal{L}_{F_7}$, no $\mathcal{O}_{K_5}$, and no $b(\mathcal{O}_{K_5})$ minor.

Consider a clutter $\mathcal{H}$ and an arbitrary element $t \notin E(\mathcal{H})$. We write $\mathcal{H}^+$ for the clutter with $E(\mathcal{H}^+) = E(\mathcal{H}) \cup \{t\}$ and $\mathcal{H}^+ = \{S \cup \{t\} : S \in \mathcal{H}\}$. The clutter Q_6 is defined as follows: $E(Q_6)$ is the set of edges of the complete graph K_4 and Q_6 is the set of triangles of K_4. The clutter Q_7 is defined as follows:

$$A(Q_7) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Note that the first six columns of $A(Q_7)$ form the matrix $A\big(b(Q_6)\big)$.

The main result of this paper is that Seymour's Conjecture holds for the class of clutters that do not have Q_6^+ and Q_7^+ minors.

Theorem 1.1. *A binary clutter is ideal if it does not have $\mathcal{L}_{F_7}$, $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, Q_6^+ or Q_7^+ as a minor.*

Since the blocker of an ideal binary clutter is also a ideal, we can restate Theorem 1.1 as follows.

Corollary 1.2. *A binary clutter is ideal if it does not have $\mathcal{L}_{F_7}$, $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, $b(Q_7^+)$, or $b(Q_6^+)$ as a minor.*

We say that $\mathcal{H}$ is the *clutter of odd circuits* of a graph G if $E(\mathcal{H})$ is the set of edges of G and $\mathcal{H}$ the set of odd circuits of G. A graph is said to be *weakly bipartite* if the clutter of its odd circuits is ideal. This class of graphs has a nice excluded minor characterization.

Theorem 1.3 (Guenin [10])**.** *A graph is weakly bipartite if and only if its clutter of odd circuits has no $\mathcal{O}_{K_5}$ minor.*

The class of clutters of odd circuits is closed under minor taking (Remark 8.2). Moreover, one can easily check that $\mathcal{O}_{K_5}$ is the only clutter of odd circuits among the five excluded minors of Theorem 1.1 (see Remark 8.3 and [20]). It follows that Theorem 1.1 implies Theorem 1.3. It does not provide a new proof of Theorem 1.3 however, as we shall use Theorem 1.3 to prove Theorem 1.1.

Consider a graph G and a subset T of its vertices of even cardinality. A *T-join* is an inclusion-wise minimal set of edges J such that T is the set of vertices of odd degree of the edge-induced subgraph $G[J]$.

A T-*cut* is an inclusion-wise minimal set of edges $\delta(U) := \{(u,v) : u \in U, v \notin U\}$, where U is a set of vertices of G that satisfies $|U \cap T|$ odd. T-joins and T-cuts generalize many interesting special cases. If $T = \{s,t\}$, then the T-joins (resp. T-cuts) are the st-paths (resp. inclusion-wise minimal st-cuts) of G. If $T = V$, then the T-joins of size $|V|/2$ are the perfect matchings of G. The case where T is identical to the set of odd-degree vertices of G is known as the Chinese postman problem [6, 12]. The families of T-joins and T-cuts form a blocking pair of clutters.

Theorem 1.4 (Edmonds and Johnson [6])**.** *The clutters of T-cuts and T-joins are ideal.*

The class of clutters of T-cuts is closed under minor taking (Remark 8.2). Moreover, it is not hard to check that none of the five excluded minors of Theorem 1.1 are clutters of T-cuts (see Remark 8.3 and [20]). Thus Theorem 1.1 implies that the clutter of T-cuts is ideal, and thus that its blocker, the clutter of T-joins, is ideal. Hence Theorem 1.1 implies Theorem 1.4. However, we shall also rely on this result to prove Theorem 1.1.

The paper is organized as follows. Section 2 considers representations of binary clutters in terms of signed matroids and matroid ports. Section 3 reviews the notions of lifts and sources, which are families of binary clutters associated to a given binary matroid [20, 29]. Connections between multicommodity flows and ideal clutters are discussed in Section 4. The material presented in Sections 2, 3 and 4 is not all new. We present it here for the sake of completeness and in order to have a unified framework for the remainder of the paper. In Sections 5, 6, 7 we show that minimally nonideal clutters do not have small separations. The proof of Theorem 1.1 is given in Section 8. Finally, Section 9 presents an intriguing example of an ideal binary clutter.

2. Binary matroids and binary clutters

We assume that the reader is familiar with the basics of matroid theory. For an introduction and all undefined terms, see for instance Oxley [21]. Given a matroid M, the set of its elements is denoted by $E(M)$ and the set of its circuits by $\Omega(M)$. The dual of M is written M^*. The *deletion* minor $M \setminus e$ of M is the matroid defined as follows: $E(M \setminus e) = E(M) - \{e\}$ and $\Omega(M \setminus e) = \{C : e \notin C \in \Omega(M)\}$. The *contraction* minor M/e of M is defined as $(M^* \setminus e)^*$. Contractions and deletions can be performed sequentially, and the result does not depend on the order. A matroid obtained from M by a set of deletions J_d and a set of contractions J_c is a *minor* of M and is denoted by $M \setminus J_d/J_c$.

A matroid M is *binary* if there exists a $0,1$ matrix A with column set $E(M)$ such that the independent sets of M correspond to independent sets of columns of A over the two element field. We say that A is a *representation* of M. Equivalently, a $0,1$ matrix A is a representation of a binary matroid M if the rows of A span the circuit space of M^*. If C_1 and C_2 are two cycles of a binary matroid then $C_1 \triangle C_2$ is also a cycle of M. In particular this implies that every cycle of M can be partitioned into circuits. Let M be a binary matroid and $\Sigma \subseteq E(M)$. The pair (M,Σ) is called a *signed matroid*, and Σ is called the *signature* of M. We say that a circuit C of M is *odd* (resp. *even*) if $|C \cap \Sigma|$ is odd (resp. even).

The results in this section are fairly straightforward and have appeared explicitly or implicitly in the literature [8, 13, 20, 23]. We include some of the proofs for the sake of completeness.

Proposition 2.1 (Lehman [13])**.** *The following statements are equivalent for a clutter: (i) $\mathcal{H}$ is binary; (ii) for every $S \in \mathcal{H}$ and $T \in b(\mathcal{H})$, $|S \cap T|$ is odd; (iii) for every $S_1, \dots, S_k \in \mathcal{H}$ where k is odd, $S_1 \triangle S_2 \triangle \dots S_k$ contains, or is equal, to an element of $\mathcal{H}$.*

Proposition 2.2. *The odd circuits of a signed matroid (M, Σ) form a binary clutter.*

Proof. Let C_1, C_2, C_3 be three odd circuits of (M, Σ). Then $L := C_1 \triangle C_2 \triangle C_3$ is a cycle of M. Since each of C_1, C_2, C_3 intersects Σ with odd parity, so does L. Since M is binary, L can be partitioned into a family of circuits. One of these circuits must be odd since $|L \cap \Sigma|$ is odd. The result now follows from the definition of binary clutters (see Section 1). □

Proposition 2.3. *Let $\mathcal{F}$ be a clutter such that $\emptyset \notin \mathcal{F}$. Consider the following properties: (i) for all $C_1, C_2 \in \mathcal{F}$ and $e \in C_1 \cap C_2$ there exists $C_3 \in \mathcal{F}$ such that $C_3 \subseteq C_1 \cup C_2 - \{e\}$. (ii) for all $C_1, C_2 \in \mathcal{F}$ there exists $C_3 \in \mathcal{F}$ such that $C_3 \subseteq C_1 \triangle C_2$. If property (i) holds then $\mathcal{F}$ is the set of circuits of a matroid. If property (ii) holds then $\mathcal{F}$ is the set of circuits of a binary matroid.*

Property (i) is known as the *circuit elimination axiom*. Circuits of matroids satisfy this property. Note that property (ii) implies property (i). Both results are standard, see Oxley [21].

Proposition 2.4. *Let $\mathcal{H}$ be a binary clutter such that $\emptyset \notin \mathcal{H}$. Let $\mathcal{F}$ be the clutter consisting of all inclusion-wise minimal, non-empty sets obtained by taking the symmetric difference of an arbitrary number of sets of $\mathcal{H}$. Then $\mathcal{H} \subseteq \mathcal{F}$ and $\mathcal{F}$ is the set of circuits of a binary matroid.*

Proof. By definition, $\mathcal{F}$ satisfies property (ii) in Proposition 2.3. Thus $\mathcal{F}$ is the set of circuits of a binary matroid M. Suppose for a contradiction there is $S \in \mathcal{H} - \mathcal{F}$. Then there exists $S' \in \mathcal{F}$ such that $S' \subset S$. Thus S' is the symmetric difference of a family of, say t, sets of $\mathcal{H}$. If t is odd then, Proposition 2.1 implies that S' contains a set of $\mathcal{H}$. If t is even then, Proposition 2.1 implies that $S' \triangle S$ contains a set of $\mathcal{H}$. Thus S is not inclusion-wise minimal, a contradiction. □

Consider a binary clutter $\mathcal{H}$ such that $\emptyset \notin \mathcal{H}$. The matroid defined in Proposition 2.4 is called the *up matroid* and is denoted by $u(\mathcal{H})$. Proposition 2.1 implies that every circuit of $u(\mathcal{H})$ is either an element of $\mathcal{H}$ or the symmetric difference of an even number of sets of $\mathcal{H}$. Since $\mathcal{H}$ is a binary clutter, sets of $b(\mathcal{H})$ intersect with odd parity exactly the circuits of $u(\mathcal{H})$ that are elements of $\mathcal{H}$. Hence,

Remark 2.5. A binary clutter $\mathcal{H}$ such that $\emptyset \notin \mathcal{H}$ is the clutter of odd circuits of $(u(\mathcal{H}), \Sigma)$ where $\Sigma \in b(\mathcal{H})$.

Moreover, this representation is essentially unique.

Proposition 2.6. *Let $\mathcal{H}$ be the clutter of odd circuits of the signed matroid (M, Σ). If $\mathcal{H}$ is not trivial and N is connected, then $N = u(\mathcal{H})$.*

To prove this, we use the following result (see Oxley [21] Theorem 4.3.2).

Theorem 2.7 (Lehman [13])**.** *Let e be an element of a connected binary matroid M. The circuits of M not containing e are of the form $C_1 \triangle C_2$ where C_1 and C_2 are circuits of M containing e.*

We shall also need the following observation which follows directly from Proposition 2.3.

Proposition 2.8. *Let (M, Σ) be a signed matroid and e an element not in $E(M)$. Let $\mathcal{F} := \{C \cup \{e\} : C \in \Omega(M), |C \cap \Sigma|$ odd$\} \cup \{C : C \in \Omega(M), |C \cap \Sigma|$ even$\}$. Then $\mathcal{F}$ is the set of circuit of a binary matroid.*

Proof of Proposition 2.6. Let N and N' be connected matroids and suppose that the clutters of odd circuits of (N, Σ) and (N', Σ') are the same and are not trivial. Let M (resp. M') be the matroid constructed from (N, Σ) (resp. (N', Σ)) as in Proposition 2.8. By construction the circuits of M and M' using e are the same. Since N is connected and $\mathcal{H}$ is not trivial, M and M' are connected. It follows from Theorem 2.7 that $M = M'$ and in particular $N = M/e = M'/e = N'$. By the same argument and Remark 2.5, $N = u(\mathcal{H})$. □

In a binary matroid, any circuit C and cocircuit D have an even intersection. So, if D is a cocircuit, the clutter of odd circuits of (M, Σ) and $(M, \Sigma \triangle D)$ are the same (see Zaslavsky [28]). Let $e \in E(M)$. The *deletion* $(M, \Sigma) \setminus e$ of (M, Σ) is defined as $(M \setminus e, \Sigma - \{e\})$. The *contraction* $(M, \Sigma)/e$ of (M, Σ) is defined as follows: if $e \notin \Sigma$ then $(M, \Sigma)/e := (M/e, \Sigma)$; if $e \in \Sigma$ and e is not a loop then there exists a cocircuit D of M with $e \in D$ and $(M, \Sigma)/e := (M/e, \Sigma \triangle D)$. Note if $e \in \Sigma$ is a loop of M, then $\mathcal{H}/e$ is a trivial clutter. A *minor* of (M, Σ) is any signed matroid which can be obtained by a sequence of deletions and contractions. A minor of (M, Σ) obtained by a sequence of J_c contraction and J_d deletions is denoted $(M, \Sigma)/J_c \setminus J_d$.

Remark 2.9. Let $\mathcal{H}$ be a the clutter of odd circuits of a signed matroid (M, Σ). If J_c does not contain an odd circuit, then $\mathcal{H}/J_c \setminus J_d$ is the clutter of odd circuits of the signed matroid $(M, \Sigma)/J_c \setminus J_d$.

Let M be a binary matroid and e an element of M. The clutter $Port(M, e)$, called a *port of* M, is defined as follows: $E\big(Port(M, e)\big) := E(M) - \{e\}$ and $Port(M, e) := \{S - \{e\} : e \in S \in \Omega(M)\}$.

Proposition 2.10. *Let M be a binary matroid, then $Port(M, e)$ is a binary clutter.*

Proof. By definition $S \in Port(M, e)$ if and only if $S \cup \{e\}$ is an odd circuit of the signed matroid $(M, \{e\})$. We may assume $Port(M, e)$ is nontrivial, hence in particular, e is not a loop of M. Therefore, there exists a cocircuit D that contains e. Thus $Port(M, e)$ is the clutter of odd circuits of the signed matroid $(M/e, D \triangle \{e\})$. Proposition 2.2 states that these odd circuits form a binary clutter. □

Proposition 2.11. *Let $\mathcal{H}$ be a binary clutter. Then there exists a binary matroid M with element $e \in E(M) - E(\mathcal{H})$ such that $Port(M, e) = \mathcal{H}$.*

Proof. If $\emptyset \in \mathcal{H}$, define M to have element e as a loop. If $\emptyset \notin \mathcal{H}$, we can represent $\mathcal{H}$ as the set of odd circuits of a signed matroid (N, Σ) (see Remark 2.5). Construct a binary matroid M from (N, Σ) as in Proposition 2.8. Then $Port(M, e) = \mathcal{H}$. □

Proposition 2.12 (Seymour [23]). *$Port(M, e)$ and $Port(M^*, e)$ form a blocking pair.*

Proof. Proposition 2.10 implies that $Port(M, e)$ and $Port(M^*, e)$ are both binary clutters. Consider $T \in Port(M^*, e)$. Then $T \cup \{e\}$ is a circuit of M. For all $S \in Port(M, e)$, $S \cup \{e\}$ is a circuit of M^*. Since $T \cup \{e\}$ and $S \cup \{e\}$ have an even intersection, $|S \cap T|$ is odd. Thus we proved: for all $T \in Port(M^*, e)$, there is $T' \in b(Port(M, e))$ where $T' \subseteq T$. To complete the proof it suffices to show: for all $T' \in b(Port(M, e))$, there is $T \in Port(M^*, e)$ where $T \subseteq T'$. Since $Port(M, e)$ is binary, for every $S \in Port(M, e)$, $|S \cap T'|$ is odd (Proposition 2.1). Thus $T' \cup \{e\}$ intersects every circuit of M using e with even parity. It follows from

Theorem 2.7 that $T' \cup \{e\}$ is orthogonal to the space spanned by the circuits of M, i.e. $T' \cup \{e\}$ is a cycle of M^*. It follows that there is a circuit of M^* of the form $T \cup \{e\}$ where $T \subseteq T'$. Hence, $T \in Port(M^*, e)$ as required. □

3. Lifts and sources

Let N be a binary matroid. For any binary matroid M with element e such that $N = M/e$, the binary clutter $Port(M, e)$ is called a *source* of N. Note that $\mathcal{H}$ is a source of its up matroid $u(\mathcal{H})$. For any binary matroid M with element e such that $N = M \setminus e$, the binary clutter $Port(M, e)$ is called a *lift* of N. Note that a source or a lift can be a trivial clutter.

Proposition 3.1. *Let N be a binary matroid. $\mathcal{H}$ is a lift of N if and only if $b(\mathcal{H})$ is a source of N^*.*

Proof. Let $\mathcal{H}$ be a lift of N, i.e. there is a binary matroid M with $M \setminus e = N$ and $\mathcal{H} = Port(M, e)$. By Proposition 2.12, $b(\mathcal{H}) = Port(M^*, e)$. Since $M^*/e = (M \setminus e)^* = N^*$ we have that $b(\mathcal{H})$ is a source of N^*. Moreover, the implications can be reversed. □

It is useful to relate a description of $\mathcal{H}$ in terms of excluded *clutter minors* to a description of $u(\mathcal{H})$ in terms of excluded *matroid minors*.

Theorem 3.2. *Let $\mathcal{H}$ be a binary clutter such that its up matroid $u(\mathcal{H})$ is connected, and let N be a connected binary matroid. Then $u(\mathcal{H})$ does not have N as a minor if and only if $\mathcal{H}$ does not have $\mathcal{H}_1$ or $\mathcal{H}_2^+$ as a minor, where $\mathcal{H}_1$ is a source of N and $\mathcal{H}_2$ is a lift of N.*

To prove this we will need the following result (see Oxley [21] Proposition 4.3.6).

Theorem 3.3 (Brylawski [3], Seymour [25])**.** *Let M be a connected matroid and N a connected minor of M. For any $i \in E(M) - E(N)$, at least one of $M \setminus i$ or M/i is connected and has N as a minor.*

Proof of Theorem 3.2. Let $M := u(\mathcal{H})$ and let $\Sigma \in b(\mathcal{H})$. Remark 2.5 states that $\mathcal{H}$ is the clutter of odd circuits of (M, Σ). Suppose first that $\mathcal{H}$ has a minor $\mathcal{H}_1$ that is a source of N. Remark 2.9 implies that $\mathcal{H}_1$ is the clutter of odd circuits of a signed minor (N', Σ') of (M, Σ). Since N is connected, $\mathcal{H}_1$ is nontrivial and therefore Proposition 2.6 implies $N = N'$. In particular N is a minor of M. Suppose now that $\mathcal{H}$ has a minor $\mathcal{H}_2^+$ where $\mathcal{H}_2$ is a lift of N. Let e be the element of $E(\mathcal{H}_2^+) - E(\mathcal{H}_2)$. Remark 2.9 implies that $\mathcal{H}_2^+$ is the clutter of odd circuits of a signed minor $(\hat{M}, \hat{\Sigma})$ of (M, Σ). Since $\mathcal{H}_2$ is a lift of N there is a connected matroid $\hat{M}'$ with element e such that $\hat{M}' \setminus e = N$ and $Port(\hat{M}', e) = \mathcal{H}_2$. Thus $\mathcal{H}_2^+$ is the clutter of odd circuits of $(\hat{M}', \{e\})$. Proposition 2.6 implies $\hat{M} = \hat{M}'$. Thus $\hat{M}'$ is a minor of M and so is $N = \hat{M}' \setminus e$.

Now we prove the converse. Suppose that M has N as a minor and does not satisfy the theorem. Let $\mathcal{H}$ be such a counterexample minimizing the cardinality of $E(\mathcal{H})$. Clearly, N is a proper minor of M as otherwise $u(\mathcal{H}) = N$, i.e. $\mathcal{H}$ is a source of N. By Theorem 3.3, for every $i \in E(M) - E(N)$, one of $M \setminus i$ and M/i is connected and has N as a minor. Suppose M/i is connected and has N as a minor. Since i is not a loop of M, it follows from Remark 2.9 that $\mathcal{H}/i$ is nontrivial and is a signed minor $(M/i, \Sigma')$ of (M, Σ). Proposition 2.6 implies $M/i = u(\mathcal{H}/i)$. But then $\mathcal{H}/i$ contradicts the choice of $\mathcal{H}$ minimizing the cardinality of $E(\mathcal{H})$. Thus, for every $i \in E(M) - E(N)$, $M \setminus i$ is connected and has an N minor. Suppose for some $i \in E(M) - E(N)$, $\mathcal{H} \setminus i$ is nontrivial. Then because of Remark 2.9 and Proposition 2.6 $u(\mathcal{H} \setminus i) = M \setminus i$, a contradiction to the choice of $\mathcal{H}$. Thus for every $i \in E(M) - E(N)$, $\mathcal{H} \setminus i$ is trivial, or equivalently, all odd circuits of (M, Σ) use i. As $M = u(\mathcal{H})$, even circuits of M do not use i. We claim that $E(M) - E(N) = \{i\}$. Suppose not and let $j \neq i$ be an element of $E(M) - E(N)$. The set of circuits of (M, Σ) using j is exactly the set of odd circuits. It follows that the elements i, j must be in series in M. But then $M \setminus i$ is not connected, a contradiction. Therefore $E(M) - E(N) = \{i\}$ and $M \setminus i = N$. As the circuits of (M, Σ) using i are exactly the odd circuits of (M, Σ), it follows that column i of $A(\mathcal{H})$ consists of all 1's. Thus $\mathcal{H} = \mathcal{H}_2^+$ where $\mathcal{H}_2 = Port(M, i)$, i.e. $\mathcal{H}_2$ is a lift of N. □

Next we define the binary matroids F_7, F_7^* and R_{10}. For any binary matroid N, let B_N be a 0,1 matrix whose rows span the circuit space of N (equivalently B_N is a representation of the dual matroid N^*). Square identity matrices are denoted I. Observe that $R_{10}^* = R_{10}$.

$$B_{F_7} = \left[\begin{array}{c|ccc} & 0 & 1 & 1 \\ I & 1 & 0 & 1 \\ & 1 & 1 & 0 \\ & 1 & 1 & 1 \end{array}\right] \quad B_{F_7^*} = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \end{array}\right] \quad B_{R_{10}} = \left[\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 0 & 0 \\ I & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 & 1 \end{array}\right]$$

Given a binary matroid N, let M be a binary matroid with element e such that $N = M/e$. The circuit space of M is spanned by the rows of a matrix of the form $[B_N|x]$, where x is a 0,1 column vector indexed by e. Assuming M is connected, we have (up to isomorphism), the following possible columns x for each of the three aforementioned matroids N:

(1) F_7^*:
$x_a = (1,1,1)^T$.
(2) F_7:
$x_a = (0,1,1,1)^T$, $x_b = (1,1,1,0)^T$ and $x_c = (1,1,1,1)^T$.
(3) R_{10}:
$x_a = (1,0,1,0,0)^T$, $x_b = (1,0,1,0,1)^T$, $x_c = (1,0,1,1,0)^T$,
$x_d = (1,0,1,1,1)^T$, $x_e = (1,1,0,0,0)^T$, $x_f = (1,1,1,1,1)^T$.

Note that (1),(2) are easy and (3) can by found in [24] (p. 357). The rows of the matrix $[B_{F_7}|x_b]$ (resp. $[B_{F_7}|x_c]$) span the circuit space of a matroid known as $AG(3,2)$ (resp. S_8). If $[B_N|x]$ is a matrix whose rows span the circuits of M, then by definition of sources, $Port(M, e)$ is a source of N. Thus,

Remark 3.4.

- F_7^* has a unique source, namely Q_6^+.
- F_7 has three sources: $b(Q_6)^+$ (when $x = x_a$), $\mathcal{L}_{F_7}$ (when $x = x_b$), and $b(Q_7)$ (when $x = x_c$).
- R_{10} has six sources including $b(\mathcal{O}_{K_5})$ (when $x = x_f$).

Luetolf and Margot [16] have enumerated all minimally nonideal clutters with at most 10 elements (and many more). Using Remark 3.4, we can then readily check the following.

Proposition 3.5. *Let $\mathcal{H}$ be the clutter of odd circuits of a signed matroid (M, Σ).*

- *If $M = R_{10}$, then either $\mathcal{H} = b(\mathcal{O}_{K_5})$ or $\mathcal{H}$ is ideal.*
- *If $M = F_7$, then either $\mathcal{H} = \mathcal{L}_{F_7}$ or $\mathcal{H}$ is ideal.*
- *If $M = F_7^*$, then $\mathcal{H}$ is ideal.*

4. MULTICOMMODITY FLOWS

In this section, we show that a binary clutter $\mathcal{H}$ is ideal exactly when certain multicommodity flows exist in the matroid $u(\mathcal{H})$. This equivalence will be used in Sections 6 and 7 to show that minimally nonideal binary clutters do not have small separations. Given a set S, a function $p : S \to \mathbb{Q}_+$, and $T \subseteq S$, we write $p(T)$ for $\sum_{i \in T} p(i)$. Consider a signed matroid (M, F). The set of circuits of M that have exactly one element in common with F, is denoted Ω_F. Let $p : E(M) \to \mathbb{Q}_+$ be a cost function on the elements of M. Seymour [26] considers the following two statements about the triple (M, F, p).

For any cocircuit D of M:

$$p(D \cap F) \leq p(D - F). \tag{4.1}$$

There exists a function $\eta : \Omega_F \to \mathbb{Q}_+$ such that:

$$\sum_{C : e \in C \in \Omega_F} \eta(C) \quad \begin{cases} \geq p(e) & \text{if } e \in F \\ \leq p(e) & \text{if } e \in E - F. \end{cases} \tag{4.2}$$

We say that *the cut condition holds* if inequality (4.1) holds for all cocircuits D. We say that M is *F-flowing with costs* p if statement (4.2) holds; the corresponding solution is an *F-flow satisfying costs* p. M is *F-flowing* [26] if, for every p for which the cut condition holds, M is F-flowing with costs p. Elements in F (resp. $E - F$) are called *demand* (resp. *capacity*) elements. It is helpful to illustrate the aforementioned definitions in the case where M is a graphic matroid [9]. For a demand edge f, $p(f)$ is the amount of flow required between its endpoints. For a capacity edge e, $p(e)$ is the maximum amount of flow that can be carried by e. Then M is F-flowing with costs p when a multicommodity flow meeting all demands and satisfying all capacity constraints exists. The cut condition requires that for every cut the demand across the cut does not exceed its capacity. When F consists of a single edge f and when M is graphic then M is f-flowing [7].

The cut condition states $p(D \cap F) \leq p(D - F) = p(D) - p(D \cap F)$. Adding $p(F) - p(D \cap F)$ to both sides, we obtain $p(F) \leq p(D) - p(D \cap F) + p(F) - p(D \cap F) = p(D \triangle F)$. Hence,

Remark 4.1. The cut condition holds if and only if $p(F) \leq p(D \triangle F)$ for all cocircuits D.

Let $\mathcal{H}$ be the clutter of odd circuits of (M, F). We define:

$$\tau(\mathcal{H}, p) = \min\{\sum_{e \in E(M)} p(e) x_e : \sum_{e \in S} x_e \geq 1, \forall S \in \mathcal{H}, x_e \in \{0, 1\}, \forall e \in E(M)\} \tag{a}$$

$$\nu^*(\mathcal{H}, p) = \max\{\sum_{C \in \mathcal{H}} y_C : \sum_{C : e \in C \in \mathcal{H}} y_C \leq p(e), \forall e \in E(M), y_C \geq 0, \forall C \in \mathcal{H}\} \tag{b}$$

By linear programming duality we have: $\tau(\mathcal{H}, p) \geq \nu^*(\mathcal{H}, p)$. When $p(e) = 1$ for all $e \in E(M)$ then we write $\tau(\mathcal{H})$ for $\tau(\mathcal{H}, p)$ and $\nu^*(\mathcal{H})$ for $\nu^*(\mathcal{H}, p)$.

Proposition 4.2. *Let $\mathcal{H}$ be the clutter of odd circuits of a signed matroid (M, F) and let $p : E(M) \to \mathbb{Q}_+$.*

(i) $\tau(\mathcal{H}, p) = p(F)$ if and only if the cut condition holds.
(ii) $\nu^*(\mathcal{H}, p) = p(F)$ if and only if M is F-flowing with costs p.
(iii) If $\eta(C) > 0$ for a solution to (4.2), then $C \in \Omega_F$ for all $F \in b(\mathcal{H})$ with $p(F) = \tau(\mathcal{H}, p)$.

Proof. We say that a set $X \subseteq E(M)$ is a (feasible) solution for (a) if its characteristic vector is. Consider (i). Suppose $\tau(\mathcal{H}, p) = p(F)$. We can assume that F is an inclusion-wise minimal solution of (a) and thus

$F \in b(\mathcal{H})$. Let D be any cocircuit of M and consider any $S \in \mathcal{H}$. Since S is a circuit of M, $|D \cap S|$ is even and since $\mathcal{H}$ is binary, $|F \cap S|$ is odd. Thus $|(D \triangle F) \cap S|$ is odd. It follows that $D \triangle F$ is a transversal of $\mathcal{H}$. Therefore, $D \triangle F$ is a feasible solution to (a) and we have $p(F) \leq p(D \triangle F)$. Hence, by Remark 4.1, the cut condition holds. Conversely, assume the cut condition holds and consider any set X that is feasible for (a). We need to show $p(F) \leq p(X)$. We can assume that X is inclusion-wise minimal, i.e. that $X \in b(\mathcal{H})$. Observe that F and X intersect circuits of M with the same parity. Thus $D := F \triangle X$ is a cocycle of M. Since the cut condition holds, by Remark 4.1, $p(F) \leq p(D \triangle F) = p(X)$.

Consider (ii). Suppose $\nu^*(\mathcal{H}, p) = p(F)$. Since $\nu^*(\mathcal{H}, p) \leq \tau(\mathcal{H}, p) \leq p(F)$, it follows from linear programming duality that F is an optimal solution to (a). Let y be an optimal solution to (b). Complementary slackness states: if $|F \cap C| > 1$, then the corresponding dual variable $y_C = 0$. Thus $\sum_{C: e \in C \in \mathcal{H}} y_C = \sum_{C: e \in C \in \Omega_F} y_C$, for all $e \in E(M)$. Complementary slackness states: if $e \in F$, then $\sum_{C: e \in C \in \Omega_F} y_C = p(e)$. Hence, choosing $\eta(C) = y_C$ for every $C \in \Omega_F$ satisfies (4.2). Conversely, suppose η is a solution to (4.2). For each $e \in F$ such that $\sum_{C:\, e \in C \in \Omega_F} \eta_C > p(e)$, reduce the values η_C on the left hand side until equality holds. Since C contains no element of F other than e, we can get equality for every $e \in F$. So we may assume $\sum_{C: e \in C \in \Omega_F} \eta(C) \leq p(e)$ for all $e \in E(M)$. Set $y_C = 0$ if $C \notin \Omega_F$ and $y_C = \eta(C)$ if $C \in \Omega_F$. Now y is a feasible solution to (b) and F, y satisfy all complementary slackness conditions. Thus F and y must be a pair of optimal solutions to (a) and (b) respectively.

Finally, consider (iii). From (ii) we know there is an optimal solution y to (b) with $y_C > 0$. By complementary slackness, it follows that $|F \cap C| = 1$ for all F that are optimal solutions to (a). □

The last proposition implies in particular that, if M is F-flowing with costs p, then the cut condition is satisfied. We say that a cocircuit D is *tight* if the cut condition (4.1) holds with equality, or equivalently (Remark 4.1) if $p(F) = p(D \triangle F)$.

Proposition 4.3. *Suppose M is F-flowing with costs p and let D be a tight cocircuit. If C is a circuit with $\eta(C) > 0$, then $C \cap D = \emptyset$ or $C \cap D = \{e, f\}$ where $e \in E(M) - F$ and $f \in F$.*

Proof. We may assume $C \cap D \neq \emptyset$. As $C \in \Omega_F$, it follows that $C \cap F = \{f\}$. Moreover, $C \cap D \neq \{f\}$, since M is binary. To complete the proof, it suffices to show that there is no pair of elements $e, e' \in (C \cap D) - F$. Suppose for a contradiction that we have such a pair and let $F' = D \triangle F$. As D is tight, $p(F) = p(D \triangle F) = p(F')$. It follows from Proposition 4.2(iii) that $C \in \Omega_{F'}$. But $e, e' \in F'$, a contradiction. □

Corollary 4.4. *Let $\mathcal{H}$ be the clutter of odd circuits of a signed matroid (M, F). (i) If $\mathcal{H}$ is ideal then M is F-flowing with costs p, for all $p : E(M) \to \mathbb{Q}_+$ where (M, F, p) satisfies the cut condition. (ii) If $\mathcal{H}$ is nonideal then M is not F'-flowing with costs p, for some $p : E(M) \to \mathbb{Q}_+$ and some $F' \in b(\mathcal{H})$ that minimizes $p(F')$.*

Proof. Consider (i). Proposition 4.2 states $\tau(\mathcal{H}, p) = p(F)$. Because $\mathcal{H}$ is ideal, $\tau(\mathcal{H}, p) = \nu^*(\mathcal{H}, p)$, i.e. $p(F) = \nu^*(\mathcal{H}, p)$. This implies by Proposition 4.2(ii) that M is F-flowing with costs p. Consider (ii). If $\mathcal{H}$ is nonideal then for some $p : E(M) \to \mathbb{Z}_+$, $\tau(\mathcal{H}, p) > \nu^*(\mathcal{H}, p)$ [5]. Let F' be an optimal solution to (a). Then $p(F') = \tau(\mathcal{H}, p)$ and $F' \in b(\mathcal{H})$. Proposition 4.2(ii) states M is not F'-flowing with costs p. □

We leave the next result as an easy exercise.

Corollary 4.5. *A binary clutter $\mathcal{H}$ is ideal if and only if $u(\mathcal{H})$ is F-flowing for every $F \in b(\mathcal{H})$.*

Consider the case where $\mathcal{H} = \mathcal{O}_{K_5}$. Let F be a set of edges of K_5 such that $E(K_5) - F$ induces a $K_{2,3}$. Then $F \in b(\mathcal{H})$ and $u(\mathcal{H})$ (the graphic matroid of K_5) is not F-flowing.

5. Connectivity, preliminaries

Let E_1, E_2 be a partition of the elements E of a matroid M and let $r : 2^{|E|} \to \mathbb{Z}_+$ be the rank function. M is said to have a *k-separation* E_1, E_2 if $r(E_1) + r(E_2) - r(E) \leq k - 1$ and $|E_1|, |E_2| \geq k$. If $|E_1|, |E_2| > k$, then the separation is said to be *strict*. A matroid M has a k-separation E_1, E_2 if and only if its dual M^* does (Oxley [21], 4.2.7). A matroid is *k-connected* if it has no $(k-1)$-separation and is *internally k-connected* if it has no strict $(k-1)$-separation. A 2-connected matroid is simply said to be *connected*. We now follow Seymour [24] when presenting k-sums. Let M_1, M_2 be binary matroids whose element sets $E(M_1), E(M_2)$ may intersect. We define $M_1 \triangle M_2$ to be the binary matroid on $E(M_1) \triangle E(M_2)$ where the cycles are all the subsets of $E(M_1) \triangle E(M_2)$ of the form $C_1 \triangle C_2$ where C_i is a cycle of M_i, $i = 1, 2$. The following special cases will be of interest to us:

Definition 5.1.

(1) $E(M_1) \cap E(M_2) = \emptyset$. Then $M_1 \triangle M_2$ is the *1-sum* of M_1, M_2.
(2) $E(M_1) \cap E(M_2) = \{f\}$ and f is not a loop of M_1 or M_2. Then $M_1 \triangle M_2$ is the *2-sum* of M_1, M_2.
(3) $|E(M_1) \cap E(M_2)| = 3$ and $E(M_1) \cap E(M_2)$ is a circuit of both M_1 and M_2. Then $M_1 \triangle M_2$ is the *3-sum* of M_1, M_2.

We denote the k-sum of M_1 and M_2 as $M_1 \otimes_k M_2$. The elements in $E(M_1) \cap E(M_2)$ are called the *markers* of M_i ($i = 1, 2$). As an example, for $k = 1, 2, 3$, the k-sum of two graphic matroids corresponds to taking two graphs, choosing a k-clique from each, identifying the vertices in the clique pairwise and deleting the edges in the clique. The markers are the edges in the clique. We have the following connection between k-separations and k-sums.

Theorem 5.2 (Seymour [24])**.** *Let M be a k-connected binary matroid and $k \in \{1, 2, 3\}$. Then M has a k-separation if and only if it can be expressed as $M_1 \otimes_k M_2$. Moreover, M_1 (resp. M_2) is a minor of M obtained by contracting and deleting elements in $E(M_2) - E(M_1)$ (resp. $E(M_1) - E(M_2)$).*

We say that a binary clutter $\mathcal{H}$ has a (strict) *k-separation* if $u(\mathcal{H})$ does.

Remark 5.3. $\mathcal{H}$ has a 1-separation if and if $A(\mathcal{H})$ is a block diagonal matrix. Moreover, $\mathcal{H}$ is ideal if and only if the minors corresponding to each of the blocks are ideal.

Recall (Proposition 2.11) that every binary clutter $\mathcal{H}$ can be expressed as $Port(M, e)$ for some binary matroid M with element e. So we could define the connectivity of $\mathcal{H}$ to be the connectivity of the associated matroid M. The two notions of connectivity are not equivalent as the clutter $\mathcal{L}_{F_7}$ illustrates. The matroid $AG(3, 2)$ has a strict 3-separation while F_7 does not, but $Port(AG(3, 2), t) = \mathcal{L}_{F_7}$ and $\mathcal{L}_{F_7}$ is the clutter of odd circuits of the signed matroid $\big(F_7, E(F_7)\big)$.

Chopra [4] gives composition operations for matroid ports and sufficient conditions for maintaining idealness. This generalizes earlier results of Bixby [1]. Other compositions for ideal (but not necessarily binary clutters) can be found in [19, 17, 18]. Novick-Sebö [20] give an outline on how to show that mni binary clutters do not have 2-separations, the argument is similar to that used by Seymour [26](7.1) to show that k-cycling matroids are closed under 2-sums. We will follow the same strategy (see Section 6). Proving that mni binary clutters do not have 3-separations is more complicated and requires a different approach (see Section 7). In closing observe that none of $\mathcal{L}_{F_7}, \mathcal{O}_{K_5}$ and $b(\mathcal{O}_{K_5})$ have strict 4-separations. So if Seymour's Conjecture holds, then mni binary clutters are internally 5-connected.

6. 2-SEPARATIONS

Let (M, F) be a signed matroid with a 2-separation E_1, E_2, i.e. $M = M_1 \otimes_2 M_2$ and $E_1 = E(M_1) - E(M_2)$, $E_2 = E(M_2) - E(M_1)$. We say that (M_i, F_i) (for $i = 1, 2$) is a *part* of (M, F) if it is a signed minor of (M, F). It is not hard to see that at most two choices of F_i can give distinct signed matroids (M_i, F_i). Therefore (M, F) can have at most four distinct parts. In light of Remark 2.5 we can identify binary clutters with signed matroids. The main result of this section is the following.

Proposition 6.1. *A binary clutter with a 2-separation is ideal if and only if all its parts are ideal.*

To prove this, we shall need the following results.

Proposition 6.2 (Seymour [24]). *If $M = M_1 \otimes_2 M_2$, then M is connected if and only if M_1 and M_2 are connected.*

Proposition 6.3 (Seymour [24]). *Let M be a binary matroid with a 2-separation E_1, E_2 and let C_1, C_2 be two circuits of M. If $C_1 \cap E_i \subseteq C_2 \cap E_i$, then $C_1 \cap E_i = C_2 \cap E_i$ (for $i = 1, 2$).*

Proposition 6.4 (Seymour [24]). *Let $M = M_1 \otimes_2 M_2$. Then choose any circuit C of M such that $C \cap E_1 \neq \emptyset$ and $C \cap E_2 \neq \emptyset$. Let $i, j = 1, 2$ and $i \neq j$. For any $f \in C \cap E_j$, $M_i = M \setminus (E_j - C)/(E_j \cap C - \{f\})$.*

Proof of Proposition 6.1. Let $\mathcal{H}$ be a binary clutter with a 2-separation, $M = u(\mathcal{H})$ and $F \in b(\mathcal{H})$. Assume without loss of generality that M is connected. Remark 2.5 states that $\mathcal{H}$ is the clutter of odd circuits of (M, F). If $\mathcal{H}$ is ideal, then so are all its parts by Remark 2.9. Conversely, suppose all parts of (M, F) are ideal. Consider any $p : E(M) \to \mathbb{Z}_+$ and assume $F \in b(\mathcal{H})$ minimizes $p(F)$. Because of Corollary 4.4(ii), it suffices to show that M is F-flowing with costs p. Observe that the cut condition is satisfied because of Proposition 4.2(i).

Since M has a 2-separation, it can be expressed as $M_1 \otimes_2 M_2$. Throughout this proof, i, j will always denote arbitrary distinct elements of $\{1, 2\}$. Define $F_i = F \cap E_i$ and let f_i be the marker of M_i. Since f_i is not a loop, there is a cocircuit D_i of M_i using f_i. Let α_i denote the smallest value of

$$p[D_i - (F_i \cup \{f_i\})] - p(D_i \cap F_i). \qquad (*)$$

where D_i is any cocircuit of M_i using f_i. In what follows, we let D_i denote some cocircuit where the minimum is attained. Expression (*) gives the difference between the sum of the capacity elements and the sum of the demand elements in D_i, excluding the marker f_i. Thus $\alpha_1 + \alpha_2 = p([D_1 \triangle D_2] - F) - p([D_1 \triangle D_2] \cap F)$. Since $D_1 \triangle D_2$ is a cocycle of M and the cut condition is satisfied, we must have:

$$\alpha_1 + \alpha_2 \geq 0.$$

Claim 1. If $\alpha_i > 0$, then there is an even circuit of (M_i, F_i) that uses marker f_i.

Proof of Claim: Suppose for a contradiction that all circuits C of M_i that use f_i, satisfy $|C \cap F_i|$ odd. Then $D = F_i \cup \{f_i\}$ intersects all these circuits with even parity. By hypothesis M is connected and, because of Proposition 6.2, so is M_i. We know from Theorem 2.7 that all circuits that do not use the marker f_i are the symmetric difference of two circuits that do use f_i. It follows that D intersects all circuits of M_i with even parity. Thus D is a cocycle of M_i. But expression (*) is nonpositive for cocycle D. D can be partitioned into cocircuits. Because the cut condition holds, expression (*) is nonpositive for the cocircuit that uses f_i, a contradiction as $\alpha_i > 0$. ◇

Claim 2. If $\alpha_i < 0$, then there is an odd circuit of (M_i, F_i) that uses marker f_i.

Proof of Claim: Suppose, for a contradiction, that all circuits C of M_i that use f_i, satisfy $|C \cap F_i|$ even. By the same argument as in Claim 1, we know that in fact so do all circuits of M_i. This implies that F and F_j intersect each circuit of M with the same parity. As F is inclusion-wise minimal ($F \in b(\mathcal{H})$) we must have $F = F_j$, i.e. $F_i = \emptyset$. But this implies that expression (*) is non negative, a contradiction. ◇

Claim 3. If $\alpha_j < 0$ (resp. $\alpha_j > 0$), then $(M_i, F_i \cup \{f_i\})$ (resp. (M_i, F_i)) is a part of (M, F).

Proof of Claim: From Claim 2 (resp. Claim 1), there is an odd (resp. even) circuit C using f_j of (M_j, F_j). Proposition 6.3 implies that elements $C \cap E$ are in series in $M \setminus (E_j - C)$. Proposition 6.4 implies that M_i is obtained from $M \setminus (E_j - C)$ by replacing series elements of $C \cap E_j$ by a unique element f_j. The required signed minor is $(M, F) \setminus (E_j - C)/(C - \{f\})$ where f is any element of $C \cap E_j$. ◇

Because $\alpha_1 + \alpha_2 \geq 0$, it suffices to consider the following cases.

Case 1: $\alpha_1 \geq 0, \alpha_2 \geq 0$.

We know from Proposition 6.4 that M_i is a minor of M (where no loop is contracted) say $M \setminus J_d/J_c$. For $i = 1, 2$, let $(M_i, \hat{F}_i)$ be the signed minor $(M, F) \setminus J_d/J_c$. Since $(M_i, \hat{F}_i)$ is a part of (M, F), it is ideal. So in particular $(M_i, \hat{F}_i) \setminus f_i = (M_i \setminus f_i, F_i)$ are ideal. Let $p^i : E(M_i) - f_i \to \mathbb{Z}_+$ be defined as follows: $p^i(e) = p(e)$ if $e \in E_i$. Let D be a cocircuit of $M_i \setminus f_i$. The inequality $p(D \cap F_i) \leq p(D - F_i)$ follows from $\alpha_i \geq 0$ when $D \cup f_i$ is a cocircuit of M_i and it follows from the fact that the cut condition holds for (M, F) when D is a cocircuit of M_i. Therefore the cut condition holds for $(M_i \setminus f_i, F_i)$. It follows from Corollary 4.4(i) that each of these signed matroids has an F_i-flow satisfying costs p^i. Let $\eta_i = \Omega_{F_i} \to \mathbb{Q}_+$ be the corresponding function satisfying (4.2). By scaling p, we may assume $\eta_i(C) \in \mathbb{Z}_+$ for each circuit in Ω_{F_i}. Let $\mathcal{L}_i$ be the multiset where each circuit C in Ω_{F_i} appears $\eta_i(C)$ times. Define $\mathcal{L}_j$ similarly. The union (with repetition) of all circuits in $\mathcal{L}_i$ and $\mathcal{L}_j$ correspond to an F-flow of M satisfying costs p.

Case 2: $\alpha_i < 0, \alpha_j > 0$.

Because of Claim 3, there are parts (M_i, F_i) and $(M_j, F_j \cup \{f_j\})$ of (M, F). Let $p^i : E(M_i) \to \mathbb{Z}_+$ be defined as follows: $p^i(f_i) = \alpha_j$ and $p^i(e) = p(e)$ for $e \in E_i$. Let $p^j : E(M_j) \to \mathbb{Z}_+$ be defined as follows: $p^j(f_j) = -\alpha_i$ and $p^j(e) = p(e)$ for $e \in E_j$. Since we can scale p, we can assume that the F_i-flow of M_i satisfying costs p^i is a multiset $\mathcal{L}^i$ of circuits and that the $F_j \cup \{f_j\}$-flow of M_j satisfying costs p^j is a multiset $\mathcal{L}^j$. For $l = 1, 2$, $\mathcal{L}^l$ can be partitioned into $\mathcal{L}^l_0 := \{C \in \mathcal{L}^l : f_l \notin C\}$, and $\mathcal{L}^l_1 := \{C \in \mathcal{L}^l : f_l \in C\}$. Since f_j is a demand element for the flow $\mathcal{L}^j$, $|\mathcal{L}^j_1| = p^j(f_j) = -\alpha_i$. Since f_i is a capacity element for the flow $\mathcal{L}^i$, $|\mathcal{L}^i_1| \leq p^i(f_i) = \alpha_j$. Because $\alpha_1 + \alpha_2 \geq 0$, $|\mathcal{L}^j_1| \leq |\mathcal{L}^i_1|$. Let us define a collection of circuits of M

as follows: include all circuits of $\mathcal{L}_0^i \cup \mathcal{L}_0^j$. Pair each circuit $C_j \in \mathcal{L}_1^j$ with a different circuit $C_i \in \mathcal{L}_1^i$, and add to the collection the circuit included in $C_i \triangle C_j$ that contains the element of F. The resulting collection corresponds to a F-flow of M satisfying costs p. □

7. 3-SEPARATIONS

The main result of this section is the following,

Proposition 7.1. *A minimally nonideal binary clutter $\mathcal{H}$ has no strict 3-separation.*

The proof follows from two lemmas, stated next and proved in sections 7.1 and 7.2 respectively.

Lemma 7.2. *Let $\mathcal{H}$ be a minimally nonideal binary clutter with a strict 3-separation E_1, E_2. There exists a set $F \in b(\mathcal{H})$ of minimum cardinality such that $F \subseteq E_1$ or $F \subseteq E_2$.*

Let (M, F) be a signed matroid with a strict 3-separation E_1, E_2, i.e. $M = M_1 \otimes_3 M_2$ and $E_1 = E(M_1) - E(M_2)$, $E_2 = E(M_2) - E(M_1)$. Let $C_0 = E(M_1) \cap E(M_2)$ be the triangle common to both M_1 and M_2. Let $\bar{M}_i$ (with $i = 1, 2$) be obtained by deleting from M_i a (possibly empty) set of elements of C_0. We call $(\bar{M}_i, F_i)$ a *part* of (M, F) if it is a signed minor of (M, F).

Lemma 7.3. *Let (M, F) be a connected signed matroid with a strict 3-separation E_1, E_2 and suppose $F \subseteq E_1$. Then M is F-Flowing with costs p if the cut condition is satisfied and all parts of (M, F) are ideal.*

Proof of Proposition 7.1. Suppose $\mathcal{H}$ is a mni binary clutter that is connected with a strict 3-separation. Remark 2.5 states that $\mathcal{H}$ is the clutter of odd circuits of a signed matroid (M, F). Consider $p : E(M) \to \mathbb{Z}_+$ defined by $p(e) = 1$ for all $e \in E(M)$. We know (see Remark 7.5) that $\tau(\mathcal{H}, p) > \nu^*(\mathcal{H}, p)$. From Lemma 7.2 and Remark 2.5, we may assume $F \subseteq E_1$ and $p(F) = \tau(\mathcal{H}, p)$. It follows from Proposition 4.2(i) that the cut condition holds. Since the separation of (M, F) is strict, all parts of (M, F) are proper minors, and hence ideal. It follows therefore from Lemma 7.3 that M is F-flowing with costs p. Hence, because of Proposition 4.2(ii), $\nu^*(\mathcal{H}, p) = p(F)$, a contradiction. □

7.1. **Separations and blocks.** In this section, we shall prove Lemma 7.2. But first let us review some results on minimally nonideal clutters. For every clutter $\mathcal{H}$, we can associate a $0, 1$ matrix $A(\mathcal{H})$. Hence we shall talk about mni $0, 1$ matrices, blocker of $0, 1$ matrices, and binary $0, 1$ matrices (when the associated clutter is binary). The next result on mni 0,1 matrices is due to Lehman [15] (see also Padberg [22], Seymour [27]). We state it here in the binary case.

Theorem 7.4. *Let A be a minimally nonideal binary 0,1 matrix with n columns. Then $B = b(A)$ is minimally nonideal binary as well, the matrix A (resp. B) has a square, nonsingular row submatrix $\bar{A}$ (resp. $\bar{B}$) with r (resp. s) nonzero entries in every row and columns, $rs > n$. Rows of A (resp. B) not in $\bar{A}$ (resp. $\bar{B}$) have at least $r + 1$ (resp. $s + 1$) nonzero entries. Moreover, $\bar{A}\bar{B}^T = J + (rs - n)I$, where J denotes an $n \times n$ matrix filled with ones.*

It follows that $(\frac{1}{r}, \ldots, \frac{1}{r})$ is a fractional extreme point of the polyhedron $\{x \in \mathbb{R}_+^n : Ax \geq \mathbf{1}\}$. Hence,

Remark 7.5. If $\mathcal{H}$ is a minimally nonideal binary clutter, then $\tau(\mathcal{H}) > \nu^*(\mathcal{H})$.

The submatrix $\bar{A}$ is called the *core* of A. Given a mni clutter $\mathcal{H}$ with $A = A(\mathcal{H})$, we define the core of $\mathcal{H}$ to be the clutter $\bar{\mathcal{H}}$ for which $A(\bar{\mathcal{H}}) = \bar{A}$. Let $\mathcal{H}$ and $\mathcal{G} = b(\mathcal{H})$ be binary and mni. Since $\mathcal{H}, \mathcal{G}$ are binary, for all $S \in \bar{\mathcal{H}}$ and $T \in \bar{\mathcal{G}}$, we have $|S \cap T|$ odd. As $\bar{A}\bar{B}^T = J + (rs - n)I$, for every $S \in \bar{\mathcal{H}}$, there is exactly one set $T \in \bar{\mathcal{G}}$ called the *mate* of S such that $|S \cap T| = 1 + (rs - n)$. Note that if A is binary then $rs - n + 1 \geq 3$.

Proposition 7.6. *Let A be a mni binary matrix. Then no column of $\bar{A}$ is in the union of two other columns.*

Proof. Bridges and Ryser [2] proved that square $0,1$ matrices $\bar{A}, \bar{B}$ that satisfy $\bar{A}\bar{B}^T = J + (rs - n)I$ commute, i.e. $\bar{A}^T\bar{B} = J + (rs - n)I$. Thus $col(\bar{A}, i)col(\bar{B}, i) = rs - n + 1 \geq 3$ for every $i \in \{1, \ldots, n\}$. Hence there is no $j, k \in \{1, \ldots, n\} - \{i\}$ such that $col(\bar{A}, j) \cup col(\bar{A}, k) \supseteq col(\bar{A}, i)$, for otherwise $col(\bar{A}, j)col(\bar{B}, i) > 1$ or $col(\bar{A}, k)col(\bar{B}, i) > 1$, contradicting the equation $\bar{A}^T\bar{B} = J + (rs - n)I$. ☐

Proposition 7.7 (Guenin [10])**.** *Let $\mathcal{H}$ be a mni binary clutter and $e \in E(\mathcal{H})$. There exists $S_1, S_2, S_3 \in \bar{\mathcal{H}}$ such that $S_1 \cap S_2 = S_2 \cap S_3 = S_1 \cap S_3 = \{e\}$.*

Proposition 7.8 (Guenin [10])**.** *Let $\mathcal{H}$ be a mni binary clutter and $S_1, S_2 \in \bar{\mathcal{H}}$. If $S \subseteq S_1 \cup S_2$ and $S \in \mathcal{H}$ then either $S = S_1$ or $S = S_2$.*

Proposition 7.9. *Let $\mathcal{H}$ be a mni binary clutter and let $S, S' \in \bar{\mathcal{H}}$. Then $|S - S'| \geq 2$.*

Proof. Let T be the mate of S. Then $|T \cap S| \geq 3$ and $|T \cap S'| = 1$. ☐

Proposition 7.10 (Luetolf and Margot [16])**.** *Let $\mathcal{H}$ be a mni binary clutter. Then $\tau(\mathcal{H}) = \tau(\bar{\mathcal{H}})$. Furthermore, if T is a transversal of $\bar{\mathcal{H}}$ and $|T| = \tau(\bar{\mathcal{H}})$, then T is a transversal of $\mathcal{H}$.*

We shall also need,

Proposition 7.11 (Seymour [24])**.** *Let M be a binary matroid with 3-separation E_1, E_2. Then there exist circuits C_1, C_2 such that every circuit of M can be expressed as the symmetric difference of a subset of circuits in $\{C \in \Omega(M) : C \subseteq E_1$ or $C \subseteq E_2\} \cup \{C_1, C_2\}$.*

Throughout this section, we shall consider a signed matroid (M, F) with a 3-separation E_1, E_2 and C_1, C_2 will denote the corresponding circuits of Proposition 7.11. Let $\mathcal{H}$ be the clutter of odd circuits of (M, F). We shall partition $b(\mathcal{H})$ into sets B_1, B_2, B_3, B_4 as follows:

$$B_1 = \{S \in b(\mathcal{H}) : |S \cap C_1 \cap E_1| \text{ even}, |S \cap C_2 \cap E_1| \text{ even}\}$$
$$B_2 = \{S \in b(\mathcal{H}) : |S \cap C_1 \cap E_1| \text{ even}, |S \cap C_2 \cap E_1| \text{ odd}\}$$
$$B_3 = \{S \in b(\mathcal{H}) : |S \cap C_1 \cap E_1| \text{ odd}, |S \cap C_2 \cap E_1| \text{ even}\}$$
$$B_4 = \{S \in b(\mathcal{H}) : |S \cap C_1 \cap E_1| \text{ odd}, |S \cap C_2 \cap E_1| \text{ odd}\}.$$

Proposition 7.12. *If $S_1, S_2 \in B_i$ where $i \in \{1, \ldots, 4\}$, then $(S_1 \cap E_1) \cup (S_2 \cap E_2)$ contains a set of $b(\mathcal{H})$.*

Proof. Let $S' := (S_1 \cap E_1) \cup (S_2 \cap E_2)$. Note that since $S_1, S_2 \in b(\mathcal{H})$ for all circuits C of M, $|S_1 \cap C|$ and $|S_2 \cap C|$ have the same parity. This implies that if C is a circuit where $C \subseteq E_k$ $(k \in \{1, 2\})$ then C intersects S' and S_1 with the same parity. It also implies, together with the definition of B_i, that S' intersects C_k $(k \in \{1, 2\})$ with the same parity as S_1. It follows from Proposition 7.11 that S' and S_1 intersect all circuits of M with the same parity. ☐

Proof of Lemma 7.2. Let $\mathcal{G}$ denote the blocker of $\mathcal{H}$ and let B_1, B_2, B_3, B_4 be the sets partitioning $\mathcal{G}$. We will denote by $\bar{\mathcal{G}}$ the core of $\mathcal{G}$. It follows that $\bar{\mathcal{G}}$ can be partitioned into sets $\bar{B}_i$ with $i \in \{1, \dots, 4\}$ and $\bar{B}_i \subseteq B_i$. Assume for a contradiction that for all $S \in \bar{\mathcal{G}}$, $S \cap E_1 \neq \emptyset \neq S \cap E_2$. We will say that a set $\bar{B}_i$ with $i \in \{1, \dots, 4\}$ forms an E_1-*block* if, for all pairs of sets $S, S' \in \bar{B}_i$, we have $S \cap E_1 = S' \cap E_1 \neq \emptyset$. Similarly we define E_2-blocks.

Claim 1. For $i \in \{1, \dots, 4\}$, each nonempty $\bar{B}_i$ is either an E_1- or an E_2-block.

Proof of Claim: Consider S_1, S_2 in $\bar{B}_i$. Proposition 7.12 states that $(S_1 \cap E_1) \cup (S_2 \cap E_2)$ contains a set $S' \in \mathcal{G}$. Proposition 7.8 implies that $S' = S_1$ or $S' = S_2$. If $S' = S_1$ then $(S_1 \cap E_2) = (S_2 \cap E_2)$. If $S' = S_2$ then $(S_1 \cap E_1) = (S_1 \cap E_2)$. Moreover, by hypothesis neither $S' \cap E_1$ nor $S' \cap E_2$ is empty. Since S, S' were chosen arbitrarily, the result follows. ◇

For any nonempty $\bar{B}_i$, consider any $S \in \bar{B}_i$. We define $E(\bar{B}_i)$ to be equal to $S \cap E_1$ if $\bar{B}_i$ is an E_1-block, and to $S \cap E_2$ if $\bar{B}_i$ is an E_2-block. Let r (resp. s) be the cardinality of the members of $\bar{\mathcal{H}}$ (resp $\bar{\mathcal{G}}$) and $n = |E(\bar{\mathcal{H}})|$. As $\mathcal{H}$ is binary $r \geq 3$ and $s \geq 3$.

Claim 2. Let $U \subseteq E(\bar{\mathcal{G}})$ be a set that intersects $E(\bar{B}_i)$ for each nonempty $\bar{B}_i$. Then U is a transversal of $\bar{\mathcal{G}}$ and $|U| \geq \tau(\mathcal{G}) = r$.

Proof of Claim: Clearly U is a transversal of $\bar{\mathcal{G}}$, thus $|U| \geq \tau(\bar{\mathcal{G}})$. Proposition 7.10 states $\tau(\bar{\mathcal{G}}) = \tau(\mathcal{G})$. ◇

Claim 3. Let U, U' be distinct transversals of $\bar{\mathcal{G}}$. If $\tau(\bar{\mathcal{G}}) = |U| = |U'|$ then $|U - U'| \geq 2$.

Proof of Claim: Proposition 7.10 imply that U and U' are minimum transversals of $\mathcal{G}$. Hence, $U, U' \in \bar{\mathcal{H}}$. The result now follows from Corollary 7.9. ◇

Claim 4. None of the $\bar{B}_i$ is empty.

Proof of Claim: Let U be a minimum cardinality set that intersects $E(\bar{B}_i)$ for each nonempty $\bar{B}_i$. Since $r \geq 3$, it follows from Claim 2 that at most one of the $\bar{B}_i$ can be empty. Assume for a contradiction that one of the $\bar{B}_i$, say $\bar{B}_4$, is empty. It follows from Claim 2 and the choice of U that each of $E(\bar{B}_1)$, $E(\bar{B}_2)$, $E(\bar{B}_3)$ are pairwise disjoint (otherwise U contains an element common to at least 2 of $E(\bar{B}_1)$, $E(\bar{B}_2)$, $E(\bar{B}_3)$ and $|U| \leq 2$). If $|E(\bar{B}_1)| > 1$, then let t_1, t'_1 be distinct elements of $E(\bar{B}_1)$. Let $t_2 \in E(\bar{B}_2)$ and $t_3 \in E(\bar{B}_3)$. Then $U = \{t_1, t_2, t_3\}$ and $U' = \{t'_1, t_2, t_3\}$ contradict Claim 3. Thus $|E(\bar{B}_1)| = 1$ and similarly, $|E(\bar{B}_2)| = |E(\bar{B}_3)| = 1$. As $|E_1| > 3$ and $|E_2| > 3$, $\bar{B}_1, \bar{B}_2, \bar{B}_3$ are not all E_1-blocks and not all E_2-blocks. Thus w.l.o.g. we may assume $\bar{B}_1, \bar{B}_2$ are E_1-blocks and $\bar{B}_3$ is an E_2-block. Let t_1 be any element in $E_1 - E(\bar{B}_1) - E(\bar{B}_2)$ and t_2 be the unique element in $E(\bar{B}_3)$. Then the column of $A(\bar{\mathcal{G}})$ indexed by t_1 is included in the column of $A(\bar{\mathcal{G}})$ indexed by t_2, a contradiction to Proposition 7.6. ◇

Consider first the case where every $\bar{B}_i$ is an E_1-block. Suppose that no two $E(\bar{B}_i)$ intersect. Then $A(\bar{\mathcal{G}})$ has four columns that add up to the vector of all ones. By Theorem 7.4, each of these columns has s ones and therefore $n = 4s$. Furthermore the four elements that index these columns form a transversal of $\bar{\mathcal{G}}$ and therefore $r \leq 4$ (see Claim 2). This contradicts Theorem 7.4 stating that $rs > n$. Thus two $E(\bar{B}_i)$ intersect, say $\bar{B}_1$ and $\bar{B}_2$. For otherwise $n = 4s$, a contradiction to $rs > n$. Let t be any element of $E(\bar{B}_1) \cap E(\bar{B}_2)$ and let g_3 (resp. g_4) be any element of $E(\bar{B}_3)$ (resp. $E(\bar{B}_4)$). Let $U = \{t, g_3, g_4\}$. It follows from Claim 2 that $r = 3$. It follows from Claim 3 that each of $E(\bar{B}_3)$ and $E(\bar{B}_4)$ have cardinality one, and

$E(\bar{B}_1) \cap E(\bar{B}_2)$ contains a unique element e. Since there are no dominated columns in $A(\bar{\mathcal{G}})$ we have that $E(\bar{B}_1) - \{e\} = E(\bar{B}_2) - \{e\} = \emptyset$. Thus $|E_1| \leq 3$, a contradiction to the hypothesis that the 3-separation is strict.

Consider now the case where $\bar{B}_1, \bar{B}_2$ are E_1-blocks and $\bar{B}_3, \bar{B}_4$ are E_2-blocks. Suppose there exists $e \in E(\mathcal{H})$ that is not in any of $E(\bar{B}_i)$ for $i \in \{1, \ldots, 4\}$. Assume without loss of generality that $e \in E_1$. Then column e of $A(\bar{\mathcal{G}})$ is included in the union of any two columns $f_1 \in E(\bar{B}_3)$ and $f_2 \in E(\bar{B}_4)$, a contradiction to Proposition 7.6. Thus every element of $E(\mathcal{H})$ is in $E(\bar{B}_i)$ for some $i \in \{1, \ldots, 4\}$. Suppose there is $e \in E(\bar{B}_1) \cap E(\bar{B}_2)$. Let $U = \{e, f_1, f_2\}$. Then Claim 2 implies $r = 3$ and Claim 3 implies that $|E(\bar{B}_3)| = |E(\bar{B}_4)| = 1$. Hence $|E_2| = 2$, a contradiction. Thus E_1 is partitioned into $E(\bar{B}_1), E(\bar{B}_2)$, and E_2 is partitioned into $E(\bar{B}_3), E(\bar{B}_4)$. If $r = 4$, then we can use Claim 3 to show that for each $i \in \{1, \ldots, 4\}$, $|E(\bar{B}_i)| = 1$. A contradiction as then $|E_1| = |E_2| = 2$. Thus $r = 3$ and let $T = \{u, v, w\}$ be a minimum transversal of $\bar{\mathcal{G}}$. Suppose both $u, v \in E(\bar{B}_i)$ for some $i \in \{1, \ldots, 4\}$, say $i = 3$. If $T \subseteq E_2$, then $w \in E(\bar{B}_4)$, as T is a transversal. It then follows that T intersects all sets of $\bar{B}_3$ with even parity, a contradiction as $\mathcal{H}$ is binary. Thus we may assume $w \in E(\bar{B}_1)$ and w intersects all sets in $\bar{B}_4$. It follows that, for any $x \in E(\bar{B}_2), y \in E(\bar{B}_3)$, the set $\{w, x, y\}$ is a transversal of $\bar{\mathcal{G}}$, a contradiction to Claim 3. Hence for any transversal $T = \{u, v, w\}$ each element of T is in a different $E(\bar{B}_i)$. We may assume $u \in E(\bar{B}_1), v \in E(\bar{B}_3), w \in E(\bar{B}_4)$. It follows that for any $x \in E(\bar{B}_1)$, $\{x, v, w\}$ is a transversal and thus by Claim 3 $E(\bar{B}_1)$ contains a unique element t. Since $|E_1| > 2$, we cannot have a transversal $u \in E(\bar{B}_2), v \in E(\bar{B}_3), w \in E(\bar{B}_4)$, as this would imply $|E(\bar{B}_2)| = 1$. Hence every minimum transversal contains t, a contradiction to Theorem 7.4.

Finally, consider the case where $\bar{B}_1, \bar{B}_2, \bar{B}_3$ are E_1-blocks and $\bar{B}_4$ is an E_2-block. Note that every $t \in E_1$ is in some $E(\bar{B}_i)$ for $i \in \{1, 2, 3\}$. Otherwise the corresponding column t of $A(\bar{\mathcal{G}})$ is dominated by any column $t' \in E(\bar{B}_4)$. Suppose there is $t \in E(\bar{B}_i) - E(\bar{B}_j) - E(\bar{B}_k)$ where i, j, k are distinct elements in $\{1, 2, 3\}$. Proposition 7.7 states there exist three sets of $\bar{\mathcal{G}}$ that intersect exactly in t. This implies $|E(\bar{B}_i)| = 1$. Now since $E(\bar{B}_j) \neq E(\bar{B}_k)$, there is a column in say $E(\bar{B}_j) - E(\bar{B}_i) - E(\bar{B}_k)$. Thus $|E(\bar{B}_j)| = 1$. Similarly, $|E(\bar{B}_k)| = 1$, a contradiction to $|E_1| > 3$. Thus $E(\bar{B}_i) \subseteq E(\bar{B}_j) \cup E(\bar{B}_k)$ for all distinct $i, j, k \in \{1, 2, 3\}$ and therefore either (1) for some distinct $i, j, k \in \{1, 2, 3\}$, $E(\bar{B}_j), E(\bar{B}_k)$ is a partition of $E(\bar{B}_i)$ or (2) $E(\bar{B}_i) \cap E(\bar{B}_j) \neq \emptyset$, for each distinct $i, j \in \{1, 2, 3\}$. By considering sets U containing one element of $E(\bar{B}_4)$ and intersecting each of $E(\bar{B}_1), E(\bar{B}_2)$, and $E(\bar{B}_3)$ we can use Claim 3 to show that $|E_1| \leq 2$ in Case (1) and $|E_1| \leq 3$ in Case (2), a contradiction. □

7.2. **Parts and minors.** In this section, we prove Lemma 7.2. Consider the matroid with exactly three elements $1, 2, 3$ which form a circuit C_0. Let I_0, I_1 be disjoint subsets of C_0. We say that a signed matroid (N, Γ) is a *fat triangle* (I_0, I_1) if $\Gamma = I_1$ and N is obtained from C_0 by adding a parallel element for every $i \in I_0 \cup I_1$. Let (M, Σ) be a signed binary matroid with a circuit $C_0 = \{1, 2, 3\}$ where $C_0 \cap \Sigma = \emptyset$ and let $i \in C_0$. A circuit C_i of M is a *simple circuit* of type i if $C_i \cap C_0 = \{i\}$ and $|C_i \cap \Sigma| = 1$. We say that a cocircuit D has a *small intersection* with a simple circuit C if either: $D \cap C = \emptyset$; or $|D \cap C| = 2$ and the unique element in $C \cap \Sigma$ is in D.

Lemma 7.13. *Let (M, Σ) be a signed binary matroid with a circuit $C_0 = \{1, 2, 3\}$ such that $C_0 \cap \Sigma = \emptyset$.*

(1) *Let $I \subseteq C_0$ be such that for all $i \in I$ there is a simple circuit C_i of type i. Suppose for all distinct $i, j \in C_0$ we have a cocircuit D_{ij} where $D_{ij} \cap C_0 = \{i, j\}$ and D_{ij} has a small intersection with the simple circuits in $\{C_t : t \in I\}$. Then the fat triangle $(\emptyset, I)$ is a minor of (M, Σ).*

(2) *Let C_1 be a simple circuit of type 1. Suppose we have a cocircuit D_{12} where $D_{12} \cap C_0 = \{1, 2\}$ and D_{12} has a small intersection with C_1. If $C_1 - \{1\} \cup \{2\}$ is dependent then $C_1 - \{1\} \cup \{2\}$ contains an odd circuit using 2 and the fat triangle $(\{3\}, \{2\})$ is a minor of (M, Σ).*

(3) *Suppose for each $i = 1, 2$ we have a simple circuit C_i of type i. Suppose we have a cocircuit D_{12} where $D_{12} \cap C_0 = \{1, 2\}$ and D_{12} has a small intersection with C_1 and C_2. If both $C_1 - \{1\} \cup \{2\}$ and $C_2 - \{2\} \cup \{1\}$ are independent, then the fat triangle $(\emptyset, \{1, 2\})$ is a minor of (M, Σ).*

Proof. Throughout the proof i, j, k will denote distinct elements of C_0.

Let us prove (1). For each $i \in I$ let f_i be the unique element in $C_i \cap \Sigma$. For each D_{jk} either: $D_{jk} \cap C_i = \emptyset$ or $D_{jk} \cap C_i = \{f_i, g_i\}$ where g_i is an element not in Σ. Let E_0 be the set of elements in C_0 or in any of C_i where $i \in I$. Observe,

(a) If g_i exists then f_i is in each of D_{12}, D_{13}, D_{23} and g_i is in D_{jk} but not D_{ij}, D_{ik}.

(b) If g_i does not exists but f_i does then f_i is in D_{ij}, D_{ik} but not in D_{jk}.

Define $\Gamma := (\Sigma \triangle D_{12} \triangle D_{13} \triangle D_{23}) \cap E_0$. Observe that (a) and (b) imply respectively (a') and (b').

(a') If g_i exists then $f_i \notin \Gamma$ and $g_i \in \Gamma$.

(b') If g_i does not exist then $f_i \in \Gamma$.

Let (N, Γ) be the minor of (M, Σ) obtained by deleting the elements not in E_0 and then contracting the elements not in $C_0 \cup \Gamma$. It follows from (a') and (b') that if C_0 is a circuit then (N, Γ) is the fat triangle $(\emptyset, I)$. Otherwise some element $i \in C_0$ is a loop of N, say $i = 1$. Then there is a circuit C of M such that $C \subseteq E_0, C \cap C_0 = \{1\}$, and $C \cap \Gamma = \emptyset$. Clearly $C \cap C_0$ does not intersect D_{12} and D_{23} with the same parity. Consider any $e \in C - C_0$ such that e is in some cocircuit D_{ij}. Since $e \notin \Gamma$, it follows from (a') and (b') that $e = f_i$ for some $i \in I$ and that g_i exists. But then (a) implies that $e \in D_{12} \cap D_{13} \cap D_{23}$. It follows that C cannot intersect D_{12} and D_{23} with the same parity, a contradiction.

Let us prove (2). Let f be the unique element in $C_1 \cap \Sigma$. By hypothesis there is a circuit C in $C_1 - \{1\} \cup \{2\}$. Since C_1 is a circuit $2 \in C$. Since D_{12} has a small intersection $C_1 \cap D_{12} = \{1, f\}$. It follows that $C \cap D_{12} = \{2, f\}$. Let C' be the circuit using 3 in $C_1 \triangle C \triangle C_0$. Since C_0 is a circuit, 3 is not a loop, hence $C' - \{3\}$ contains at least one element say g. Let $(N, \Gamma) = (M, \Sigma) \backslash (E(M) - C_0 - C_1)/(C_1 - \{f, g\})$. Observe that $\{2, f\}$ is an odd cycle of (N, Γ) and that $\{3, g\}$ and C_0 are even cycles of (N, Γ). Hence, if C_0 is a circuit of N then (N, Γ) is the fat triangle $(\{3\}, \{2\})$. Because D_{12} is a cocircuit of M, $\{1, 2, f\}$ is a cocycle of N, in particular $1, 2, f$ are not loops. If 3 is a loop of N then there is a circuit $S \subseteq C_1 - \{1, 2, f, g\} \cup \{3\}$ of (M, Σ). But $C' \triangle S$ is a cycle and $C' \triangle S \subset C_1$, a contradiction as C_1 is a circuit.

Let us prove (3). Let M' be obtained from M by deleting all elements not in $C_0 \cup C_1 \cup C_2$ and let $\Sigma' := (\Sigma \triangle D_{12}) \cap E(M')$. Since D_{12} has a small intersection with C_1 and C_2 we have $\Sigma' = \{1, 2\}$. Then $(M', \{1, 2\})$ is a signed minor of (M, Σ). Choose a minor N of M' which is minimal and satisfies the following properties:

(i) C_0 is a circuit of N,

(ii) for $i = 1, 2$ there exist circuits C_i of N such that $C_i \cap C_0 = \{i\}$,

(iii) $C_1 - \{1\} \cup \{2\}$ is independent,
(iv) $C_2 - \{2\} \cup \{1\}$ is independent.

Note that by hypothesis M satisfies properties (i)-(iv) and thus so does M'. Hence N is well defined. We will show that $|C_1| = |C_2| = 2$ in N. Then $(N, \{1,2\})$ is a minor of (M, Σ) and after resigning on the cocircuit containing $1, 2$ we obtain the fat triangle $(\emptyset, \{1,2\})$. There is no circuit $S \subseteq C_1 - \{1\} \cup \{3\}$ of N, for otherwise there exists a cycle $C_1 \triangle S \triangle C_0 \subseteq C_1 - \{1\} \cup \{2\}$, a contradiction with (iii). Hence,

Claim 1. $C_1 - \{1\} \cup \{3\}$ is independent.

Claim 2. $C_1 \cap C_2 = \emptyset$.

Proof of Claim: Otherwise define $N' := N/(C_1 \cap C_2)$. Note that N' satisfies (ii)-(iv). Suppose (i) does not hold for N', i.e. C_0 is a cycle but not a circuit of N'. Then 3 is a loop of N'. Thus there is $S \subseteq C_1 \cap C_2$ such that $S \cup \{3\}$ is a circuit of N, contradicting Claim 1. ◇

Assume for a contradiction $|C_i| > 2$ for some $i \in \{1,2\}$, say $i = 1$.

Claim 3. There exists a circuit $S \subseteq C_1 \cup C_2 - \{1,2\}$ of N.

Proof of Claim: Let $e \in C_1 - \{1\}$ and consider $(N', \Gamma') := (N, \Gamma)/e$. Suppose C_0 is not a circuit of N'. Then 2 or 3 is a loop of N. But then either $\{2, e\}$ or $\{3, e\}$ is a circuit of N. In the former case it contradicts (iii), in the latter it contradicts Claim 1. Hence (i) holds for N'. Trivially (iii) holds for N' as well. Suppose (ii) does not hold, then C_2 is not a circuit of N'. It implies there exists a circuit $S \subseteq C_2 \cup \{e\} - \{2\}$ of N. Then S is the required circuit. Suppose (iv) does not hold. Then there is a circuit $S \subseteq C_2 \cup \{e, 1\} - \{2\}$ of N, and $S \triangle C_1$ contains the required circuit. ◇

Let S be the circuit in the previous claim. Since C_1, C_2 are circuits, $S \cap C_1, S \cap C_2$ are non-empty. Let C_2' be the circuit in $C_2 \triangle S$ which uses 2. Note that $N \setminus (E(N) - C_0 - C_1 - C_2')$ satisfies properties (i)-(iii) using C_2' instead of C_2. Thus, by minimality, (iv) is not satisfied for C_2', i.e. $C_2' - \{2\} \cup \{1\}$ contains a circuit C_1'. Since C_2' is a circuit, $1 \in C_1'$. By the same argument as above, (iii) is not satisfied for C_1', i.e. $C_1' - \{1\} \cup \{2\}$ contains a circuit C_2'' using 2. Since $C_2'' \subseteq C_2'$ it follows that $C_2'' = C_2'$ (since C_2' is a circuit). Therefore, $C_1' = C_2' - \{2\} \cup \{1\}$. But the cycle $C_1' \triangle C_2' = \{1,2\}$ contradicts the fact that C_0 is a circuit. □

Lemma 7.14. *Let (M, Σ) be an ideal signed binary matroid with a circuit $C_0 = \{1,2,3\}$ where $C_0 \cap \Sigma = \emptyset$. Suppose we have $p : E(M) \to Q_+$ such that the cut condition is satisfied. Then there exists $p' : E(M) \to Q_+$ which satisfies the following properties: (i) p' satisfies the cut condition; (ii) $p'(e) = p(e)$ for all $e \notin C_0$; and (iii) $p'(i) + p'(j) \leq p(i) + p(j)$ for all distinct $i, j \in C_0$. Let $I = \{i \in C_0 : p'(i) > 0\}$. There is a Σ-flow $\eta : \Omega_\Sigma \to Q_+$ with costs p'. Moreover, either:*

(1) *The fat triangle $(\emptyset, I)$ is a signed minor of (M, Σ) and*
(2) *$|C \cap C_0| \leq 1$ for all odd circuits C such that $\eta(C) > 0$.*

Or after possibly relabeling elements of C_0 we have $p'(3) = 0$ and $p'(2) \leq p(2) + p(3)$. Moreover,

(3) *The fat triangle $(\{3\}, \{2\})$ is a signed minor of (M, Σ) and*
(4) *for all odd circuits C with $\eta(C) > 0$ and $C \cap C_0 \neq \emptyset$ either $C \cap C_0 = \{2\}$, or $C \cap C_0 = \{1\}$ and $C - \{1\} \cup \{2\}$ contains an odd circuit using 2.*

Proof.

Claim 1. We can assume that there exists $p' : E(M) \to Q_+$ such that properties (i)-(iii) hold. For distinct $i, j \in C_0$ let α_{ij} be the minimum of $p'(D - \Sigma) - p'(D \cap \Sigma)$ where $D \cap C_0 = \{i, j\}$. We then have (after possibly relabeling the elements of C_0) the following cases, either: (a) $\alpha_{12} = \alpha_{13} = \alpha_{23} = 0$; or (b) $p'(3) = 0, p'(2) \leq p(2), p'(1) \leq p(1) + p(3)$ and $\alpha_{12} = 0$.

Proof of Claim: Choose $p' : E(M) \to Q_+$ which minimizes $p'(C_0)$ and which satisfies the following properties: the cut condition holds for p'; $p'(e) = p(e)$ for all $e \notin C_0$; and $p'(i) \leq p(i)$ for all $i \in C_0$. Clearly, (i)-(iii) holds for p'. Suppose (a) does not hold. Then we may assume (after relabeling) that $\alpha_{23} > 0$ and that $p'(3) \leq p'(2)$.

Consider first the case where $\alpha_{12} > 0$. Then 2 is in no tight cocircuit, it follows from the choice of p' that $p'(2) = 0$. Hence $p'(3) = 0$. Suppose $\alpha_{13} > 0$, then $p'(1) = 0$. But then for all circuits C such that $\eta(C) > 0$ we have $C \cap C_0 = \emptyset$ and (2) holds. Moreover, (1) is satisfied since $(M, \Sigma) \backslash (E(M) - C_0)$ is the $(\emptyset, \emptyset)$ fat triangle. Thus we may assume $\alpha_{13} = 0$. But by relabeling 2 and 3 we satisfy (b).

Hence we can assume $\alpha_{12} = 0$. If $\alpha_{13} > 0$ then 3 is in no tight cocircuit, thus $p'(3) = 0$, and (b) holds. Thus we may assume $\alpha_{13} = 0$. Let $\epsilon = \min\{\alpha_{23}/2, p'(3)\}$. Let $\hat{p} : E(M) \to Q_+$ be defined as follows: $\hat{p}(e) = p'(e)$ if $e \notin C_0$ and $\hat{p}(1) = p'(1) + \epsilon, \hat{p}(2) = p'(2) - \epsilon, \hat{p}(3) = p'(3) - \epsilon$. Note, (i)-(iii) hold for $\hat{p}$. Suppose $\epsilon = p'(3)$. Then $\hat{p}(3) = 0$, and (b) holds with $\hat{p}$ since $\hat{p}(2) \leq p'(2) \leq p(2)$ and $\hat{p}(1) = p'(1) + \epsilon \leq p(1) + p(3)$. Thus we may assume $\epsilon = \alpha_{32}/2$. Then for each distinct $i, j \in C_0$ there is a cocircuit D where $D \cap C_0 = \{i, j\}$ which is tight for $\hat{p}$. It follows that (a) holds. $\diamond$

Throughout the proof i, j, k will denote distinct elements of C_0. Let p' be the costs given in Claim 1. Since (M, Σ) is ideal, Corollary 4.4(i) implies that there is a Σ-flow, $\eta : \Omega_\Sigma \to Q_+$ for M with costs p'. Let D_{ij} be the cocircuits of M for which $D_{ij} \cap C_0 = \{i, j\}$ and $p(D_{ij} - \Sigma) - p(D_{ij} \cap \Sigma) = \alpha_{ij}$.

Consider first case (a) of Claim 1, i.e. D_{ij} is tight for all distinct $i, j \in C_0$. We will show that (1) and (2) hold. Let C be any circuit with $\eta(C) > 0$. Then $|C \cap \Sigma| = 1$. Suppose there is an element i in $C_0 \cap C$. Proposition 4.3 states $C \cap D_{ij} = \{i, f\}$ and $C \cap D_{ik} = \{i, f\}$ where f is the unique element in $C \cap \Sigma$. Thus $C \cap C_0 = \{i\}$ and (2) holds. Every element $i \in C_0$ is in a tight cocircuit, thus if $p'(i) > 0$ then there is a circuit C_i with $i \in C_i$ and $\eta(C_i) > 0$. Moreover, (2) implies that C_i is a simple circuit of type i. Proposition 4.3 implies that D_{12}, D_{13}, D_{23} all have small intersections with each of the simple circuits. Then (1) follows from Proposition 7.13(1).

Consider case (b) of Claim 1, i.e. $p'(3) = 0, p'(2) \leq p(2), p'(1) \leq p(1) + p(3)$ and $\alpha_{12} = 0$.

Claim 2. Let C be a circuit with $\eta(C) > 0$. If $i \in C \cap \{1, 2\}$, then C_i is a simple circuit of type i.

This follows from the fact that $3 \notin C$ (as $p'(3) = 0$) and that $|C \cap \{1, 2\}| = 1$ (because of Proposition 4.3 and the fact that D_{12} is tight). The case where $p'(i) = 0$ for all $i \in C_0$ has already been considered (see proof of Claim 1). Suppose for some $i \in \{1, 2\}, p'(3 - i) = 0$. Then $p'(i) > 0$ and let f be the unique element in $C_i \cap \Sigma$. The minor $(M, \Sigma) \backslash (E(M) - C_0 - C_i)/(C_i - \{i, f\})$ is the fat triangle $(\emptyset, \{i\})$ and both (1) and (2) hold. Thus $p'(1) > 0, p'(2) > 0$. Suppose now for all $i \in \{1, 2\}$ there exists a circuit C_i with $\eta(C_i) > 0$ and $i \in C_i$ such that $C_i - \{i\} \cup \{3 - i\}$ is independent. Claim 2 states that these circuits are simple circuits of type i. Then (2) holds and Proposition 7.13(3) implies that (M, Σ) contains the fat triangle $(\emptyset, \{1, 2\})$, i.e.

(1) holds. Thus we may assume, for some $i \in \{1,2\}$ that for all circuits C_i such that $\eta(C_i) > 0$ and $i \in C_i$, $C_i - \{i\} \cup \{3-i\}$ is dependent. If $i = 2$ interchange the labels 2 and 1. Since we had $p'(1) \leq p(1) + p(3)$ we get in that case $p'(2) \leq p(2) + p(3)$. Otherwise (if $i = 1$) we have $p'(2) \leq p(2) \leq p(2) + p(3)$. Proposition 7.13(2) implies that for all circuits C_1 with $\eta(C_1) > 0$ and $1 \in C_1$, $C_1 - \{1\} \cup \{2\}$ contains an odd circuit using 1 and that (M, Σ) contains the fat triangle $(\{3\}, \{2\})$ as a minor. Together with Claim 2 this implies (3) and (4) hold. □

We are now ready for the proof of the main lemma.

Proof of Lemma 7.3. Since M has a strict 3-separation, $M = M_1 \otimes_3 M_2$ where $C_0 = E(M_1) \cap E(M_2)$ is a triangle. Throughout this proof i, j, k will denote distinct elements of C_0. Recall that $F \subseteq E_1$. Let α^1_{ij} denote the smallest value of

$$p(D_{ij} - F - C_0) - p(D_{ij} \cap F) \qquad (*)$$

where D_{ij} is some cocircuit of M_1 with $D_{ij} \cap C_0 = \{i,j\}$. Expression (*) gives the difference between the sum of the capacity elements and the sum of the demand elements in D_{ij}, excluding the marker C_0. Denote by D^1_{ij} the cocircuit for which the minimum is attained in (*). Let α^2_{ij} denote the smallest value of $p(D_{ij} - C_0)$, where D_{ij} is some cocircuit of M_2 with $D_{ij} \cap C_0 = \{i,j\}$. In what follows, we let D^2_{ij} denote the cocircuit for which $p(D^2_{ij} - C_0) = \alpha^2_{ij}$. For each $i \in C_0$ define:

$$\beta_i = \frac{1}{2}(\alpha^2_{ij} + \alpha^2_{ik} - \alpha^2_{jk}).$$

Claim 1. $\beta_i \geq 0$ for all $i \in C_0$.

Proof of Claim: We have $\alpha^2_{ij} + \alpha^2_{ik} = p(D^2_{ij} - C_0) + p(D^2_{ik} - C_0) \geq p\big((D^2_{ij} \triangle D^2_{ik}) - C_0\big) \geq \alpha^2_{jk}$. Thus $(\alpha^2_{ij} + \alpha^2_{ik}) - \alpha^2_{jk} \geq 0$ and $\beta_i \geq 0$. ◇

Claim 2. $\alpha^1_{ij} + \alpha^2_{ij} \geq 0$.

Proof of Claim: $\alpha^1_{ij} + \alpha^2_{ij} = p(D^1_{ij} - F - C_0) - p(D^1_{ij} \cap F) + p(D^2_{ij} - C_0) = p((D^1_{ij} \triangle D^2_{ij}) - F) - p((D^1_{ij} \triangle D^2_{ij}) \cap F)$. But the last expression is non negative since the cut condition holds for (M, F, p). ◇

Claim 3. (M_1, F) is a signed minor of (M, F).

Proof of Claim: Theorem 5.2 implies that M_1 is a minor of M obtained by contracting and deleting elements in E_2 only. Since $F \subseteq E_1$ the result follows. ◇

Define $p_1 : E(M_1) \to \mathbb{Q}_+$ as follows: $p_1(e) = p(e)$ for all $e \in E_1$ and $p_1(i) = \beta_i$ for all $i \in C_0$.

Claim 4. The cut condition is satisfied for (M_1, F, p_1).

Proof of Claim: Since the cut condition holds for (M, F, p) the cut condition is satisfied for all cocircuits of M_1 disjoint from C_0. Let D be a cocircuit of M_1 such that $D \cap C_0 = \{i,j\}$. Then $p_1(D - F) - p_1(D \cap F) = p(D - F - C_0) - p(D \cap F) + p_1(i) + p_1(j) \geq \alpha^1_{ij} + p_1(i) + p_1(j) = \alpha^1_{ij} + \beta_i + \beta_j = \alpha^1_{ij} + \alpha^2_{ij}$. It follows from Claim 2 that the previous expression is non-negative. ◇

Claim 3 implies that (M_1, F) is a part of (M, F) and hence its clutter of odd circuits is ideal. Together with Claim 4 it implies that (M_1, F) and p_1 satisfy the hypothesis of Lemma 7.14. It follows that M_1 is F-flowing with costs p'_1 (where p'_1 is as described in the lemma) and either case 1 or case 2 occurs.

Case 1: Statements (1) and (2) hold.

We define $I := \{i \in C_0 : p'_1(i) > 0\}$ and let M'_2 denote $M_2 \setminus (C_0 - I)$.

Claim 5. (M'_2, I) is a signed minor of (M, F).

Proof of Claim: Statement (1) says that the fat triangle $(\emptyset, I)$ is a signed minor of (M_1, F), i.e. it is equal to $(M_1, F) \setminus J_d/J_c$ for some $J_d, J_c \subseteq E_1$. Seymour [24] showed that $(M_1 \otimes_3 M_2) \setminus J_d/J_c = (M_1 \setminus J_d/J_c) \otimes_3 M_2$. Thus $(M'_2, I) = (M, F) \setminus J_d/J_c$. ◇

Define $p_2 : E(M'_2) \to \mathbb{Q}_+$ as follows: $p_2(e) = p(e)$ for all $e \in E_2$ and $p_2(i) = p'_1(i)$ for all $i \in I$.

Claim 6. The cut condition is satisfied for (M'_2, I, p_2).

Proof of Claim: It suffices to show the cut condition holds for cocircuits D that intersect C_0. Suppose $D \cap C_0 = \{i, j\}$. Lemma 7.14 states that $p'_1(i)+p'_1(j) \leq p_1(i)+p_1(j)$. Moreover, $p_1(i)+p_1(j) = \beta_i+\beta_j = \alpha^2_{ij}$. Thus $p_2(D - I) - p_2(D \cap I) = p(D - C_0) - (p'_1(i) + p'_1(j)) \geq \alpha^2_{ij} - \alpha^2_{ij} = 0$. ◇

Claim 5 implies that (M'_2, I) is a part of (M, F) and hence its clutter of odd circuits is ideal. It follows from Claim 6 and Corollary 4.4(i) that M'_2 is I-flowing with costs p_2. Since we can scale p (and hence p'_1 and p_2) we may assume that the F-flow of M_1 satisfying costs p'_1 is a multiset $\mathcal{L}^1$ of circuits and that the I-flow of M'_2 satisfying costs p_2 is a multiset $\mathcal{L}^2$ of circuits. Because of Statement (2), $\mathcal{L}^1$ can be partitioned into $\mathcal{L}^1_0$ and $\mathcal{L}^1_i$ for all $i \in I$ where: $\mathcal{L}^1_0 = \{C \in \mathcal{L}^1 : C \cap C_0 = \emptyset\}$ and $\mathcal{L}^1_i = \big\{C \in \mathcal{L}^1 : C \cap C_0 = \{i\}\big\}$. Because $C \in \mathcal{L}^2$ implies $C \in \Omega_I$, $\mathcal{L}^2$ can be partitioned into $\mathcal{L}^2_i$ for all $i \in I$ where: $\mathcal{L}^2_i = \big\{C \in \mathcal{L}^2 : C \cap C_0 = \{i\}\big\}$. Since $p_2(i) = p'_1(i)$ for each $i \in I$, $|\mathcal{L}^1_i| \leq |\mathcal{L}^2_i|$ for each $i \in I$. Let us define a collection of circuits of M as follows: include all circuits of $\mathcal{L}^1_0$, and for every $i \in I$ pair each circuit $C_1 \in \mathcal{L}^1_i$ with a different circuit $C_2 \in \mathcal{L}^2_i$ and add to the collection the circuit included in $C_1 \triangle C_2$ that contain the element of F. The resulting collection corresponds to a F-flow of M satisfying costs p.

Case 2: $p'_1(3) = 0, p'_1(2) \leq p(2) + p(3)$ and statements (3) and (4) hold (after possibly relabeling C_0).

Let M'_2 denote $M_2 \setminus 1$. Statement (3) says that the fat triangle $(\{3\}, \{2\})$ is a signed minor of (M_1, F). Proceeding as in the proof of Claim 5 we obtain the following.

Claim 7. $(M'_2, \{2\})$ is a signed minor of (M, F).

Define $p_2 : E(M'_2) \to \mathbb{Q}_+$ as follows: $p_2(e) = p(e)$ for all $e \in E(M_2)$; $p_2(2) = p'_1(2) + p'_1(1)$ and $p_2(3) = p'_1(1)$.

Claim 8. The cut condition is satisfied for $(M'_2, \{2\}, p_2)$.

Proof of Claim: Consider first a cocircuit D of M'_2 such that $D \cap C_0 = \{2, 3\}$. Let us check D does not violate the cut condition. The following expression should be non negative: $p_2(D - \{2\}) - p_2(D \cap \{2\}) = p(D - C_0) + p_2(3) - p_2(2) = p(D - C_0) + p'_1(1) - p'_1(1) - p'_1(2) = p(D - C_0) - p'_1(2)$. Lemma 7.14 states $p'_1(2) \leq p_1(2) + p_1(3) = \beta_2 + \beta_3$. Since $p(D - C_0) \geq \alpha^2_{23} = \beta_2 + \beta_3$ it follows that $p(D - C_0) - p'_2(1) \geq 0$. Consider a cocircuit D of M'_2 such that $2 \in D$ but $3 \notin D$. Let us check D does not violate the cut condition. The following expression should be non negative: $p_2(D - \{2\}) - p_2(D \cap \{2\}) = p(D - C_0) - p_2(2) = p(D - C_0) - (p'_1(1) + p'_1(2))$. Lemma 7.14 states $p'_1(1) + p'_1(2) \leq p_1(1) + p_1(2) = \beta_1 + \beta_2$. Since $D \cup \{1\}$ is a cocircuit of M_2, $p_2(D - C_0) \geq \alpha^2_{12} = \beta_1 + \beta_2$. It follows that $p(D - C_0) - (p'_1(1) + p'_1(2)) \geq 0$. ◇

Claim 7, Claim 8, and Corollary 4.4(i) imply that M_2' is $\{2\}$-flowing with costs p_2. We may assume that the F-flow of M_1 satisfying costs p_1' is a multiset $\mathcal{L}^1$ of circuits. Because of Statement (4), $\mathcal{L}^1$ can be partitioned into $\mathcal{L}_0^1, \mathcal{L}_1^1, \mathcal{L}_2^1$ where: $\mathcal{L}_0^1 = \{C \in \mathcal{L}^1 : C \cap C_0 = \emptyset\}$, $\mathcal{L}_1^1 = \{C \in \mathcal{L}^1 : C \cap C_0 = \{1\}\}$, $\mathcal{L}_2^1 = \{C \in \mathcal{L}^1 : C \cap C_0 = \{2\}\}$. We may assume that the $\{2\}$-flow of M_2' satisfying costs p_2 is a multiset $\mathcal{L}^2$ of circuits. Since $C \in \mathcal{L}^2$ implies $C \in \Omega_{\{2\}}$ and since $1 \notin E(M_2')$, $\mathcal{L}^2$ can be partitioned into $\mathcal{L}_1^2, \mathcal{L}_2^2$ where: $\mathcal{L}_1^2 = \{C \in \mathcal{L}^2 : C \cap C_0 = \{2, 3\}\}$, and $\mathcal{L}_2^2 = \{C \in \mathcal{L}^2 : C \cap C_0 = \{2\}\}$.

Claim 9. (i) $|\mathcal{L}_2^1| \leq |\mathcal{L}_2^2|$ and (ii) $|\mathcal{L}_1^1| + |\mathcal{L}_2^1| \leq |\mathcal{L}_1^2| + |\mathcal{L}_2^2|$.

Proof of Claim: Let us prove (i). 2 is a demand element for the flow $\mathcal{L}^2$, thus $|\mathcal{L}_2^2| + |\mathcal{L}_1^2| = p_2(2) = p_1'(1) + p_1'(2)$. 3 is a capacity element for the flow $\mathcal{L}^2$, thus $|\mathcal{L}_1^2| \leq p_2(3) = p_1'(1)$. Hence, $|\mathcal{L}_2^2| \geq p_1'(2) \geq |\mathcal{L}_2^1|$ where the last inequality follows from the fact that 2 is a capacity element for the flow $\mathcal{L}^1$. Let us prove (ii). $|\mathcal{L}_2^1| + |\mathcal{L}_1^1| \leq p_1'(2) + p_1'(1) = p_2(2) = |\mathcal{L}_2^2| + |\mathcal{L}_1^2|$. ◇

Let us define a collection of circuits of M as follows: (a) include all circuits of $\mathcal{L}_0^1$; (b) pair every circuit $C_1 \in \mathcal{L}_2^1$ with a different circuit $C_2 \in \mathcal{L}_2^2$ - such a pairing exists because of Claim 9(i) - and add to the collection $C_1 \triangle C_2$; (c) pair as many circuits C_1 of $\mathcal{L}_1^1$ to as many different circuits C_2 of $\mathcal{L}_1^2$ as possible, and add to the collection $C_1 \triangle C_2$; (d) pair all remaining circuits C_1 of $\mathcal{L}_1^1$ to circuits of $\mathcal{L}_2^2$ not already used in (b). Such a pairing exists because of Claim 9(ii). Statement (4) says that $C_1 - \{1\} \cup \{2\}$ contains an odd circuit C_1'. For every pair C_1, C_2 add to the collection the cycle $C_1' \triangle C_2$; (e) for each cycle C in the collection only keep the circuit included in C that contains the element of F. The resulting collection corresponds to an F-flow of M satisfying costs p. □

8. Sufficient conditions for idealness

We will prove Theorem 1.1 in this section, i.e. that a binary clutter is ideal if it has none of the following minors: $\mathcal{L}_{F_7}$, $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, Q_6^+ and Q_7^+. The next result is fairly straightforward.

Proposition 8.1 (Novick and Sebö [20])**.**

- *$\mathcal{H}$ is a clutter of odd circuits of a graph if and only if $u(\mathcal{H})$ is graphic.*
- *$\mathcal{H}$ is a clutter of T-cuts if and only if $u(\mathcal{H})$ is cographic.*

Remark 8.2. The class of clutters of odd circuits and the class of clutters of T-cuts is closed under taking minors.

This follows from the previous proposition, Remark 2.9 and the fact that the classes of graphic and cographic matroid are closed under taking (matroid) minors. We know from Remark 3.4 that $b(Q_6)^+$ (a minor of Q_7^+) is a source of F_7, and Q_6^+ is a source of F_7^*. Thus Proposition 8.1 implies,

Remark 8.3. Q_7^+ and Q_6^+ are not clutters of odd circuits or clutters of T-cuts.

We use the following two decomposition theorems.

Theorem 8.4 (Seymour [24])**.** *Let M be a 3-connected and internally 4-connected regular matroid. Then $M = R_{10}$ or M is graphic or M is cographic.*

Theorem 8.5 (Seymour [24, 26])**.** *Let M be a 3-connected binary matroid with no F_7^* (resp. F_7) minor. Then M is regular or $M = F_7$ (resp. F_7^*).*

Corollary 8.6. *Let $\mathcal{H}$ be a binary clutter such that $u(\mathcal{H})$ has no F_7^* minor. If $\mathcal{H}$ is 3-connected and internally 4-connected, then $\mathcal{H}$ is one of $b(Q_7), \mathcal{L}_{F_7}, b(Q_6)^+$, or one of the 6 lifts of R_{10}, or a clutter of odd circuits or a clutter of T-cuts.*

Proof. Since $\mathcal{H}$ is 3-connected, $u(\mathcal{H})$ is 3-connected. So, by Theorem 8.5, $u(\mathcal{H})$ is regular or $u(\mathcal{H}) = F_7$. In the latter case, Remark 3.4 implies that $\mathcal{H}$ is one of $b(Q_7), \mathcal{L}_{F_7}, b(Q_6)^+$. Thus we can assume that $u(\mathcal{H})$ is regular. By hypothesis, $u(\mathcal{H})$ is internally 4-connected and therefore, by Theorem 8.4, $u(\mathcal{H}) = R_{10}$ or $u(\mathcal{H})$ is graphic or $u(\mathcal{H})$ is cographic. Now the corollary follows from Proposition 8.1 and Remark 3.4. □

We are now ready for the proof of the main result of this paper.

Proof of Theorem 1.1. We need to prove that, if $\mathcal{H}$ is nonideal, then it contains $\mathcal{L}_{F_7}$, $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, Q_6^+ or Q_7^+ as a minor. Without loss of generality we may assume that $\mathcal{H}$ is minimally nonideal. It follows from Remark 5.3 and propositions 6.1 and 7.1 that $\mathcal{H}$ is 3-connected and internally 4-connected. Consider first the case where $u(\mathcal{H})$ has no F_7^* minor. Then, by Corollary 8.6 either: (i) $\mathcal{H}$ is one of $b(Q_7), \mathcal{L}_{F_7}, b(Q_6)^+$, or (ii) $\mathcal{H}$ is one of the 6 lifts of R_{10}, or (iii) $\mathcal{H}$ is a clutter of odd circuits, or (iv) $\mathcal{H}$ is a clutter of T-cuts. Since $\mathcal{H}$ is minimally nonideal, it follows from Proposition 3.5 that if (i) occurs then $\mathcal{H} = \mathcal{L}_{F_7}$ and if (ii) occurs then $\mathcal{H} = b(\mathcal{O}_{K_5})$. If (iii) occurs then, by Theorem 1.3, $\mathcal{H} = \mathcal{O}_{K_5}$; (iv) cannot occur because of Theorem 1.4.

Now consider the case where $u(\mathcal{H})$ has an F_7^* minor. It follows by Theorem 3.2 that $\mathcal{H}$ has a minor $\mathcal{H}_1$ or $\mathcal{H}_2^+$, where $\mathcal{H}_1$ is a source of F_7^* and $\mathcal{H}_2$ is a lift of F_7^*. Proposition 3.1 states that the lifts of F_7^* are the blockers of the sources of F_7. Remark 3.4 states that the sources of F_7 are $b(Q_7)$, $\mathcal{L}_{F_7}$ or $b(Q_6)^+$, and that F_7^* has only one source, namely Q_6^+. This implies that $\mathcal{H}_1 = Q_6^+$ and $\mathcal{H}_2^+ = Q_7^+$ or $b(\mathcal{L}_{F_7})^+$ or $b\big(b(Q_6^+)\big)^+$. Since $b(\mathcal{L}_{F_7})^+$ has an $\mathcal{L}_{F_7}$ minor and $b\big(b(Q_6^+)\big)^+$ has a Q_6^+ minor, the proof of the theorem is complete. □

One can obtain a variation of Theorem 1.1 by modifying Corollary 8.6 as follows: Let $\mathcal{H}$ be a binary clutter such that $u(\mathcal{H})$ has no F_7 minor. If $\mathcal{H}$ is 3-connected and internally 4-connected, then $\mathcal{H}$ is $b(Q_7)$, $\mathcal{L}_{F_7}$, $b(Q_6^+)$ or one of the 6 lifts of R_{10} or a clutter of odd circuits or a clutter of T-cuts. Following the proof of Theorem 1.1, this yields: A binary clutter is ideal if it does not have an $\mathcal{L}_{F_7}$, $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, $b(Q_7)$ or $b(Q_6^+)$ minor. But this result is weaker than Corollary 1.2. Other variations of Theorem 1.1 can be obtained by using Seymour's Splitter Theorem [24] which implies, since $u(\mathcal{H})$ is 3-connected and $u(\mathcal{H}) \neq F_7^*$, that $u(\mathcal{H})$ has either S_8 or $AG(3,2)$ as a minor. Again, by using Proposition 3.2, we can obtain a list of excluded minors that are sufficient to guarantee that $\mathcal{H}$ is ideal.

9. Some Additional Comments

Corollary 8.6 implies the following result, using the argument used in the proof of Theorem 1.1.

Theorem 9.1. *Let $\mathcal{H}$ be an ideal binary clutter such that $u(\mathcal{H})$ has no F_7^* minor. If $\mathcal{H}$ is 3-connected and internally 4-connected, then $\mathcal{H}$ is one of $b(Q_7), b(Q_6)^+$, or one of the 5 ideal lifts of R_{10}, or a clutter of odd circuits of a weakly bipartite graph, or a clutter of T-cuts.*

A possible strategy for resolving Seymour's Conjecture would be to generalize this theorem by removing the assumption that $u(\mathcal{H})$ has no F_7^* minor, while allowing in the conclusion the possibility for $\mathcal{H}$ to also be a clutter of T-joins or the blocker of a clutter of odd circuits in a weakly bipartite graph. However, this is not possible as illustrated by the following example.

Let T_{12} be the binary matroid with the following partial representation.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matroid first appeared in [11]. It is self dual and satisfies the following properties:

(i) For every element t of T_{12}, T_{12}/t is 3-connected and internally 4-connected.
(ii) For every element t of T_{12}, T_{12}/t is not regular.

We are indebted to James Oxley (personal communication) for bringing to our attention the existence of the matroid T_{12} and pointing out that it satisfies properties (i) and (ii). Let t be any element of T_{12} and let $\mathcal{H} = Port(T_{12}, t)$. Because of (i), $T_{12}/t = u(\mathcal{H})$ is 3-connected and internally 4-connected and thus so is $\mathcal{H}$. Because of (ii), $T_{12}/t = u(\mathcal{H})$ is not graphic or cographic thus Proposition 8.1 implies that $\mathcal{H}$ is not a clutter of T-cuts and not a clutter of odd circuits. We know from Proposition 2.12 that $b(\mathcal{H}) = Port(T_{12}^*, t) = Port(T_{12}, t)$. Thus, $b(\mathcal{H})$ is also 3-connected, internally 4-connected, and $\mathcal{H}$ is not the clutter of T-joins or the blocker of the clutter of odd circuits. However, it follows from the results of Luetolf and Margot [16] that the clutter $\mathcal{H}$ is ideal.

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