Sufficiency of Cut-Generating Functions*

Gérard Cornuéjols[†1], Laurence Wolsey[‡2], and Sercan Yıldız[§1]

[1]Tepper School of Business, Carnegie Mellon University, 5000 Forbes Ave, Pittsburgh, PA 15213, United States
[2]CORE, Université Catholique de Louvain, Voie du Roman Pays 34, B-1348 Louvain-la-Neuve, Belgium

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Abstract

The concept of cut-generating function has its origin in the work of Gomory and Johnson from the 1970s. It has received renewed attention in the past few years. Recently Conforti, Cornuéjols, Daniilidis, Lemaréchal, and Malick proposed a general framework for studying cut-generating functions. However, they gave an example showing that not all cuts can be produced by cut-generating functions in this framework. They conjectured a natural condition under which cut-generating functions might be sufficient. This note settles this open problem.

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1 Introduction

We consider sets of the form

$$X = X(R,S) := \{x \in \mathbb{R}^n_+ : \, Rx \in S\}, \tag{1a}$$

$$\text{where } \begin{cases} R = [r_1, \dots, r_n] \text{ is a real } q \times n \text{ matrix,} \\ S \subset \mathbb{R}^q \text{ is a nonempty closed set with } 0 \notin S. \end{cases} \tag{1b}$$

This model has been studied in [Joh81] and [CCD+13]. It arises in integer programming when studying Gomory's corner relaxation [Gom69, GJ72] or the relaxation proposed by Andersen, Louveaux, Weismantel, and Wolsey [ALWW07]. It also arises in other optimization problems such as complementarity problems [JSRF06]. In framework (1) the goal is to generate inequalities that are valid for X but not for the origin. Such cutting planes are well-defined [CCD+13, Lemma 2.1] and can be written as

$$c^\top x \geqslant 1. \tag{2}$$

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†gc0v@andrew.cmu.edu
‡laurence.wolsey@uclouvain.be
§syildiz@andrew.cmu.edu

Let $S \subset \mathbb{R}^q$ be a given nonempty, closed set with $0 \notin S$. The set S is assumed to be fixed in this paragraph. [CCD+13] introduces the notion of a *cut-generating function*: This is any function $\rho : \mathbb{R}^q \mapsto \mathbb{R}$ that produces coefficients $c_j := \rho(r_j)$ of a cut (2) valid for $X(R,S)$ for any choice of n and $R = [r_1, \dots, r_n]$. It is shown in [CCD+13] that cut-generating functions enjoy significant structure, generalizing earlier work in integer programming [DW10, BCCZ10]. For instance, the minimal ones are sublinear and are closely related to S-free neighborhoods of the origin. We say that a closed, convex set is *S-free* if it contains no point of S in its interior. For any minimal cut-generating function ρ, there exists a closed, convex, S-free set $V \subset \mathbb{R}^q$ such that $0 \in \operatorname{int} V$ and $V = \{r \in \mathbb{R}^q : \rho(r) \leqslant 1\}$. A cut (2) with coefficients $c_j := \rho(r_j)$ is called an *S-intersection cut*.

Now assume that both S and R are fixed. Noting $X \subset \mathbb{R}^n_+$, we say that a cutting plane $c^\top x \geqslant 1$ *dominates* $b^\top x \geqslant 1$ if $c_j \leqslant b_j$ for $j \in [n]$. (In this note we use the notation $[n] := \{1, \dots, n\}$.) A natural question is whether every cut (2) valid for X is dominated by an S-intersection cut. [CCZ10] proves that this is true for Gomory's corner relaxation. However, [CCD+13] gives an example showing that this is not always the case in the more general framework (1). This example has the peculiarity that S contains points that cannot be obtained as Rx for any $x \in \mathbb{R}^n_+$. [CCD+13] proposes the following open problem: Assuming $S \subset \operatorname{cone} R$, is it true that every cut (2) valid for $X(R,S)$ is dominated by an S-intersection cut? Our main theorem shows that this is indeed the case. This generalizes the main result of [CCZ10] as well as Theorem 1 in [Zam09] and Theorem 6.3 in [CCD+13], all of which consider the case where $c \in \mathbb{R}^n_+$.

Theorem 1.1. *Suppose $S \subset \operatorname{cone} R$. Then any valid inequality $c^\top x \geqslant 1$ separating the origin from X is dominated by an S-intersection cut.*

2 Proof of the Main Theorem

Our proof of Theorem 1.1 will use several lemmas. We first introduce some notation and terminology. Given a set $W \subset \mathbb{R}^d$, let $\operatorname{conv} W$, $\operatorname{cone} W$, and $\operatorname{span} W$ denote the convex, conical, and linear hull of W, and let $\operatorname{lin} W$ and $\operatorname{rec} W$ denote the lineality space and recession cone of W, respectively. Given a set $W \subset \mathbb{R}^d$, let $W^\circ := \{u \in \mathbb{R}^d : u^\top w \leqslant 0, \forall w \in W\}$ and $W^* := -W^\circ$ denote the *polar* and *dual cone* of W, respectively. Let $\sigma_W(u) := \sup_{w \in W} u^\top w$ be the *support function* of a set $W \subset \mathbb{R}^d$. A function $\rho : \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}$ is said to be *positively homogeneous* if $\rho(\lambda u) = \lambda \rho(u)$ for all $\lambda > 0$ and $u \in \mathbb{R}^d$ and *subadditive* if $\rho(u_1) + \rho(u_2) \geqslant \rho(u_1 + u_2)$ for all $u_1, u_2 \in \mathbb{R}^d$. Moreover, ρ is *sublinear* if it is both positively homogeneous and subadditive. Sublinear functions are known to be convex, and it is not difficult to show that support functions are sublinear and satisfy $\sigma_W = \sigma_{\operatorname{conv} W}$ (see, e.g., [HUL04, Chapter C]). Given a closed, convex neighborhood V of the origin, a *representation of* V is any sublinear function $\rho : \mathbb{R}^q \mapsto \mathbb{R}$ such that $V = \{r \in \mathbb{R}^q : \rho(r) \leqslant 1\}$. Minkowski's gauge function is a representation of V, but there can be other representations when V is unbounded. S-intersection cuts are generated via representations of closed, convex, S-free neighborhoods of the origin.

Throughout this section we assume that $X \neq \emptyset$ and $c^\top x \geqslant 1$ is a valid inequality separating the origin from X.

Lemma 2.1. *If $u \in \mathbb{R}^n_+$ and $Ru = 0$, then $c^\top u \geqslant 0$. Equivalently, $c \in \mathbb{R}^n_+ + \operatorname{Im} R^\top$.*

Proof. Let $\overline{x} \in X$. Note that $R(\overline{x} + tu) = R\overline{x} \in S$ and $\overline{x} + tu \geqslant 0$ for all $t \geqslant 0$. By the validity of c, we have $c^\top(\overline{x} + tu) \geqslant 1$ for all $t \geqslant 0$. Observing $tc^\top u \geqslant 1 - c^\top \overline{x}$ and letting $t \to +\infty$ implies

$c^\top u \geqslant 0$ as desired. Because u is an arbitrary vector in $\mathbb{R}^n_+ \cap \operatorname{Ker} R$, we can write $c \in (\mathbb{R}^n_+ \cap \operatorname{Ker} R)^*$. The equality $(\mathbb{R}^n_+ \cap \operatorname{Ker} R)^* = \mathbb{R}^n_+ + \operatorname{Im} R^\top$ follows from the facts $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$, $(\operatorname{Ker} R)^* = \operatorname{Im} R^\top$, and $\mathbb{R}^n_+ + \operatorname{Im} R^\top$ is closed (see, e.g., [Roc70, Cor. 16.4.2]). ■

Given the valid inequality $c^\top x \geqslant 1$, we now construct a sublinear function $h_c : \mathbb{R}^q \mapsto \mathbb{R} \cup \{+\infty\}$ that produces a valid inequality $\sum_{j=1}^n h_c(r_j) x_j \geqslant 1$ dominating $c^\top x \geqslant 1$, i.e., the coefficients of the inequality satisfy $h_c(r_j) \leqslant c_j$, $j \in [n]$. Let

$$h_c(r) := \min \quad c^\top x \quad Rx = r, \quad x \geqslant 0. \tag{3}$$

Remark 2.2.

1. $h_c(r_j) \leqslant c_j$ *for all* $j \in [n]$.
2. $h_c(\overline{r}) \geqslant 1$ *for all* $\overline{r} \in S$.

Proof. The first claim follows directly from the observation that the j^{th} unit vector is feasible to the linear program (3) associated with $r = r_j$. To prove the second claim, let $\overline{r} \in S$. If the linear program (3) associated with $r = \overline{r}$ is infeasible, $h_c(\overline{r}) = +\infty \geqslant 1$. Otherwise, any feasible solution $\overline{x}$ to this linear program satisfies $\overline{x} \in X$ and $c^\top \overline{x} \geqslant 1$ by the validity of $c^\top x \geqslant 1$. Hence, $h_c(\overline{r}) \geqslant 1$. ■

Lemma 2.3. *h_c is a piecewise-linear, sublinear function which is finite on* $\operatorname{cone} R$.

Proof. The linear program (3) is feasible if and only if $r \in \operatorname{cone} R$. Hence, $h_c(r) < +\infty$ for $r \in \operatorname{cone} R$ and $h_c(r) = +\infty$ for $r \in \mathbb{R}^q \setminus \operatorname{cone} R$. The dual of (3) is

$$\max \quad r^\top y \quad R^\top y \leqslant c. \tag{4}$$

Let $P := \{y \in \mathbb{R}^q : R^\top y \leqslant c\}$. By Lemma 2.1, $c = c' + c''$ where $c' \in \mathbb{R}^n_+$ and $c'' \in \operatorname{Im} R^\top$. Because $c'' \in \operatorname{Im} R^\top$, there exists $y'' \in \mathbb{R}^q$ such that $R^\top y'' = c'' \leqslant c$. Hence, $y'' \in P$ which shows that the dual linear program is always feasible, strong duality holds, and $h_c = \sigma_P > -\infty$. This shows that h_c is a sublinear function and finite on $\operatorname{cone} R$.

Now let $\overline{r} \in \operatorname{cone} R$. Let W be a finite set of points for which $P = \operatorname{conv} W + \operatorname{rec} P$. Observe that $\operatorname{rec} P = (\operatorname{cone} R)^\circ$ and $\overline{r}^\top u \leqslant 0$ for all $u \in \operatorname{rec} P$. Thus, $\overline{r}^\top (w + u) \leqslant \overline{r}^\top w$ for all $w \in \operatorname{conv} W$ and $u \in \operatorname{rec} P$, which implies

$$\sigma_P(\overline{r}) := \sup_{p \in P} \overline{r}^\top p \leqslant \sigma_{\operatorname{conv} W}(\overline{r}) := \sup_{w \in \operatorname{conv} W} \overline{r}^\top w = \sigma_W(\overline{r}).$$

Since $W \subset P$ implies $\sigma_W \leqslant \sigma_P$, we have $\sigma_P(\overline{r}) = \sigma_W(\overline{r})$. Therefore, $h_c(\overline{r}) = \sigma_P(\overline{r}) = \sigma_W(\overline{r}) = \max_{w \in W} \overline{r}^\top w$ where the last equality follows from the finiteness of W. This and the fact that $\operatorname{cone} R$ is polyhedral imply that h_c is piecewise-linear. ■

Lemma 2.3 implies in particular that $h_c(0) = 0$.

Proposition 2.4. *Theorem 1.1 holds when* $\operatorname{cone} R = \mathbb{R}^q$.

Proof. In this case h_c is finite everywhere. Let $V_c := \{r \in \mathbb{R}^q : h_c(r) \leqslant 1\}$. V_c is a closed, convex neighborhood of the origin because h_c is sublinear and finite everywhere, and $h_c(0) = 0$. Because the Slater condition is satisfied with $h_c(0) = 0$, we have $\operatorname{int} V_c = \{r \in \mathbb{R}^q : h_c(r) < 1\}$ (see, e.g., [HUL04, Prop. D.1.3.3]). Then V_c is also S-free since $h_c(\overline{r}) \geqslant 1$ for all $\overline{r} \in S$ by Remark 2.2 (ii). The function h_c is a cut-generating function because it represents the closed, convex, S-free neighborhood of the origin V_c by definition, and $\sum_{j=1}^n h_c(r_j)x_j \geqslant 1$ is an S-intersection cut that can be obtained from V_c. By Remark 2.2 (i), $h_c(r_j) \leqslant c_j$ for all $j \in [n]$. This shows that the S-intersection cut $\sum_{j=1}^n h_c(r_j)x_j \geqslant 1$ dominates $c^\top x \geqslant 1$. ■

We now consider the case where $\operatorname{cone} R \subsetneq \mathbb{R}^q$. We want to extend the definition of h_c to the whole of $\mathbb{R}^q$ and show that this extension is a cut-generating function. We will first construct a function h'_c such that 1) h'_c is finite everywhere on $\operatorname{span} R$, 2) h'_c coincides with h_c on $\operatorname{cone} R$. If $\operatorname{rank}(R) < q$, we will further extend h'_c to the whole of $\mathbb{R}^q$ by letting $h'_c(r) = h'_c(r')$ for all $r \in \mathbb{R}^q$, $r' \in \operatorname{span} R$, $r'' \in (\operatorname{span} R)^\perp$ such that $r = r' + r''$. Our proof of Theorem 1.1 will show that this procedure yields a function h'_c that is the desired extension of h_c.

Let $r_0 \in -\operatorname{ri}(\operatorname{cone} R)$ where $\operatorname{ri}(\cdot)$ denotes the relative interior. Note that this guarantees $\operatorname{cone}(R \cup \{r_0\}) = \operatorname{span} R$ since there exist $\epsilon > 0$ and $d := \operatorname{rank}(R)$ linearly independent vectors $a_1, \ldots, a_d \in \operatorname{span} R$ such that $-r_0 \pm \epsilon a_i \in \operatorname{cone} R$ for all $i \in [d]$ which implies $\pm a_i \in \operatorname{cone}(R \cup \{r_0\})$. Now we define c_0 as

$$c_0 := \sup_{\substack{r \in \operatorname{cone} R \\ \alpha > 0}} \frac{h_c(r) - h_c(r + \alpha(-r_0))}{\alpha}. \tag{5}$$

Lemma 2.5. *c_0 is finite.*

Proof. Any pair $\overline{r} \in \operatorname{cone} R$ and $\overline{\alpha} > 0$ yields a lower bound on c_0: Our choice of r_0 ensures $\overline{r} + \overline{\alpha}(-r_0) \in \operatorname{cone} R$ and $c_0 \geqslant \frac{h_c(\overline{r}) - h_c(\overline{r} + \overline{\alpha}(-r_0))}{\overline{\alpha}}$. To get an upper bound on c_0, consider the linear programs (3) and (4). Let $\tilde{r} \in \operatorname{cone} R$ and $\tilde{\alpha} \geqslant 0$. Observe that $\tilde{r} + \tilde{\alpha}(-r_0) \in \operatorname{cone} R$ and as in the proof of Lemma 2.3, one can show that both linear programs are feasible when we plug in $\tilde{r} + \tilde{\alpha}(-r_0)$ for r. Therefore, strong duality holds and $h_c(\tilde{r} + \tilde{\alpha}(-r_0)) = \sigma_P(\tilde{r} + \tilde{\alpha}(-r_0))$ where $P := \{y \in \mathbb{R}^q : R^\top y \leqslant c\}$ is the feasible region of (4). Let W be a finite set of points for which $P = \operatorname{conv} W + \operatorname{rec} P$. Because $\operatorname{rec} P = (\operatorname{cone} R)^\circ$, we have $(\tilde{r} + \tilde{\alpha}(-r_0))^\top u \leqslant 0$ for all $u \in \operatorname{rec} P$. This implies $\sigma_P(\tilde{r} + \tilde{\alpha}(-r_0)) = \sigma_W(\tilde{r} + \tilde{\alpha}(-r_0))$, and we can write

$$\begin{aligned} c_0 &= \sup_{\substack{r \in \operatorname{cone} R \\ \alpha > 0}} \frac{\sigma_W(r) - \sigma_W(r + \alpha(-r_0))}{\alpha} \\ &\leqslant \sup_{\substack{r \in \operatorname{cone} R \\ \alpha > 0}} \frac{\sigma_W(\alpha r_0)}{\alpha} \\ &= \sigma_W(r_0) \end{aligned}$$

where we have used the sublinearity of σ_W in the inequality and the second equality. The conclusion follows now from the fact that W is a finite set. ■

Remark 2.6. *If we scale r_0 by a positive scalar λ, c_0 is scaled by λ as well.*

Proof. This follows from $\frac{h_c(r)-h_c(r+\alpha(-\lambda r_0))}{\alpha} = \lambda\frac{h_c(r/\lambda)-h_c(r/\lambda+\alpha(-r_0))}{\alpha}$ (positive homogeneity of h_c) and the fact that $r \in \operatorname{cone} R$ if and only if $r/\lambda \in \operatorname{cone} R$. ■

Proposition 2.7. $c_0x_0 + c^\top x \geqslant 1$ *is a valid inequality for* $X([r_0, R], S)$.

Proof. Let $(\overline{x}_0, \overline{x}) \in X([r_0, R], S)$ and $\overline{r} := r_0\overline{x}_0 + R\overline{x} \in S$. Then

$$c_0\overline{x}_0 + c^\top\overline{x} \geqslant c_0\overline{x}_0 + \sum_{j=1}^{n} h_c(r_j)\overline{x}_j \geqslant c_0\overline{x}_0 + h_c(R\overline{x}) = c_0\overline{x}_0 + h_c(\overline{r} + \overline{x}_0(-r_0))$$

where the first inequality follows from Remark 2.2 (i) and the second from the sublinearity of h_c. Using the definition of c_0 and applying Remark 2.2 (ii), we conclude $c_0\overline{x}_0 + c^\top\overline{x} \geqslant c_0\overline{x}_0 + h_c(\overline{r} + \overline{x}_0(-r_0)) \geqslant h_c(\overline{r}) \geqslant 1$. ■

We define the function h'_c on $\operatorname{span} R$ by

$$h'_c(r) := \min \quad \begin{array}{l} c_0x_0 + c^\top x \\ r_0x_0 + Rx = r, \\ x_0 \geqslant 0,\, x \geqslant 0. \end{array} \tag{6}$$

The function h'_c is real-valued, piecewise-linear, and sublinear on $\operatorname{span} R$ as a consequence of Lemma 2.3 applied to the matrix $[r_0, R]$ and the inequality $c_0x_0 + c^\top x \geqslant 1$ which is valid for $X([r_0, R], S)$ by Proposition 2.7.

Lemma 2.8. *The function* h'_c *coincides with* h_c *on* $\operatorname{cone} R$.

Proof. It is clear from the definitions (3) and (6) that $h'_c \leqslant h_c$ on $\operatorname{span} R$. Let $\overline{r} \in \operatorname{cone} R$ and suppose $h'_c(\overline{r}) < h_c(\overline{r})$. Then there exists $(\overline{x}_0, \overline{x})$ satisfying $r_0\overline{x}_0 + R\overline{x} = \overline{r}$, $\overline{x} \geqslant 0$, $\overline{x}_0 > 0$ and $c_0\overline{x}_0 + c^\top\overline{x} < h_c(\overline{r})$. Rearranging the terms and using Remark 2.2 (i), we obtain

$$c_0 < \frac{h_c(\overline{r}) - c^\top\overline{x}}{\overline{x}_0} \leqslant \frac{h_c(\overline{r}) - \sum_{j=1}^{n} h_c(r_j)\overline{x}_j}{\overline{x}_0}.$$

Finally, the sublinearity of h_c and the observation that $R\overline{x} = \overline{r} - r_0\overline{x}_0$ give

$$c_0 < \frac{h_c(\overline{r}) - \sum_{j=1}^{n} h_c(r_j)\overline{x}_j}{\overline{x}_0} \leqslant \frac{h_c(\overline{r}) - h_c(R\overline{x})}{\overline{x}_0} = \frac{h_c(\overline{r}) - h_c(\overline{r} - r_0\overline{x}_0)}{\overline{x}_0}.$$

This contradicts the definition of c_0 and proves the claim. ■

Lemma 2.8 and Remark 2.2 yield the following corollary.

Corollary 2.9.

1. $h'_c(r_j) \leqslant c_j$ *for all* $j \in [n]$.
2. *Suppose* $S \subset \operatorname{cone} R$. *Then* $h'_c(\overline{r}) \geqslant 1$ *for all* $\overline{r} \in S$.

If $\operatorname{rank}(R) < q$, we extend the function h'_c defined in (6) to the whole of $\mathbb{R}^q$ by letting

$$h'_c(r) = h'_c(r') \text{ for all } r \in \mathbb{R}^q,\ r' \in \operatorname{span} R,\ r'' \in (\operatorname{span} R)^\perp \text{ such that } r = r' + r''. \tag{7}$$

Note that this extension preserves the sublinearity of h'_c.

Proof of Theorem 1.1. Let h'_c be defined as in (6) and (7) and let $V'_c := \{r \in \mathbb{R}^q : h'_c(r) \leqslant 1\}$. Observe that V'_c is a closed, convex neighborhood of the origin because h'_c is sublinear and finite everywhere, and $h'_c(0) = 0$. Furthermore, $\operatorname{int} V'_c = \{r \in \mathbb{R}^q : h'_c(r) < 1\}$ by the Slater property $h'_c(0) = 0$. This implies that V'_c is also S-free since $h'_c(\bar{r}) \geqslant 1$ for all $\bar{r} \in S$ by Corollary 2.9 (ii). The function h'_c is a cut-generating function because it represents V'_c, and $\sum_{j=1}^n h'_c(r_j)x_j \geqslant 1$ is an S-intersection cut. By Corollary 2.9 (i), $h'_c(r_j) \leqslant c_j$ for all $j \in [n]$. This shows that the S-intersection cut $\sum_{j=1}^n h'_c(r_j)x_j \geqslant 1$ dominates $c^\top x \geqslant 1$. ■

3 Constructing the S-Free Convex Neighborhood of the Origin

Here we give a geometric interpretation for the proof of Theorem 1.1 and explicitly describe the S-free neighborhood of the origin $V'_c := \{r \in \mathbb{R}^q : h'_c(r) \leqslant 1\}$ in terms of the vectors $r_1, \ldots, r_n$.

As in Section 1, we let $c^\top x \geqslant 1$ be a valid inequality separating the origin from X. Assume without any loss of generality that the vectors $r_1, \ldots, r_n$ have been normalized so that $c_j \in \{0, \pm 1\}$ for all $j \in [n]$. Define the sets $J_+ := \{j \in [n] : c_j = +1\}$, $J_- := \{j \in [n] : c_j = -1\}$ and $J_0 := \{j \in [n] : c_j = 0\}$. Let $C := \operatorname{conv}(\{0\} \cup \{r_j : j \in J_+\})$ and $K := \operatorname{cone}(\{r_j : j \in J_0 \cup J_-\} \cup \{r_j + r_i : j \in J_+, i \in J_-\})$. Let $Q := C + K$. See Figure 1(a) for an illustration. Defining h_c as in (3), one can show $Q = \{r \in \mathbb{R}^q : h_c(r) \leqslant 1\}$.

When $\operatorname{cone} R \neq \mathbb{R}^q$, the origin lies on the boundary of Q. This happens in the example of Figure 1. In the proof of Theorem 1.1, we overcame the difficulty occurring when $\operatorname{cone} R \neq \mathbb{R}^q$ by extending h_c into a function h'_c which is defined on the whole of $\mathbb{R}^q$ and coincides with h_c on $\operatorname{cone} R$. The geometric counterpart is to extend the set Q into a set Q' that contains the origin in its interior. Let $r_0 \in -\operatorname{ri}(\operatorname{cone} R)$ and let c_0 be as defined in (5). When $c_0 \neq 0$, scale r_0 so that $c_0 \in \{\pm 1\}$ (this is possible by Remark 2.6). Introduce r_0 into the relevant subset of $[n]$ according to the sign of c_0: If $c_0 = +1$, let $J'_+ := J_+ \cup \{0\}$, $J'_0 := J_0$ and $J'_- := J_-$; if $c_0 = 0$, let $J'_+ := J_+$, $J'_0 := J_0 \cup \{0\}$ and $J'_- := J_-$; and if $c_0 = -1$, let $J'_+ := J_+$, $J'_0 := J_0$ and $J'_- := J_- \cup \{0\}$. Finally, let $C' := \operatorname{conv}(\{0\} \cup \{r_j : j \in J'_+\})$, $K' := \operatorname{cone}(\{r_j : j \in J'_0 \cup J'_-\} \cup \{r_j + r_i : j \in J'_+, i \in J'_-\})$ and $Q' := C' + K' + (\operatorname{span} R)^\perp$. Figures 1(b) and 1(c) illustrate examples of this procedure with $c_0 = +1$ and $c_0 = -1$, respectively.

The following proposition shows that the function h'_c defined in (6) and (7) represents the set Q' defined above.

Proposition 3.1. *$Q' = \{r \in \mathbb{R}^q : h'_c(r) \leqslant 1\}$ where h'_c is defined as in* (6) *and* (7)*.*

Proof. Let $V'_c := \{r \in \mathbb{R}^q : h'_c(r) \leqslant 1\}$. Note that V'_c is convex by the sublinearity of h'_c. We have $h'_c(r_j) \leqslant c_j = 1$ for all $j \in J'_+$, $h'_c(r_j) \leqslant c_j \leqslant 0$ for all $j \in J'_0 \cup J'_-$ and $h'_c(r_j + r_i) \leqslant h'_c(r_j) + h'_c(r_i) \leqslant c_j + c_i = 0$ for all $j \in J'_+$ and $i \in J'_-$. Moreover, $h'_c(r) = h'_c(r + r')$ for all $r \in \mathbb{R}^q$ and $r' \in (\operatorname{span} R)^\perp$ by the definition of h'_c. Hence, $C' \subset V'_c$, $K' \subset \operatorname{rec} V'_c$, and $(\operatorname{span} R)^\perp \subset \operatorname{lin} V'_c$ which together give us $Q' = C' + K' + (\operatorname{span} R)^\perp \subset V'_c$.

To prove the converse, let $\bar{r} \in \mathbb{R}^q$ be such that $h'_c(\bar{r}) \leqslant 1$. We need to show $\bar{r} \in Q'$. We consider two distinct cases: $h'_c(\bar{r}) \leqslant 0$ and $0 < h'_c(\bar{r}) \leqslant 1$. First, let us suppose $h'_c(\bar{r}) \leqslant 0$. Then the definition of h'_c implies that there exist $(\bar{x}_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n$ and $\bar{r}' \in (\operatorname{span} R)^\perp$ such that $(\bar{x}_0, \bar{x}) \geqslant 0$, $\sum_{j \in J'_+} \bar{x}_j - \sum_{i \in J'_-} \bar{x}_i \leqslant 0$, and $r_0\bar{x}_0 + R\bar{x} = \bar{r} - \bar{r}'$. Consider the cone $\Gamma := \{(\bar{x}_0, \bar{x}) \geqslant 0 : \sum_{j \in J'_+} \bar{x}_j - \sum_{i \in J'_-} \bar{x}_i \leqslant 0\}$ defined by the first two sets of inequalities. The extreme rays of Γ have all their components equal to 0 except for one or two components. Therefore, it is easy to verify by inspection that Γ is generated by the rays $\{e_j : j \in J'_0 \cup J'_-\} \cup \{e_j + e_i : j \in J'_+, i \in J'_-\}$. This shows $\overline{r} \in K' + (\operatorname{span} R)^\perp \subset Q'$. Now suppose $0 < h'_c(\overline{r}) \leqslant 1$. Then there exist $(\overline{x}_0, \overline{x}) \in \mathbb{R} \times \mathbb{R}^n$ and $\overline{r}' \in (\operatorname{span} R)^\perp$ such that $(\overline{x}_0, \overline{x}) \geqslant 0$, $0 < \sum_{j \in J'_+} \overline{x}_j - \sum_{i \in J'_-} \overline{x}_i \leqslant 1$, and $r_0 \overline{x}_0 + R\overline{x} = \overline{r} - \overline{r}'$. Define $\overline{x}_i^j := \overline{x}_i \frac{\overline{x}_j}{\sum_{j \in J'_+} \overline{x}_j}$ for all $i \in J'_-$ and $j \in J'_+$. These values are well-defined since $0 \leqslant \sum_{i \in J'_-} \overline{x}_i < \sum_{j \in J'_+} \overline{x}_j$. Observe that $\sum_{j \in J'_+} \overline{x}_i^j = \overline{x}_i$ and $r_0 \overline{x}_0 + R\overline{x} = \sum_{j \in J'_+} (\overline{x}_j - \sum_{i \in J'_-} \overline{x}_i^j) r_j + \sum_{i \in J'_-} \sum_{j \in J'_+} \overline{x}_i^j (r_i + r_j) + \sum_{j \in J'_0} \overline{x}_j r_j$. We have $\sum_{j \in J'_+} (\overline{x}_j - \sum_{i \in J'_-} \overline{x}_i^j) = \sum_{j \in J'_+} \overline{x}_j - \sum_{i \in J'_-} \overline{x}_i \leqslant 1$ together with $\overline{x}_j - \sum_{i \in J'_-} \overline{x}_i^j > 0$ which is true for all $j \in J'_+$ because $\sum_{i \in J'_-} \overline{x}_i^j = \overline{x}_j \frac{\sum_{i \in J'_-} \overline{x}_i}{\sum_{j \in J'_+} \overline{x}_j} < \overline{x}_j$. Hence, $\sum_{j \in J'_+} (\overline{x}_j - \sum_{i \in J'_-} \overline{x}_i^j) r_j \in C'$. Moreover, $\sum_{i \in J'_-} \sum_{j \in J'_+} \overline{x}_i^j (r_i + r_j) + \sum_{j \in J'_0} \overline{x}_j r_j \in K'$. These yield $\overline{r} \in C' + K' + (\operatorname{span} R)^\perp = Q'$. ∎

As a consequence, the set Q' can be used to generate an S-intersection cut that dominates $c^\top x \geqslant 1$. Indeed, the proof of Theorem 1.1 shows that $V'_c := \{r \in \mathbb{R}^q : h'_c(r) \leqslant 1\}$ is a closed, convex, S-free neighborhood of the origin. Proposition 3.1 shows that $Q' = V'_c$. Therefore, $\sum_{j=1}^n h'_c(r_j) x_j \geqslant 1$ is an S-intersection cut obtained from Q'.

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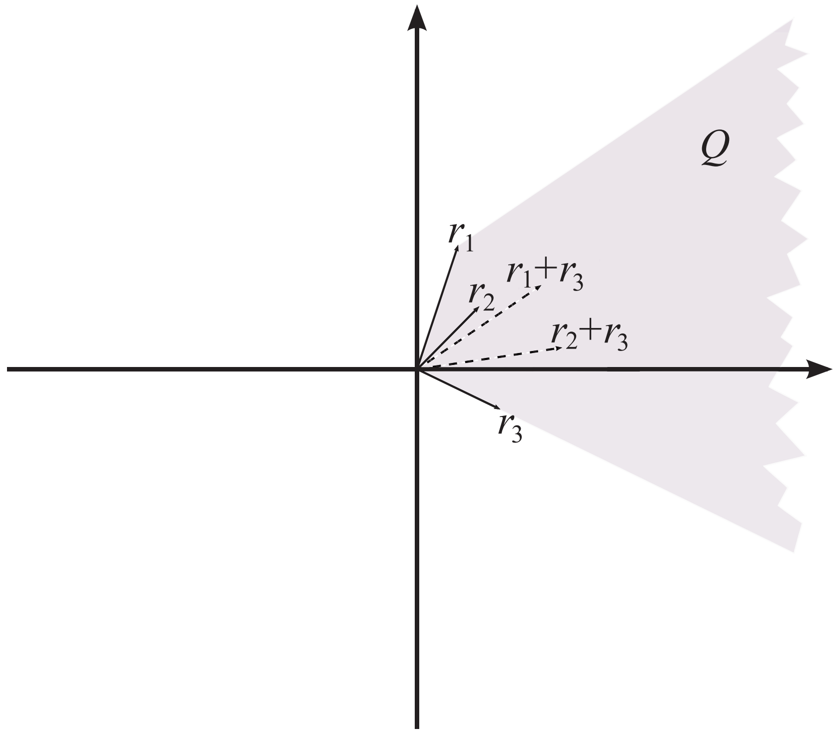
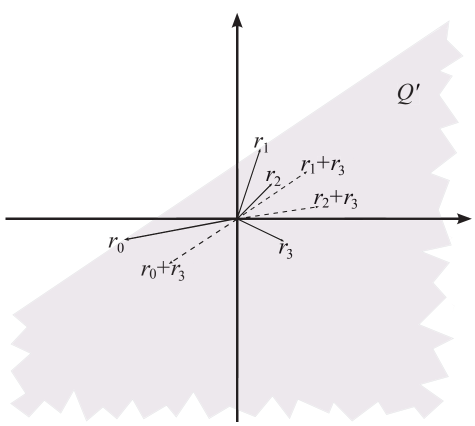
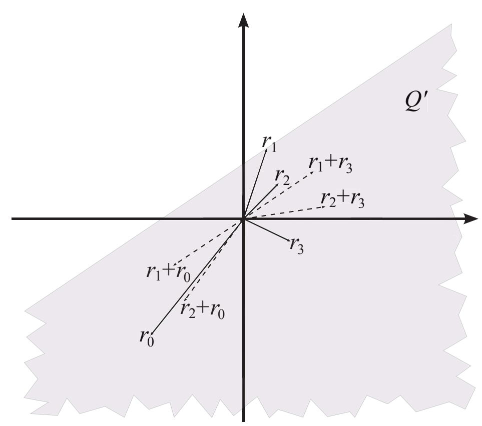


(a)



(b)



(c)

Figure 1: The vectors $r_1 = (1, 3)$, $r_2 = (1.5, 1.5)$, and $r_3 = (2, -1)$ have cut coefficients $c_1 = c_2 = +1$ and $c_3 = -1$. The shaded region in (a) is the set Q. In (b) we add the vector $r_0 = (-5, -1)$ to the collection of vectors $\{r_1, r_2, r_3\}$. The new vector r_0 has $c_0 = +1$. Its addition expands Q to the set Q' that is depicted. In (c) we add the vector $r_0 = (-4, -5)$ with $c_0 = -1$ to the original collection and again obtain Q'.