CUBOIDS, A CLASS OF CLUTTERS
EXTENDED ABSTRACT

AHMAD ABDI, GÉRARD CORNUÉJOLS, NATÁLIA GURIČANOVÁ, AND DABEEN LEE

ABSTRACT. Take an integer $n \geq 1$ and a subset S of the vertices $\{0,1\}^n$ of the hypercube $[0,1]^n$. The *cuboid of S* is the clutter over ground set $\{1,\ldots,2n\}$ whose members have incidence vectors $\{(x_1, 1-x_1, \ldots, x_n, 1-x_n) : x \in S\}$. Cuboids form an important class of clutters as several key results and conjectures about clutters, such as the $\tau = 2$ Conjecture, the Replication Conjecture, the f-Flowing Conjecture, and the classification of the binary matroids with the sums of circuits property, are equivalently formulated in terms of cuboids.

Cuboids are used as means to study the geometry of two minor-closed properties on clutters, namely idealness and the packing property. We reveal a surprising rift in the geometry of idealness versus the geometry of the packing property, the culprit being "strict polarity". We say that S is *polar* if either all of its points agree on a coordinate, or the set contains antipodal points. If every restriction of S is polar, then S is *strictly polar*. We show that strict polarity can be recognized in time polynomial in n and $|S|$.

As shown earlier by Abdi, Cornuéjols and Pashkovich [Mathematics of Operations Research, 2017], cuboids play a central role among all ideal minimally non-packing clutters. We study the spectrum of ideal minimally non-packing cuboids of bounded degree, and expose the local and global structure of the ones sitting at the end of the spectrum, which is done with the help of Mantel's Theorem and the local structure of delta free clutters. We use a computer code to generate over seven hundred new ideal minimally non-packing cuboids over at most 14 elements.

We also study three basic binary operations on cuboids, namely the *Cartesian product*, the *coproduct* and the *reflective product*, and their interplay with idealness and the packing property. This interplay reveals the starring role of the sets $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$, whose cuboids are ideal minimally non-packing, and it also brings out the importance of *strict connectivity* and *antipodal symmetry* when studying such clutters.

EXTENDED ABSTRACT

Let E be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of E, called *members*. We say that $\mathcal{C}$ is a *clutter* over *ground set* E if no member is contained in another one [8]. Two clutters are *isomorphic* if one is obtained from the other after relabeling its ground set. A *cover of* $\mathcal{C}$ is a subset of E that intersects every member, and a cover is *minimal* if it does not properly contain another cover. Balas and Padberg [2] define the *set covering polyhedron of* $\mathcal{C}$ as

$$Q(\mathcal{C}) := \left\{x \in \mathbb{R}_+^E : x(C) \geq 1 \ \forall\, C \in \mathcal{C}\right\}$$

and the *set covering polytope of* $\mathcal{C}$ as

$$P(\mathcal{C}) := \left\{x \in [0,1]^E : x(C) \geq 1 \ \forall\, C \in \mathcal{C}\right\}.$$

Here, $x(C)$ is shorthand notation for $\sum_{e\in C} x_e$.

Date: November 7, 2017.

Proposition 1 (folklore)**.** *Let $\mathcal{C}$ be a clutter. Then the integral extreme points of $Q(\mathcal{C})$ are precisely the incidence vectors of the minimal covers of $\mathcal{C}$ and the integral extreme points of $P(\mathcal{C})$ are precisely the incidence vectors of the covers of $\mathcal{C}$. Moreover, $Q(\mathcal{C})$ is an integral polyhedron if, and only if, $P(\mathcal{C})$ is an integral polytope.*

We say that $\mathcal{C}$ is *ideal* if the corresponding set covering polyhedron (or polytope) is integral [7]. Consider the primal-dual pair of linear programs

$$(P)\quad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & x(C) \geq 1 \quad C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum (y_C : e \in C \in \mathcal{C}) \leq w_e \quad e \in E \\ & y \geq \mathbf{0}. \end{array}$$

It is well-known that $\mathcal{C}$ is an ideal clutter if, and only if, the primal linear program (P) has an integral optimal solution for all $w \in \mathbb{Z}_+^E$ (see [5], Theorem 4.1). We say that (P) is *totally dual integral* if for all $w \in \mathbb{Z}_+^E$, the dual linear program (D) has an integral optimal solution. It is also well-known that if (P) is totally dual integral, then $\mathcal{C}$ is an ideal clutter ([13, 9], see also [5], Theorem 4.26). The converse however does not hold, as we will explain shortly.

Define the *covering number* $\tau(\mathcal{C})$ as the minimum cardinality of a cover, and the *packing number* $\nu(\mathcal{C})$ as the maximum number of pairwise disjoint members. As every member of a packing picks a distinct element of a cover, it follows that $\tau(\mathcal{C}) \geq \nu(\mathcal{C})$. If equality holds here, then $\mathcal{C}$ *packs*, otherwise it is *non-packing*. Observe that $\tau(\mathcal{C})$ and $\nu(\mathcal{C})$ are the integral optimal values of (P) and (D), respectively, for $w = \mathbf{1}$. Thus, if (P) is totally dual integral, then $\mathcal{C}$ must pack.

Consider the clutter over ground set $\{1, \ldots, 6\}$ whose members are

$$Q_6 := \big\{\{2,4,6\}, \{1,3,6\}, \{1,4,5\}, \{2,3,5\}\big\}.$$

Notice that Q_6 is isomorphic to the clutter of triangles (or claws) of the complete graph on four vertices. This clutter does not pack as $\tau(Q_6) = 2 > 1 = \nu(Q_6)$. This clutter was found by Lovász [16], but Seymour [23] was the person who established the significant role of Q_6 among non-packing clutters in his seminal paper on the matroids with the max-flow min-cut property. Even though Q_6 does not pack, it is an ideal clutter [23]. In fact,

Proposition 2. *Q_6 is the only ideal non-packing clutter over at most 6 elements, up to isomorphism.*

Given disjoint sets $I, J \subseteq E$, the *minor* of $\mathcal{C}$ obtained after *deleting* I and *contracting* J is the clutter

$$\mathcal{C} \setminus I/J := \text{ the minimal sets of } \{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}.$$

We say that the minor is *proper* if $I \cup J \neq \emptyset$. In terms of the set covering polyhedron, contractions correspond to restricting the corresponding coordinates to 0, while deletions correspond to projecting away the corresponding coordinates; in terms of the set covering polytope, deletions can also be thought of as restricting the corresponding coordinates to 1, which is sometimes convenient. Due to these geometric interpretations, if a clutter is ideal then so is every minor of it [23]. A clutter is *minimally non-ideal* if it is not ideal but every proper minor is ideal. In the same vein, a clutter is *minimally non-packing* if it does not pack but every proper minor packs.

A minimally non-packing clutter is either ideal or minimally non-ideal – this is a fascinating consequence of Lehman's seminal theorem on minimally non-ideal clutters [15] and was first noticed in [6].

Proposition 2 implies that Q_6 is in fact an ideal minimally non-packing clutter. Despite what Seymour [23] conjectured, Q_6 is not the only ideal minimally non-packing clutter. Schrijver [19] found an ideal minimally non-packing clutter over 9 elements, which was a minor of the clutter of dijoins of a directed graph, as a counterexample to a conjecture of Edmonds and Giles [9]. Two decades later, Cornuéjols, Guenin and Margot grew the known list to a dozen sporadic instances as well as an infinite class $\{Q_{r,t} : r \geq 1, t \geq 1\}$ of ideal minimally non-packing clutters [6]. All their examples of ideal minimally non-packing clutters, however, have covering number two, so they conjecture the following:

The $\tau = 2$ Conjecture ([6]). *Every ideal minimally non-packing clutter has covering number two.*

We will prove this conjecture for clutters over at most 8 elements. For the most part, however, we take a different perspective towards the $\tau = 2$ Conjecture. Take an integer $n \geq 1$. We will be working over $\{0,1\}^n$, the vertices of the unit n-dimensional hypercube, represented for convenience as $0,1$ strings of length n. Take a set $S \subseteq \{0,1\}^n$. The *cuboid of* S, denoted $\mathrm{cuboid}(S)$, is the clutter over ground set $[2n]$ whose members have incidence vectors

$$(x_1, 1-x_1, \ldots, x_n, 1-x_n) \quad x \in S.^1$$

Observe that every member of $\mathrm{cuboid}(S)$ has cardinality n, and that for each $i \in [n]$, $\{2i-1, 2i\}$ is a cover. For example, the cuboid of $R_{1,1} := \{000, 110, 101, 011\} \subseteq \{0,1\}^3$ is $\big\{\{2,4,6\},\{1,3,6\},\{1,4,5\},\{2,3,5\}\big\}$ which is Q_6. Thus, the smallest ideal minimally non-packing clutter is a cuboid. Abdi, Cornuéjols and Pashkovich showed that cuboids play a central role among all ideal minimally non-packing clutters [1]. They found two new ideal minimally non-packing cuboids, and observed that each clutter of $\{Q_{r,t}, r \geq 1, t \geq 1\}$ – the only known infinite class of ideal minimally non-packing clutters – is a cuboid. This was also observed by Flores, Gitler and Reyes, who referred to cuboids as 2-*partitionable* clutters [12]. However, to emphasize the fact that these clutters come from subsets of a hypercube, we refrain from this terminology. The following theorem further stresses the importance of cuboids among ideal minimally non-packing clutters:

Theorem 3. *Every minimally non-packing cuboid is ideal.*

In this paper, we will see that the $\tau = 2$ Conjecture is equivalent to a conjecture on cuboids, and on top of that, we will show how Paul Seymour's classification of the binary matroids with the sums of circuits property [21], his characterization of the binary matroids with the max-flow min-cut property [23], as well as his f-Flowing Conjecture [23, 21] translate into the world of cuboids. We will also reduce the Replication Conjecture of Conforti and Cornuéjols [4] to cuboids. After reading this paper, we hope to have convinced the reader that cuboids are an important class of clutters.

[1] For an integer $m \geq 1$, $[m] := \{1, \ldots, m\}$.

0.1. **q-Idealness.** Let $n \geq 1$ be an integer and $S \subseteq \{0,1\}^n$ an arbitrary set of vertices of the unit n-dimensional hypercube. Take a coordinate $i \in [n]$. To *twist coordinate* i is to replace S by

$$S \triangle e_i := \{x \triangle e_i : x \in S\};$$

this terminology is due to Bouchet [3]. (The symmetric difference operator $\triangle$ performs coordinatewise addition modulo 2. Novick and Sebő [17] refer to twisting as *switching*.) Observe that the cuboid of S encodes all of its twistings. If S' is obtained from S after twisting and relabeling some coordinates, then we say that S' is *isomorphic* to S and write it as $S' \cong S$. Notice that if S', S are isomorphic, then so are their cuboids.

The set obtained from $S \cap \{x : x_i = 0\}$ after dropping coordinate i is called the 0-*restriction of* S *over coordinate* i, and the set obtained from $S \cap \{x : x_i = 1\}$ after dropping coordinate i is called the 1-*restriction of* S *over coordinate* i. If S' is obtained from S after 0- and 1-restricting some coordinates, then we say that S' is a *restriction* of S. The set obtained from S after dropping coordinate i is called the *projection of* S *over coordinate* i. If S' is obtained from S after projecting away some coordinates, then we say that S' is a *projection* of S. If S' is obtained from S after a series of restrictions and projections, then we say that S' is a *minor* of S; we say that S' is a *proper* minor if at least one minor operation is applied. These minor operations can be defined directly on cuboids:

Remark 4 ([1])**.** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then, for each $i \in [n]$, the following statements hold:*

- *If S' is the 0-restriction of S over i, then* $\mathrm{cuboid}(S') = \mathrm{cuboid}(S) \setminus (2i-1)/2i$.
- *If S' is the 1-restriction of S over i, then* $\mathrm{cuboid}(S') = \mathrm{cuboid}(S)/(2i-1) \setminus 2i$.
- *If S' is the projection of S over i, then* $\mathrm{cuboid}(S') = \mathrm{cuboid}(S)/\{2i-1, 2i\}$.

If S' is a minor of S, we will say that $\mathrm{cuboid}(S')$ is a *cuboid minor* of $\mathrm{cuboid}(S)$.

Inequalities of the form $1 \geq x_i \geq 0, i \in [n]$ are called *hypercube inequalities*, and the ones of the form

$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \qquad I, J \subseteq [n], I \cap J = \emptyset$$

are called *generalized set covering inequalities*. Observe that these two classes of inequalities are closed under twistings, i.e. the change of variables $x_i \mapsto 1 - x_i, i \in [n]$.

We say that S is *q-ideal* if its convex hull $\mathrm{conv}(S)$ can be described using hypercube and generalized set covering inequalities. For instance, the set $R_{1,1} = \{000, 110, 101, 011\}$ is q-ideal as its convex hull is

$$\mathrm{conv}(R_{1,1}) = \left\{ x \in [0,1]^3 : \begin{array}{ll} (1-x_1) + x_2 + x_3 & \geq 1 \\ x_1 + (1-x_2) + x_3 & \geq 1 \\ x_1 + x_2 + (1-x_3) & \geq 1 \\ (1-x_1) + (1-x_2) + (1-x_3) & \geq 1 \end{array} \right\},$$

as illustrated in Figure 1.

Remark 5. *Take an integer $n \geq 1$ and a q-ideal set $S \subseteq \{0,1\}^n$. If S' is isomorphic to a minor of S, then S' is q-ideal.*

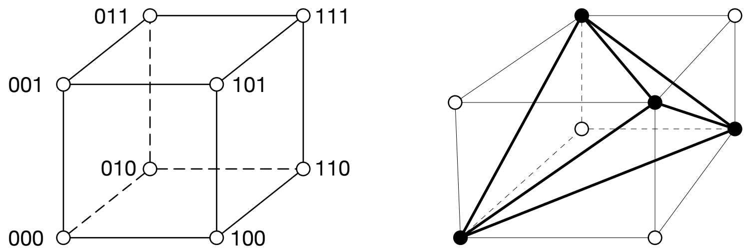

Figure 1. An illustration of the coordinate system, and the convex hull of $R_{1,1}$.

Proof. Since the hypercube and generalized set covering inequalities are closed under relabelings and the transformation $x_i \mapsto 1 - x_i, i \in [n]$, being q-ideal is closed under relabelings and twistings. It therefore suffices to prove the remark for the case when S' is obtained from S after a single minor operation. Suppose that $\mathrm{conv}(S) = \{x \in [0,1]^n : \sum_{i \in I} x_i + \sum_{j \in J}(1 - x_j) \geq 1, (I,J) \in \mathcal{V}\}$ for an appropriate $\mathcal{V}$. If S' is obtained from S after 0-restricting coordinate 1, then $\mathrm{conv}(S') = \{x \in [0,1]^{n-1} : \sum_{i \in I - \{1\}} x_i + \sum_{j \in J}(1 - x_j) \geq 1, (I,J) \in \mathcal{V}, 1 \notin J\}$. If S' is obtained from S after 1-restricting coordinate 1, then $\mathrm{conv}(S') = \{x \in [0,1]^{n-1} : \sum_{i \in I} x_i + \sum_{j \in J - \{1\}}(1 - x_j) \geq 1, (I,J) \in \mathcal{V}, 1 \notin I\}$. If S' is obtained from S after projecting away coordinate 1, then $\mathrm{conv}(S') = \{x \in [0,1]^{n-1} : \sum_{i \in I} x_i + \sum_{j \in J}(1 - x_j) \geq 1, (I,J) \in \mathcal{V}, 1 \notin I \cup J\}$. In each case, we see that S' is still q-ideal, thereby finishing the proof. $\square$

q-Idealness of subsets of a hypercube can be defined solely in terms of cuboids:

Theorem 6. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then S is q-ideal if, and only if, $\mathrm{cuboid}(S)$ is an ideal clutter.*

Using this theorem, we can use q-idealness to link idealness to another deep property. We say that S is a *vector space over $GF(2)$*, or simply a *binary space*, if $\mathbf{0} \in S$, and $a \triangle b \in S$ for all distinct points $a, b \in S$. A binary space is by definition the cycle space of a binary matroid (see [18]). For instance, $R_{1,1}$ is a binary space, and it corresponds to the cycle space of the graph on two vertices and three parallel edges. We will see that a binary space is q-ideal if, and only if, the associated binary matroid has the *sums of circuits property*. Paul Seymour introduced this rich property in [22], and after developing his splitter theorems and decomposition of regular matroids [20], he classified the binary matroids with the sums of circuits property, where he also posed the *cycle double cover conjecture* [21, 24].

Theorem 6 reduces q-idealness of subsets of a hypercube to clutter idealness; we will see a converse reduction (though with an exponential blow-up). As such, q-idealness provides a framework to interpret clutter idealness geometrically, rather than combinatorially, as foreseen by Jon Lee [14]. To this end, take a point $x \in \{0,1\}^n$. The *induced clutter of S with respect to x*, denoted $\mathrm{ind}(S \triangle x)$, is the clutter over ground set $[n]$ whose members are

$$\mathrm{ind}(S \triangle x) = \text{ the minimal sets of } \{C \subseteq [n] : \chi_C \in S \triangle x\}.$$

In words, $\mathrm{ind}(S\triangle x)$ is the clutter corresponding to the points of $S\triangle x$ of minimal support. Notice that if $S=\emptyset$ then every induced clutter is $\{\}$, and in general, if $x\in S$ then $\mathrm{ind}(S\triangle x)=\{\emptyset\}$. Observe that

$$\mathrm{ind}(S\triangle x)=\mathrm{cuboid}(S)/\{2i : i\in[n], x_i=0\}/\{2i-1 : i\in[n], x_i=1\}.$$

Hence,

Remark 7. *Take an integer $n\geq 1$ and a set $S\subseteq\{0,1\}^n$. Then the 2^n induced clutters $\mathrm{ind}(S\triangle x), x\in\{0,1\}^n$ are in correspondence with the 2^n minors of $\mathrm{cuboid}(S)$ obtained after contracting, for each $i\in[n]$, exactly one of $2i-1, 2i$.*

It therefore follows from Theorem 6 that if S is q-ideal, then all of its induced clutters are ideal. The converse of this statement is also true:

Theorem 8. *Take an integer $n\geq 1$ and a set $S\subseteq\{0,1\}^n$. Then S is q-ideal if, and only if, every induced clutter of S is ideal.*

Let $\mathcal{P}$ be a minor-closed property defined on clutters. Motivated by Theorem 8, we say that $\mathcal{P}$ is a 2*-local property* if for all integers $n\geq 1$ and sets $S\subseteq\{0,1\}^n$, the following statements are equivalent:

- $\mathrm{cuboid}(S)$ has property $\mathcal{P}$,
- the induced clutters of S have property $\mathcal{P}$.

Otherwise, we say that $\mathcal{P}$ is a *non-2-local property*. Notice that Theorems 6 and 8 imply that idealness is a 2-local property. Using the 2-locality of idealness, we will be able to use the famous result of Edmonds and Johnson on T-join polytopes [10] to prove Seymour's result that graphic matroids have the sums of circuits property [22], as well as find a new link between the binary matroids with the sums of circuits property and the f-flowing binary matroids, and formulate the famous f-Flowing Conjecture in terms of q-idealness of binary spaces.

0.2. **Strict polarity.** We say that a clutter has the *packing property* if every minor, including the clutter itself, packs. Notice that a clutter has the packing property if, and only if, it has no minimally non-packing minor. Let us consider $R_{1,1}$ again. The induced clutters of this set are isomorphic to either $\{\emptyset\}$ or $\{\{1\},\{2\},\{3\}\}$, so they all have the packing property, whereas $\mathrm{cuboid}(R_{1,1})=Q_6$ does not pack. Therefore, in contrast to idealness, the packing property is non-2-local. We will now see what causes the packing property to become 2-local.

Let $n\geq 1$ be an integer. A pair of points $a,b\in\{0,1\}^n$ are *antipodal* if $a+b=\mathbf{1}$. Take a set $S\subseteq\{0,1\}^n$. We will refer to the points in S as *feasible* points and to the points in $\{0,1\}^n-S$ as *infeasible* points. We say that S is *polar* if either there are antipodal feasible points or all the feasible points agree on a coordinate:

$$\{x,\mathbf{1}-x\}\subseteq S \text{ for some } x\in\{0,1\}^n \qquad \text{or} \qquad S\subseteq\{x : x_i=a\} \text{ for some } i\in[n] \text{ and } a\in\{0,1\}.$$

For instance, the set $R_{1,1}$ is non-polar. Notice that if a set is polar, then so is every twisting of it. Moreover,

Remark 9. *Take an integer $n\geq 1$ and a set $S\subseteq\{0,1\}^n$. Then S is polar if, and only if, $\mathrm{cuboid}(S)$ packs.*

We say that S is *strictly polar* if every restriction, including S itself, is polar.

Remark 10. *Take an integer $n \geq 1$ and a strictly polar set $S \subseteq \{0,1\}^n$. If S' is isomorphic to a minor of S, then S' is polar.*

Proof. Being polar is closed under twistings and relabelings, so it suffices to prove that every minor of S is polar. To this end, let S' be a minor of S. Then there are disjoint sets $I, J, K \subseteq [n]$ such that S' is obtained after 0-restricting I, 1-restricting J and projecting away K; among all possible I, J, K we may assume that K is minimal, so that no single projection can be replaced by a single restriction. Let R be the restriction of S obtained after 0-restricting I and 1-restricting J; notice that S' is obtained from R after projecting away K. Since S is strictly polar, it follows from the definition that R is polar. If R contains antipodal points, then the same points give antipodal points in the projection S'. Otherwise, the points in R agree on a coordinate, so by the minimality of K, the points in the projection S' also agree on the same coordinate. In either cases, we see that S' is polar, as required. $\square$

As a result, a set is strictly polar if, and only if, every cuboid minor of the corresponding cuboid packs. In particular, if $\mathrm{cuboid}(S)$ has the packing property, then S is strictly polar. We will see that once strict polarity is extracted, the non-2-local packing property becomes a 2-local property:

Theorem 11. *Let S be a strictly polar set. Then $\mathrm{cuboid}(S)$ has the packing property if, and only if, all of the induced clutters of S have the packing property.*

We will also see that strict polarity is a tractable property:

Theorem 12. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is not strictly polar,

(ii) there are distinct points $a, b, c \in S$ such that the smallest restriction of S containing them is not polar.

As a result, in time $O(n|S|^4)$ one can certify whether or not S is strictly polar.

A set is *strictly non-polar* if it is not polar and every proper restriction is polar. Theorem 12 equivalently states that every strictly non-polar set has three distinct feasible points that do not all agree on a coordinate. A set is *minimally non-polar* if it is not polar and every proper minor is polar. A minimally non-polar set is strictly non-polar, and we will see that there are strictly non-polar sets that are not minimally non-polar. Observe that a set is polar if, and only if, it has no strictly non-polar restriction if, and only if, it has no minimally non-polar minor.

0.3. **The Polarity Conjecture.** A fascinating consequence of Lehman's theorem on minimally non-ideal clutters [15] is the following:

Theorem 13 ([6])**.** *If a clutter has the packing property, then it is ideal.*

The converse however is not true, as there are ideal non-packing cuboids such as Q_6. And after all, we should not expect the two properties to be the same, because idealness is a 2-local property but the packing property is not. However, as Theorem 11 shows, strict polarity makes the packing property 2-local. We conjecture that strict polarity does far more than that:

The Polarity Conjecture. *Let S be a strictly polar set. Then* $\mathrm{cuboid}(S)$ *is ideal if, and only if,* $\mathrm{cuboid}(S)$ *has the packing property.*

Justified by Theorems 6 and 13, we may rephrase this conjecture as follows:

The Polarity Conjecture (rephrased)**.** *If a set is q-ideal and strictly polar, then its cuboid has the packing property.*

In fact,

Theorem 14. *The Polarity Conjecture is equivalent to the $\tau = 2$ Conjecture.*

Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$. We say that S is *critically non-polar* if it is strictly non-polar and, for each $i \in [n]$, both the 0- and 1-restrictions of S over coordinate i have antipodal points. We will see that critical non-polarity implies minimal non-polarity. We will also see that if the Polarity Conjecture is true, then so is the following conjecture:

Conjecture 15. *If a set is q-ideal and critically non-polar, then its cuboid is minimally non-packing.*

The Polarity Conjecture and Conjecture 15 are true for sets of *degree* at most 8 – this notion is defined later in the extended abstract.

We also study three basic binary operations on pairs of sets. Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0,1\}^{n_1}$ and $S_2 \subseteq \{0,1\}^{n_2}$. Define the *product*

$$S_1 \times S_2 := \left\{(x,y) \in \{0,1\}^{n_1} \times \{0,1\}^{n_2} : x \in S_1 \text{ and } y \in S_2\right\}$$

and the *coproduct*

$$S_1 \oplus S_2 := \left\{(x,y) \in \{0,1\}^{n_1} \times \{0,1\}^{n_2} : x \in S_1 \text{ or } y \in S_2\right\}.$$

Observe that $S_1 \oplus S_2 = \overline{\overline{S_1} \times \overline{S_2}}$, thereby justifying our terminology. We will observe that if the cuboids of two sets are ideal (resp. have the packing property), then so is (resp. does) the cuboid of their product. Moreover, by exploiting the 2-locality of idealness, and the 2-locality of the packing property once strict polarity is enforced, we show that if the cuboids of two sets are ideal (resp. have the packing property), then so is (resp. does) the cuboid of their coproduct. Define the *reflective product*

$$S_1 * S_2 := (S_1 \times S_2) \cup (\overline{S_1} \times \overline{S_2}).$$

In words, the reflective product $S_1 * S_2$ is obtained from S_1 after replacing each feasible point by a copy of S_2 and each infeasible point by a copy of $\overline{S_2}$. Observe that $S_1 * S_2 = \overline{S_1} * \overline{S_2}$ and $\overline{S_1 * S_2} = \overline{S_1} * S_2 = S_1 * \overline{S_2}$. We will see that,

Theorem 16. *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0,1\}^{n_1}$ and $S_2 \subseteq \{0,1\}^{n_2}$. If $S_1, \overline{S_1}, S_2, \overline{S_2}$ are q-ideal, then so are $S_1 * S_2, \overline{S_1 * S_2}$.*

That is, by Theorem 6, if $\mathrm{cuboid}(S_1), \mathrm{cuboid}(\overline{S_1}), \mathrm{cuboid}(S_2), \mathrm{cuboid}(\overline{S_2})$ are ideal, then so are $\mathrm{cuboid}(S_1 * S_2), \mathrm{cuboid}(\overline{S_1 * S_2})$. In contrast, the analogue of this for the packing property does not hold. For instance, let $S_1 := \{00, 11\}$ and $S_2 := \{0\}$. Then $\mathrm{cuboid}(S_1), \mathrm{cuboid}(\overline{S_1}), \mathrm{cuboid}(S_2), \mathrm{cuboid}(\overline{S_2})$ all have the packing property, while $\mathrm{cuboid}(S_1 * S_2), \mathrm{cuboid}(\overline{S_1 * S_2})$ are isomorphic to Q_6 and therefore do not pack. This phenomenon raises an intriguing question: can we build a counterexample to the Polarity Conjecture by taking the reflective product of two sets that are not counterexamples? As we will prove, the answer is no:

Theorem 17. *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0,1\}^{n_1}$ and $S_2 \subseteq \{0,1\}^{n_2}$, where $\mathrm{cuboid}(S_1)$, $\mathrm{cuboid}(\overline{S_1}), \mathrm{cuboid}(S_2), \mathrm{cuboid}(\overline{S_2})$ have the packing property. If $S_1 * S_2$ is strictly polar, then $\mathrm{cuboid}(S_1 * S_2)$ has the packing property.*

0.4. **Strictly non-polar sets.** A set is *antipodally symmetric* if a point is feasible if and only if its antipodal point is feasible. We will prove the following:

Theorem 18. *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0,1\}^{n_1}$ and $S_2 \subseteq \{0,1\}^{n_2}$, where $S_1 * S_2$ is strictly non-polar. Then the following statements hold:*

(1) $S_1, \overline{S_1}, S_2, \overline{S_2}$ are nonempty.
(2) Either $n_1 = 1$ and S_2 is antipodally symmetric, or $n_2 = 1$ and S_1 is antipodally symmetric. In particular, $S_1 * S_2 \cong \overline{S_1 * S_2}$.
(3) $S_1 * S_2$ is critically non-polar.
(4) If $\mathrm{cuboid}(S_1), \mathrm{cuboid}(\overline{S_1}), \mathrm{cuboid}(S_2), \mathrm{cuboid}(\overline{S_2})$ have the packing property, then $\mathrm{cuboid}(S_1 * S_2)$ is an ideal minimally non-packing clutter.

For an integer $k \geq 1$, let

$$R_{k,1} := \{\mathbf{0}^{k+1}, \mathbf{1}^{k+1}\} * \{0\} \subseteq \{0,1\}^{k+2}.$$

(Hereinafter, $\mathbf{0}^m, \mathbf{1}^m$ are the m-dimensional vectors all of whose entries are $0, 1$, respectively.) See Figure 2 for an illustration of $R_{1,1}$ and $R_{2,1}$. The reader can readily check that $\{R_{k,1} : k \geq 1\}$ are strictly non-polar sets, and that their cuboids are isomorphic to the ideal minimally non-packing clutters $\{Q_{k,1} : k \geq 1\}$. The following result is the second half of Theorem 18:

Theorem 19. *Take an integer $n \geq 1$ and an antipodally symmetric set $S \subseteq \{0,1\}^n$ such that $S * \{0\}$ is strictly non-polar. If $S * \{0\}$ is not isomorphic to any of $\{R_{k,1} : k \geq 1\}$, then both S and $\overline{S}$ are strictly connected.*

Take an integer $n \geq 1$. Denote by G_n the *skeleton graph of* $\{0,1\}^n$ whose vertices are the points in $\{0,1\}^n$ and two points u, v are adjacent if they differ in exactly one coordinate. A set $R \subseteq \{0,1\}^n$ is *connected* if $G_n[R]$ is connected. We say that $R \subseteq \{0,1\}^n$ is *strictly connected* if every restriction of R is connected.

For an integer $k \geq 5$, let

$$C_{k-1} := \left\{ \sum_{i=1}^{d} e_i, \mathbf{1}^{k-1} - \sum_{i=1}^{d} e_i : d \in [k-1] \right\} \subseteq \{0,1\}^{k-1}$$

$$R_k := C_{k-1} * \{0\} \subseteq \{0,1\}^k.$$

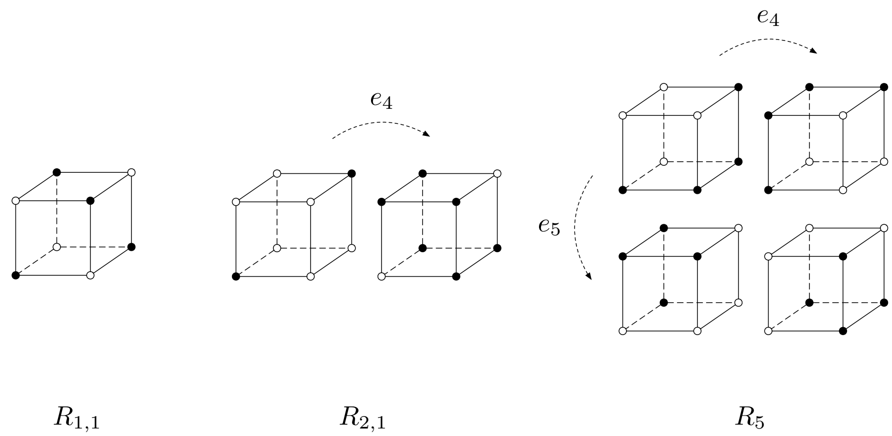

Figure 2. The strictly non-polar sets of degree at most 2. The filled-in points are feasible.

See Figure 2 for an illustration of R_5. Notice that $G_{k-1}[C_{k-1}]$ is a cycle of length $2(k-1)$. The reader can readily check that $\{R_k : k \geq 5\}$ are strictly non-polar sets, and that $C_{k-1}, \overline{C_{k-1}}$ are strictly connected, verifying Theorem 19. The cuboid of R_5 is the ideal minimally non-packing clutter Q_{10} found in [1], but the cuboids of $\{R_k : k \geq 6\}$ are not ideal and not minimally non-packing.

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. For an integer $k \in \{0, 1, \ldots, n\}$, we say that S has *degree at most* k if every infeasible point has at most k infeasible neighbors in G_n. We say that S has *degree* k if it has degree at most k and not $k-1$. As a result, given a set of degree k, every infeasible point has at most k infeasible neighbors, and there is an infeasible point achieving this bound. For each $k \geq 1$, it is known that every strictly non-polar set of degree at most k must have dimension at most $4k+1$ ([1], Theorem 1.10 (i)). It was also shown that, up to isomorphism, the strictly non-polar sets of degree at most 2 are $R_{1,1}, R_{2,1}, R_5$, as displayed in Figure 2 ([1], Theorem 1.9). We will improve the upper bound of $4k+1$ as follows:

Theorem 20. *Take an integer $k \geq 2$ and a strictly non-polar set S of degree k, whose dimension is n. Then the following statements hold:*

(1) $n \in \{k, k+1, \ldots, 2k+1\}$.

(2) If $n = k+1$, then either S is minimally non-polar, or after a possible relabeling,

$$S \subseteq \left\{x \in \{0,1\}^{k+1} : x_k = x_{k+1}\right\}$$

and the projection of S over coordinate $k+1$ is a critically non-polar set that is the reflective product of two other sets.

(3) If $n \geq k+2$, then S is critically non-polar.

(4) If $n = 2k+1$, then $|S| = 2^{n-1}$, every infeasible point has exactly k infeasible neighbors, and $\mathrm{cuboid}(S)$ *is an ideal minimally non-packing clutter.*

Notice that R_5, which is of degree 2 and dimension 5, has 16 points, every infeasible point has exactly 2 infeasible neighbors, and $\mathrm{cuboid}(R_5) = Q_{10}$ is an ideal minimally non-packing clutter. We will describe a computer code,

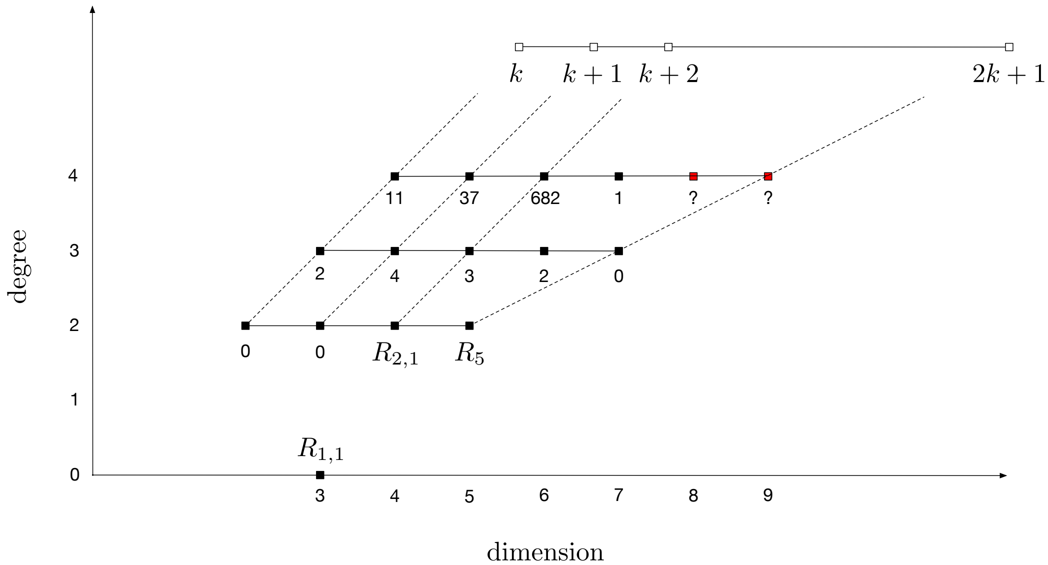

Figure 3. The spectrum of strictly non-polar sets. The number next to the filled-in dot at coordinates (x, y) indicates the number of non-isomorphic strictly non-polar sets of dimension x and degree y.

whose correctness relies on Theorem 20 (1), that generates all the non-isomorphic strictly non-polar sets of degree at most 3, as well as all the non-isomorphic strictly non-polar sets of degree 4 and dimension at most 7. As we will see, there are exactly 745 non-isomorphic strictly non-polar sets of degree at most 4 and dimension 7, summarized in Figure 3, 716 sets of which have ideal minimally non-packing cuboids.

Acknowledgements

This work was supported in parts by ONR grant 00014-15-12082, NSF grant CMMI-1560828, and an NSERC CGS D3 grant. Our code was written in Racket, Version 6.11 [11]. We would like to Christopher Calzonetti as well as the Math Faculty Computing Facility at the University of Waterloo for their computing support.

References

[1] Abdi, A., Cornuéjols, G., Pashkovich, K.: Ideal clutters that do not pack. To appear in Math. Oper. Res.

[2] Balas, E. and Padberg, M.W.: On the set-covering problem. Operations Research **20**(6), 1152–1161. (1972)

[3] Bouchet, A.: Greedy algorithm and symmetric matroids. Math. Prog. **38**(2), 147–159. (1987)

[4] Conforti, M. and Cornuéjols, G.: Clutters that pack and the max-flow min-cut property: a conjecture. The Fourth Bellairs Workshop on Combinatorial Optimization (1993)

[5] Conforti, M., Cornuéjols, G., Zambelli, G.: Integer programming. Springer (2014)

[6] Cornuéjols, G., Guenin, B., Margot, F.: The packing property. Math. Program. Ser. A **89**, 113–126 (2000)

[7] Cornuéjols, G. and Novick, B.: Ideal 0,1 matrices. J. Combin. Theory Ser. B **60**, 145–157 (1994)

[8] Edmonds, J. and Fulkerson, D.R.: Bottleneck extrema. J. Combin. Theory Ser. B **8**, 299–306 (1970)

[9] Edmonds, J. and Giles, R.: A min-max relation for submodular functions on graphs. Annals of Discrete Math. **1**, 185–204 (1977)

[10] Edmonds, J. and Johnson, E.L.: Matchings, Euler tours and the Chinese postman problem. Math. Prog. **5**, 88–124. (1973)

[11] Flatt, M. and PLT. Reference: Racket. Technical Report PLT-TR-2010-1, PLT Design Inc. (2010) https://racket-lang.org/tr1/

[12] Flores, A., Gitler, I., Reyes, E.: On 2-partitionable clutters and the MFMC property. Adv. Appl. Discrete Math. **2**(1), 59–84. (2008)

[13] Hoffman, A.J.: A generalization of max flow-min cut. Math. Prog. **6**(1), 352–359 (1974)

[14] Lee, J.: Cropped cubes. J. Combin. Optimization **7**(2), 169–178. (2003)

[15] Lehman, A.: The width-length inequality and degenerate projective planes. DIMACS Vol. **1**, 101–105 (1990)

[16] Lovász, L.: Minimax theorems for hypergraphs. Lecture Notes in Mathematics **411**, Springer-Verlag 111–126 (1972)

[17] Novick, B. and Sebő, A.: On combinatorial properties of binary spaces. Integer Programming and Combinatorial Optimization, Vol. **4**, 212–227 (1995)

[18] Oxley, J.: Matroid theory, second edition. Oxford University Press, New York (2011)

[19] Schrijver, A.: A counterexample to a conjecture of Edmonds and Giles. Discrete Math. **32**, 213–214 (1980)

[20] Seymour, P.D.: Decomposition of regular matroids. J. Combin. Theory Ser. B **28**, 305–359 (1980)

[21] Seymour, P.D.: Matroids and multicommodity flows. Europ. J. Combinatorics **2**, 257–290 (1981)

[22] Seymour, P.D.: Sums of circuits. Graph Theory and Related Topics (Bondy, J.A. and Murty, U.S.R., eds), Academic Press, New York 342–355. (1979)

[23] Seymour, P.D.: The matroids with the max-flow min-cut property. J. Combin. Theory Ser. B **23**, 189–222 (1977)

[24] Szekeres, G.: Polyhedral decomposition of cubic graphs. Bull. Austral. Math. Soc. **8**, 367–387. (1973)