Approximately Packing Dijoins via Nowhere-Zero Flows

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Abstract. In a digraph, a dicut is a cut where all the arcs cross in one direction. A dijoin is a subset of arcs that intersects each dicut. Woodall conjectured in 1976 that in every digraph, the minimum size of a dicut equals to the maximum number of disjoint dijoins. However, prior to our work, it was not even known whether at least 3 disjoint dijoins exist in an arbitrary digraph whose minimum dicut size is sufficiently large. By building connections with nowhere-zero (circular) k-flows, we prove that every digraph with minimum dicut size τ contains $\frac{\tau}{k}$ disjoint dijoins if the underlying undirected graph admits a nowhere-zero (circular) k-flow. The existence of nowhere-zero 6-flows in 2-edge-connected graphs (Seymour 1981) directly leads to the existence of $\frac{\tau}{6}$ disjoint dijoins in a digraph with minimum dicut size τ, which can be found in polynomial time as well. The existence of nowhere-zero circular $\frac{2p+1}{p}$-flows in $6p$-edge-connected graphs (Lovász et al 2013) directly leads to the existence of $\frac{\tau p}{2p+1}$ disjoint dijoins in a digraph with minimum dicut size τ whose underlying undirected graph is $6p$-edge-connected.

Keywords: Woodall's conjecture · Nowhere-zero flow · Approximation algorithm.

1 Introduction

Dicuts and Dijoins. Given a digraph $D = (V, A)$ and a subset U of its vertices with $U \neq \emptyset, V$, denote by $\delta^+_D(U)$ and $\delta^-_D(U)$ the arcs leaving and entering U, respectively. The cut induced by U is $\delta_D(U) := \delta^+_D(U) \cup \delta^-_D(U)$. We omit the subscript D if the context is clear. For an arc subset $B \subseteq A$, $\delta^+_B(U) := \delta^+_D(U) \cap B$. A *dicut* is an arc subset of the form $\delta^+(U)$ such that $\delta^-(U) = \emptyset$. A *dijoin* is a subset $J \subseteq A$ that intersects every dicut at least once. More generally, we will also work with the notion of a τ*-dijoin*, which is a subset $J \subseteq A$ that intersects every dicut at least τ times. If D is a weighted digraph with arc weights $w : A \to \mathbb{Z}_+$, we say that D can *pack* k dijoins if there exist k dijoins $J_1, ..., J_k$ such that no arc e is contained in more than $w(e)$ of these k dijoins. In this case, we say that $J_1, ..., J_k$ is a *packing* of D under weight w. In particular, when the digraph is *unweighted*, i.e. $w(e) = 1$ for every $e \in A$, D packs k dijoins if and only if D contains k disjoint dijoins. The *value* of the packing is the number k of dijoins in the packing. Edmonds and Giles [8] conjectured the following.

Conjecture 1 (Edmonds-Giles). Let $D = (V, A)$ be a digraph with arc weights $w \in \{0, 1\}^A$. If the minimum weight of a dicut is τ, then D can pack τ dijoins.

We can assume without loss of generality that $w \in \{0,1\}^A$ because we can always replace an arc e with integer weight $w(e) > 1$ by $w(e)$ parallel arcs of weight 1. Note that the weight 0 arcs cannot be removed because they, together with the weight 1 arcs, determine the dicuts. The above conjecture was disproved by Schrijver [20]. However, the following unweighted version of the Edmonds-Giles conjecture, proposed by Woodall [26], is still open.

Conjecture 2 (Woodall). In every digraph, the minimum size of a dicut equals the maximum number of disjoint dijoins.

Several weakenings of Woodall's conjecture have been made in the literature. It has been conjectured that there exists some integer $\tau \geq 3$ such that every digraph with minimum dicut size at least τ contains 3 disjoint dijoins [6]. Shepherd and Vetta [23] raised the following question. Let $f(\tau)$ be the maximum value such that every weighted digraph whose dicuts all have weight at least τ, can pack $f(\tau)$ dijoins. They conjectured that $f(\tau)$ is of order $\Omega(\tau)$. In Section 2, we give an affirmative answer to this conjecture in the *unweighted* case. The main results in this paper are the following approximate versions of Woodall's conjecture.

Theorem 1. *Every digraph $D = (V, A)$ with minimum dicut size τ contains $\lfloor \frac{\tau}{6} \rfloor$ disjoint dijoins, and such dijoins can be found in polynomial time.*

Given a digraph $D = (V, A)$, the *underlying undirected graph* is the graph with vertex set V and edge set obtained by replacing each arc $(u, v) \in A$ with an undirected edge (u, v). To exclude the cases $\tau = 0$ and $\tau = 1$, when Woodall's conjecture holds trivially, we assume $\tau \geq 2$ throughout the paper, which implies that the underlying undirected graph is 2-edge-connected.

Theorem 2. *Let p be a positive integer. Every digraph $D = (V, A)$ with minimum dicut size τ and with the property that its underlying undirected graph is $6p$-edge-connected contains $\left\lfloor \frac{\tau p}{2p+1} \right\rfloor$ disjoint dijoins.*

Nowhere-zero circular flows. Let $G = (V, E)$ be an *undirected* graph and let $k \geq 2$ be an integer. Tutte [25] introduced the notion of a *nowhere-zero k-flow* of G, which is an orientation E^+ and $f : E^+ \to \{1, 2, \ldots, k-1\}$ such that $\sum_{e \in \delta^+_{E^+}(v)} f(e) = \sum_{e \in \delta^-_{E^+}(v)} f(e)$ for every vertex $v \in V$. Goddyn et al. [10] extended the definition to allowing k to take fractional values. Let p, q be two integers such that $0 < p \leq q$. A *nowhere-zero circular $\frac{p+q}{p}$-flow* of G is an orientation E^+ and $f : E^+ \to \{1, 1+\frac{1}{p}, \ldots, \frac{q}{p}\}$, such that $\sum_{e \in \delta^+_{E^+}(v)} f(e) = \sum_{e \in \delta^-_{E^+}(v)} f(e)$. When $p = 1$ we recover Tutte's notion.

Both theorems above are consequences of the following main theorem we prove.

Theorem 3. *For a digraph $D = (V, A)$ with minimum dicut size τ, if the underlying undirected graph admits a nowhere-zero circular k-flow, where $k \geq 2$ is a rational, then D contains $\lfloor \frac{\tau}{k} \rfloor$ disjoint dijoins.*

The first ingredient of our approach to proving the above results is reducing the problem of packing dijoins in a digraph to that of packing strongly connected digraphs. This reduction is not new and it was already explored by Shepherd and Vetta [23]. Augment the input digraph D by adding reverse arcs for all input arcs and assigning weights τ to the original arcs and 1 to the newly added reverse arcs. Denote the augmented digraph by $\vec{G}$ with weight w^D. Define a *τ-strongly-connected digraph (τ-SCD)* to be a weighted digraph such that the arcs leaving every cut have weight at least τ. Note that a 1-SCD is a *strongly connected digraph (SCD)*. It is not hard to see that for a digraph D with minimum dicut size τ, the augmented digraph $\vec{G}$ with weight w^D is τ-strongly-connected. One can then show that packing $\tau' \leq \tau$ dijoins in the original digraph D is equivalent to decomposing the augmented weighted digraph $\vec{G}$ into τ' strongly connected digraphs (Proposition 2).

We then draw a connection to nowhere-zero flows. There is a rich literature on the existence of nowhere zero k-flows from which we will use two important results. Seymour [22] showed that there always exists a nowhere-zero 6-flow in 2-edge-connected graphs. Younger [27] gave a polynomial time algorithm to construct a nowhere-zero 6-flow in 2-edge-connected graphs.

Theorem 4 ([22,27]). *Every* 2*-edge-connected graph admits a nowhere-zero* 6*-flow which can be found in polynomial time.*

Lovász et al. [17] proved the following existence result for nowhere-zero circular flows under stronger connectivity requirements.

Theorem 5 ([17]). *Let p be a positive integer. Every $6p$-edge-connected graph admits a nowhere-zero circular $\frac{2p+1}{p}$-flow.*

Returning to dijoins and the augmented digraph $\vec{G}$, we need to decompose this augmented digraph into some $\tau' \leq \tau$ disjoint strongly connected digraphs. In general, decomposing a digraph into strongly connected digraphs is a notoriously hard problem. It is not known whether there exists an integer τ such that every τ-strongly-connected digraph can be decomposed into 2 disjoint strongly connected digraphs [4]. To get around this difficulty, we reduce our goal to finding *two* disjoint subdigraphs of $\vec{G}$, each of which can be decomposed into τ' in or out r-arborescences for some fixed root r. The idea of pairing up in- and out-arborescences was already used successfully by Shepherd and Vetta [23] to find a half-integral packing of dijoins of value $\frac{\tau}{2}$. Here, we crucially argue (in Theorem 6) that if the underlying undirected graph of D admits a nowhere-zero k-flow, then the digraph $\vec{G}$ with weight w^D can be decomposed into two disjoint $\lfloor\frac{\tau}{k}\rfloor$-SCD's. (Note that we do not prove this for any arbitrary τ-SCD.) Using Edmonds' disjoint arborescences theorem [7], we can now extract $\lfloor\frac{\tau}{k}\rfloor$ disjoint in r-arborescences from the first and the same number of out r-arborescences from the second. Pairing them up gives us the final set of $\lfloor\frac{\tau}{k}\rfloor$ strongly connected digraphs. Our results then follow from the prior theorems about the existence of nowhere-zero flows.

Strongly connected orientations. In Section 3, we give equivalent forms of Woodall's conjecture and of the Edmonds-Giles conjecture, respectively, in terms of packing strongly connected orientations, which are of independent interest. Given an undirected graph $G = (V, E)$, let $\vec{G} = (V, E^+ \cup E^-)$ be a digraph obtained from making two copies of each edge $e \in E$ and directing them oppositely, one arc being denoted by $e^+ \in E^+$ and the other by $e^- \in E^-$. A *τ-strongly connected orientation (τ-SCO)* of G is a multi-subset of arcs from $E^+ \cup E^-$ picking exactly τ many of e^+ and e^- (possibly with repetitions) for each e such that at least τ arcs leave every cut. In particular, a *strongly connected orientation (SCO)* of G is a 1-SCO of G. In other words, a τ-SCO is an integral vector in the polyhedron

$$P_0^\tau := \Big\{x \in \mathbb{R}^{E^+\cup E^-} \Big| \, x_{e^+} \geq 0,\ x_{e^-} \geq 0,\ \forall e \in E, \\ x_{e^+} + x_{e^-} = \tau,\ \forall e \in E, \\ x(\delta^+(U)) \geq \tau,\ \forall U \subsetneqq V, U \neq \emptyset\Big\}. \tag{1}$$

One may ask whether a τ-SCO can always be decomposed into τ disjoint SCO's. This is not the case. Indeed we prove in Theorem 9 that this question is equivalent the Edmonds-Giles conjecture.

In contrast, we define x to be a *nowhere-zero τ-SCO* if it is a τ-SCO and $x_e \geq 1$ for every arc e. In other words, a nowhere-zero τ-SCO is an integral vector in the polyhedron

$$P_1^\tau := \Big\{x \in \mathbb{R}^{E^+\cup E^-} \Big| \, x_{e^+} \geq 1,\ x_{e^-} \geq 1,\ \forall e \in E, \\ x_{e^+} + x_{e^-} = \tau,\ \forall e \in E, \\ x(\delta^+(U)) \geq \tau,\ \forall U \subsetneqq V, U \neq \emptyset\Big\}. \tag{2}$$

In Theorem 10, we prove that Woodall's conjecture is true if and only if for every undirected graph G, a *nowhere-zero* τ-SCO can be decomposed into τ disjoint SCO's.

Related Work

Shepherd and Vetta [23] raised the question of approximately packing dijoins. They also introduced the idea of adding reverse arcs to make the digraph τ-strongly-connected, then packing strongly connected subdigraphs, and finally pairing up in- and out-arborescences. Yet, this approach itself only gives a half integral packing of value $\frac{\tau}{2}$ in a digraph with minimum dicut size τ. It is conjectured by Király [15] that every digraph with minimum dicut size τ contains two disjoint $\lfloor \frac{\tau}{2} \rfloor$-dijoins, see also [1]. One might notice that if this conjecture is true, together with the approach of combining in and out r-arborescences, one can show that there exist $\lfloor \frac{\tau}{2} \rfloor$ disjoint dijoins in a digraph with minimum dicut size τ. Abdi et al. [2] proved that every digraph can be decomposed into a dijoin and a $(\tau - 1)$-dijoin. Abdi et al. [1] further showed that a digraph with minimum dicut size τ can be decomposed into a k-dijoin and a $(\tau - k)$-dijoin for every integer $k \in \{1, ..., \tau - 1\}$ under the condition that the underlying undirected graph is τ-edge-connected. Mészáros [18] proved that when the underlying undirected graph is $(q - 1, 1)$-partition-connected for some prime power q, the digraph can be decomposed into q disjoint dijoins. However, none of these approaches tell us how to decompose a digraph with minimum dicut size τ into a large number of disjoint dijoins without connectivity requirements. We also refer to the papers that view the problem from the perspective of reorienting the directions of a subset of arcs to make the graph strongly connected, such as [19,1,5]. For the context of nowhere-zero k-flow, we refer interested readers to [11,13,12,22,24,17,27] and the excellent survey by Jaeger [14]. Finally, Schrijver's unpublished notes [19] reformulate Woodall's conjecture into the problem of partitioning the arcs of the digraph into strengthenings. A *strengthening* is an arc set $J \subseteq A$ which, when flipping the orientation of the arcs in J, makes the digraph strongly connected. This inspired the reformulations in Theorem 9 and Theorem 10.

2 An Approximate Packing of Dijoins

In this section we prove our main result, Theorem 3. We begin by observing that the existence of a nowhere-zero circular flow implies that there is a nearly balanced orientation in the sense that, for each cut, the number of arcs entering it differs by a constant factor from the number of arcs leaving it. This is already pointed out in different places (e.g. see in [10], [24], [9]). We summarize this key fact in the following lemma. Since we will reuse this fact we also give the proof here. In a digraph $D = (V, A)$, denote by e^{-1} the reverse of arc $e \in A$, and by B^{-1} the arcs obtained by reversing the directions of the arcs in $B \subseteq A$.

Lemma 1. *Let $G = (V, E)$ be an undirected graph that admits a nowhere-zero circular k-flow E^+ and $f : E^+ \to [1, k-1]$, where $k \geq 2$ is a rational number. Let $E^- = (E^+)^{-1}$. Then, for every $U \subsetneq V, U \neq \emptyset$,*

$$\frac{1}{k}|\delta_G(U)| \leq |\delta^+_{E^+}(U)| \leq \frac{k-1}{k}|\delta_G(U)|,$$
$$\frac{1}{k}|\delta_G(U)| \leq |\delta^+_{E^-}(U)| \leq \frac{k-1}{k}|\delta_G(U)|.$$

Proof. By flow conservation, $f(\delta^+_{E^+}(U)) = f(\delta^-_{E^+}(U)), \forall U \subsetneq V, U \neq \emptyset$. Thus, one has $1 \cdot |\delta^+_{E^+}(U)| \leq f(\delta^+_{E^+}(U)) = f(\delta^-_{E^+}(U)) \leq (k-1) \cdot |\delta^-_{E^+}(U)|$. Similarly, one also has $1 \cdot |\delta^-_{E^+}(U)| \leq f(\delta^-_{E^+}(U)) = f(\delta^+_{E^+}(U)) \leq (k-1) \cdot |\delta^+_{E^+}(U)|$. It follows from the equality $|\delta_G(U)| = |\delta^+_{E^+}(U)| + |\delta^-_{E^+}(U)|$ that $\frac{1}{k}|\delta_G(U)| \leq |\delta^+_{E^+}(U)| \leq \frac{k-1}{k}|\delta_G(U)|$ and $\frac{1}{k}|\delta_G(U)| \leq |\delta^-_{E^+}(U)| \leq \frac{k-1}{k}|\delta_G(U)|$. By noticing that $|\delta^+_{E^-}(U)| = |\delta^-_{E^+}(U)|$, the inequality $\frac{1}{k}|\delta_G(U)| \leq |\delta^+_{E^-}(U)| \leq \frac{k-1}{k}|\delta_G(U)|$ holds. □

Let $D = (V, A)$ be a digraph. By Lemma 1, both the subdigraph consisting of the arcs that are in the same orientation as the nowhere-zero circular flow and its complement intersect every dicut in a large proportion of its size. This gives us a way to decompose the digraph into two k-dijoins with a large k (an example is in Figure 1). Recall that a k-dijoin is an arc set that intersects each dicut at least k times.

Proposition 1. *For a digraph $D = (V, A)$ with minimum dicut size τ, if the underlying undirected graph admits a nowhere-zero circular k-flow for some rational number $k \geq 2$, then D contains two disjoint $\lfloor \frac{\tau}{k} \rfloor$-dijoins.*

Proof. Let E^+ and $f : E^+ \to [1, k-1]$ be a nowhere-zero circular k-flow of the underlying undirected graph G of D. By Lemma 1, $\frac{1}{k}|\delta_G(U)| \leq |\delta^+_{E^+}(U)| \leq \frac{k-1}{k}|\delta_G(U)|$ for every $U \subsetneq V, U \neq \emptyset$. Take $J = A \cap E^+$ to be the arcs that have the same directions in A and E^+. Then, for a dicut $\delta^+_D(U)$ such that $\delta^-_D(U) = \emptyset$, we have $|J \cap \delta^+_D(U)| = |\delta^+_{E^+}(U)| \geq \frac{1}{k}|\delta_G(U)| = \frac{1}{k}|\delta^+_D(U)| \geq \frac{\tau}{k}$ and $|(A \setminus J) \cap \delta^+_D(U)| = |\delta^+_D(U)| - |\delta^+_{E^+}(U)| \geq |\delta^+_D(U)| - \frac{k-1}{k}|\delta_G(U)| = |\delta^+_D(U)| - \frac{k-1}{k}|\delta^+_D(U)| \geq \frac{\tau}{k}$. Thus, both J and $A \setminus J$ are $\lfloor \frac{\tau}{k} \rfloor$-dijoins. □

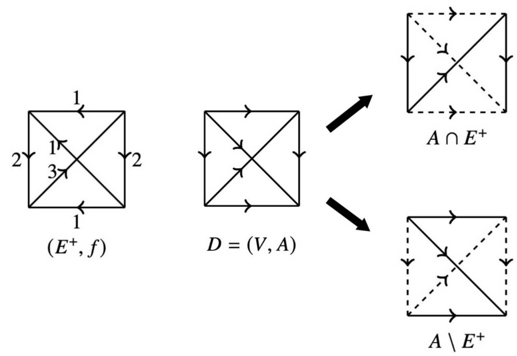

Fig. 1. (E^+, f) is a nowhere-zero 4-flow of $G = K_4$. $D = (V, A)$, whose underlying undirected graph is G, can be decomposed into a dijoin $A \cap E^+$ and a 2-dijoin $A \setminus E^+$.

In a digraph D with minimum dicut size τ, although Proposition 1 suggests that D can be decomposed into two digraphs, each being a $\lfloor \frac{\tau}{k} \rfloor$-dijoin, there is no guarantee that the new digraphs have minimum dicut size at least $\lfloor \frac{\tau}{k} \rfloor$. This is because a non-dicut in D may become a dicut when we delete arcs, which can potentially have very small size. This is a general difficulty with inductive proofs for decomposing a digraph into dijoins.

The key observation here is that, by switching to the setting of strongly connected digraphs, we can bypass this issue. Given a digraph $D = (V, A)$ with minimum dicut size τ, let G be the underlying undirected graph of D and $\vec{G} = (V, E^+ \cup E^-)$ be the digraph obtained by copying each edge of G twice and directing them oppositely. For convenience, we let $E^+ = A$ and $E^- = A^{-1}$. Define the *weights associated with* D to be $w^D \in \mathbb{Z}^{E^+ \cup E^-}$ such that $w^D_{e^+} = \tau, \forall e^+ \in E^+$ and $w^D_{e^-} = 1, \forall e^- \in E^-$. It is easy to see that $\vec{G}$ with weight w^D is τ-SCD. Indeed, for every $U \subsetneq V, U \neq \emptyset$ such that $\delta^+_D(U) \neq \emptyset$, there exists some arc $e^+ \in E^+$ such that $e^+ \in \delta^+_{\vec{G}}(U)$, and thus $w^D(\delta^+_{\vec{G}}(U)) \geq w^D_{e^+} = \tau$. Otherwise, $\delta^+_D(U) = \emptyset$ which means $\delta^-_D(U)$ is a dicut. Therefore, $w^D(\delta^+_{\vec{G}}(U)) = w^D(\delta^+_{E^-}(U)) = |\delta^-_D(U)| \geq \tau$. This means that the augmented digraph $\vec{G}$ with weight w^D is τ-strongly-connected. We first reformulate the problem of packing dijoins in D into a problem of packing strongly connected digraphs in $\vec{G}$ under weight w^D. We then prove a decomposition result into k-strongly connected digraphs with the help of nowhere-zero circular flows. The following reformulation has essentially been stated and used in [23]. We include its proof here.

Proposition 2. *For an integer $k \leq \tau$, the digraph D contains k disjoint dijoins if and only if $\vec{G}$ with weight w^D can pack k strongly connected digraphs.*

Proof. Let $F_1, ..., F_k$ be k strongly connected digraphs of G that is a packing of $\vec{G}$ under weight w^D. Define $J_i := \{e^+ \in E^+ \mid \chi_{F_i}(e^-) = 1, e^- \in E^-\}$. We claim each J_i is a dijoin of D. Suppose not. Then there exists some dicut $\delta^-_D(U)$ such that $J_i \cap \delta^-_D(U) = \emptyset$. This implies $F_i \cap \delta^+_{\vec{G}}(U) = \emptyset$, contradicting the fact that F_i is a strongly connected digraph of $\vec{G}$. Moreover, since $w^D_{e^-} = 1$, at most one of $F_1, ..., F_k$ uses e^-, $\forall e^- \in E^-$. Thus, at most one of $J_1, ..., J_k$ uses $e^+, \forall e^+ \in E^+ = A$. Therefore, $J_1, ..., J_k$ are disjoint dijoins of D.

Conversely, let $J_1, ..., J_k$ be k disjoint dijoins in D. W.l.o.g. we can assume each J_i is a minimal dijoin, that is, J_i is not contained in another dijoin. It is shown by Frank (see e.g. in [16], Chapter 6) that each minimal dijoin is a strengthening, i.e. $(A \setminus J_i) \cup J_i^{-1}$ is a strongly connected digraph. Let $F_i := (A \setminus J_i) \cup J_i^{-1}, \forall i$. The same argument as for the other direction applies to argue that $F_1, ..., F_k$ is a valid packing of strongly connected digraphs in $\vec{G}$ under weight w^D. □

Theorem 6. *Let $D = (V, A)$ be a digraph with minimum dicut size τ. If the underlying undirected graph admits a nowhere-zero circular k-flow for some rational number $k \geq 2$, then the weight w^D associated with D contains two disjoint $\lfloor \frac{\tau}{k} \rfloor$-SCD's.*

Proof. Let E^+ and $f : E^+ \to \{1, ..., k-1\}$ be a nowhere-zero k-flow of G. Let E^- be obtained by reversing the arcs of E^+. Let G be the underlying undirected graph of D and $\vec{G} = (V, E^+ \cup E^-)$. Construct $x \in \mathbb{Z}^{E^+ \cup E^-}$ as follows.

$$x_e = \begin{cases} \lceil \frac{\tau}{2} \rceil, & e \in A \cap E^+ \\ \lfloor \frac{\tau}{2} \rfloor, & e \in A \cap E^- \\ 1, & e \in A^{-1} \cap E^+ \\ 0, & e \in A^{-1} \cap E^- \end{cases}, \quad \text{and equivalently} \quad (w^D - x)_e = \begin{cases} \lfloor \frac{\tau}{2} \rfloor, & e \in A \cap E^+ \\ \lceil \frac{\tau}{2} \rceil, & e \in A \cap E^- \\ 0, & e \in A^{-1} \cap E^+ \\ 1, & e \in A^{-1} \cap E^- \end{cases}.$$

We prove that both x and $(w^D - x)$ are $\lfloor \frac{\tau}{k} \rfloor$-SCD's. We discuss two cases.

If $\delta_D(U)$ is a dicut, then $|\delta_G(U)| \geq \tau$. Since $\tau \geq 2$, we have $x_e \geq 1$, $\forall e \in \delta^+_{E^+}(U)$. Therefore, $x(\delta^+_{\vec{G}}(U)) \geq x(\delta^+_{E^+}(U)) \geq |\delta^+_{E^+}(U)| \geq \frac{1}{k}|\delta_G(U)| \geq \frac{\tau}{k}$, where the third inequality follows from Lemma 1. On the other hand, $(w^D - x)_e \geq 1$, $\forall e \in \delta^+_{E^-}(U)$. Therefore, $(w^D - x)(\delta^+_{\vec{G}}(U)) \geq (w^D - x)(\delta^+_{E^-}(U)) \geq |\delta^+_{E^-}(U)| \geq \frac{1}{k}|\delta_G(U)| \geq \frac{\tau}{k}$.

If $\delta_D(U)$ is not a dicut, then $\delta^+_{\vec{G}}(U) \cap A \neq \emptyset$. Therefore, $x(\delta^+_{\vec{G}}(U)) \geq x(\delta^+_{\vec{G}}(U) \cap A) \geq \lfloor \frac{\tau}{2} \rfloor \geq \lfloor \frac{\tau}{k} \rfloor$ since $k \geq 2$. Also, $(w^D - x)(\delta^+_{\vec{G}}(U)) \geq (w^D - x)(\delta^+_{\vec{G}}(U) \cap A) \geq \lfloor \frac{\tau}{2} \rfloor \geq \lfloor \frac{\tau}{k} \rfloor$ since $k \geq 2$. Therefore, both x and $(w^D - x)$ are $\lfloor \frac{\tau}{k} \rfloor$-SCD's. □

Proof of Theorem 3

From Proposition 2, given a digraph D, we can reduce the problem of packing dijoins of D to that of packing strongly connected digraphs of the augmented digraph $\vec{G}$ with weight w^D which is τ-strongly-connected. To achieve the goal, we recall a classical theorem about decomposing digraphs into arborescences. In a digraph $D = (V, A)$ with a fixed root r, an *out (in) r-arborescence* is a directed spanning tree such that each vertex in $V \setminus \{r\}$ has exactly one arc entering (leaving) it. If the root is not fixed it is called an *out (in) arborescence*. Edmonds' disjoint arborescences theorem [7] states that when fixing a root r, every *rooted-τ-connected* digraph, i.e. $|\delta^+_D(U)| \geq \tau, \forall U \subsetneq V, r \in U$, can be decomposed into τ disjoint out r-arborescences. Furthermore, this decomposition can be done in polynomial time.

Theorem 7 **([7]).** *Given a digraph D and a root r, if D is rooted-τ-connected, then D contains τ disjoint out r-arborescences, and such r-arborescences can be found in polynomial time.*

A τ-strongly-connected digraph is in particular rooted-τ-connected. Therefore, fixing a root $r \in V$, a τ-strongly-connected digraph contains τ disjoint out r-arborescences. If we reverse the directions of the arcs and apply Theorem 7, we see that a τ-strongly-connected digraph also contains τ disjoint in r-arborescences.

Therefore, we can decompose digraph $\vec{G}$ with weight w^D into τ in r-arborescences, or into τ out r-arborescences. Pairing each in r-arborescence with an out r-arborescence, we obtain τ strongly connected digraphs. However, each arc can be used in both in and out r-arborescences. Shepherd and Vetta [23] use this idea to obtain a half integral packing of dijoins of value $\frac{\tau}{2}$. Yet, finding disjoint in and out arborescences together is quite challenging. It is open whether there exists τ such that a τ-strongly-connected digraph can even pack one in-arborescence and one out-arborescence [3].

Theorem 6 paves the way to approximately packing disjoint in and out arborescences in our instances. Fixing a root r, if we are able to decompose the graph into two τ'-strongly-connected graphs and thereby find τ' disjoint in r-arborescences in the first graph and τ' disjoint out r-arborescences in the second graph, then we can combine them to get a strongly connected digraph.

Proof of Theorem 3. By Proposition 2, it suffices to prove that w^D can pack $\lfloor \frac{\tau}{k} \rfloor$ strongly connected digraphs. By Theorem 6, $\vec{G}$ with weight w^D can be decomposed into weighted digraphs J_1 and J_2 such that each of them is $\lfloor \frac{\tau}{k} \rfloor$-strongly-connected. Fixing an arbitrary root r, since a $\lfloor \frac{\tau}{k} \rfloor$-strongly-connected digraph is in particular rooted-$\lfloor \frac{\tau}{k} \rfloor$-connected, by Theorem 7, J_1 can be decomposed into $\lfloor \frac{\tau}{k} \rfloor$ disjoint out r-arborescences $S_1, \ldots, S_{\lfloor \frac{\tau}{k} \rfloor}$. Similarly, J_2 can be decomposed into $\lfloor \frac{\tau}{k} \rfloor$ disjoint in r-arborescences $T_1, \ldots, T_{\lfloor \frac{\tau}{k} \rfloor}$. Let $F_i := S_i \cup T_i$, for $i = 1, \ldots, \lfloor \frac{\tau}{k} \rfloor$. Each F_i is a strongly connected digraph. This is because every out r-cut $\delta^+_{\vec{G}}(U), r \in U$ is covered by S_i and every in r-cut $\delta^+_{\vec{G}}(U), r \notin U$ is covered by T_i and thus every cut $\delta^+_{\vec{G}}(U)$ is covered by F_i. Therefore, $F_1, \ldots, F_{\lfloor \frac{\tau}{k} \rfloor}$ forms a packing of strongly connected digraphs under weight w^D. □

Theorem 1 now follows by combining Theorem 3 and Theorem 4 and noting that the underlying undirected graph of a digraph with minimum dicut size $\tau \geq 2$ is 2-edge-connected. By Theorem 4, the nowhere-zero 6-flow can be found in polynomial time, and thus the decomposition described in Theorem 6 can be done in polynomial time. Moreover, further decomposing J_1 and J_2 into in and out r-arborescences can also be done in polynomial time due to Theorem 7. Thus in the end we can find $\lfloor \frac{\tau}{6} \rfloor$ disjoint dijoins in polynomial time. Theorem 2 now follows by combining Theorem 3 and Theorem 5. However, as far as we know there is no constructive version of Theorem 5, which means Theorem 2 cannot be made algorithmic directly.

3 A Reformulation of Woodall's Conjecture in terms of Strongly Connected Orientations

In this section, we discuss the relation between packing dijoins, strongly connected orientations and strongly connected digraphs. We also discuss another reformulation of Woodall's conjecture in terms of strongly connected orientations.

Given an undirected graph $G = (V, E)$, let $\vec{G} = (V, E^+ \cup E^-)$ be the digraph obtained by copying each edge of G twice and orienting them in opposite directions. Denote by $\chi_F \in \{0, 1\}^E$ the characteristic vector of F. Let

$$SCO(G) := \left\{x \in \{0,1\}^{E^+\cup E^-} \middle| x = \chi_O \text{ for some strongly connected orientation } O \text{ of } G\right\}. \tag{3}$$

Recall that strongly connected orientations (SCO's) are 0, 1 vectors in the polyhedron P_0^1 defined in (1). Recall that given a digraph $D = (V, A)$, a strengthening is a subset $J \subseteq A$ such that by flipping the orientation of the arcs in J the digraph becomes strongly connected [16]. Note that a strengthening is necessarily a dijoin.

Schrijver observed the following reformulation of Woodall's conjecture in terms of strengthenings in his unpublished note ([19], Section 2).

Theorem 8 ([19]). *Woodall's conjecture is true if and only if, in every digraph with minimum dicut size τ, the arcs can be partitioned into τ strengthenings.*

Another way to look at $SCO(G)$ is to fix a direction E^+ and view it as a lift of the set of strengthenings of $G^+ = (V, E^+)$. Indeed, given a strengthening $J \subseteq E^+$, $(E^+ \setminus J) \cup J^{-1}$ is a strongly connected orientation of G. Conversely, given a strongly connected orientation $O \subseteq E^+ \cup E^-$, $E^+ \setminus O$ is a strengthening of G^+. The characteristic vectors of the strengthenings of G^+ are the 0, 1 vectors in the following polyhedron:

$$\Big\{x \in \mathbb{R}^{E^+} \Big| \ 0 \le x_{e^+} \le 1, \ \forall e \in E,$$
$$x(\delta^-_{G^+}(U)) - x(\delta^+_{G^+}(U)) \le |\delta^-_{G^+}(U)| - 1, \forall U \subsetneq V, U \neq \emptyset\Big\}.$$

Due to the Edmonds-Giles submodular flow theorem [8], this is an integral polytope and thus it describes the convex hull of the set of strengthenings of G^+, see also [21]. Note that P_0^1 is a linear transformation of the above polyhedron and thus it is also integral, which means $\text{conv}(SCO(G)) = P_0^1$.

Recall that a τ-SCO is an integral vector in P_0^τ defined in (1). A nowhere-zero τ-SCO is an integral vector in P_1^τ defined in (2). A τ-SCO cannot always be integrally decomposed into τ SCO's, however, due to the following equivalence.

Theorem 9. *The Edmonds-Giles Conjecture 1 is true if and only if for every undirected graph G and integer $\tau > 0$, every τ-SCO can be decomposed into τ SCO's.*

The following counterexample to the Edmonds-Giles Conjecture 1 discovered by Schrijver [20] can be translated to disprove the statement that every τ-SCO can be decomposed into τ SCO's. (See Figure 2.) Let $x \in \mathbb{Z}^{E^+\cup E^-}$ be defined by $x_e = 1$ if e is solid, $x_e = 2$ if e is dashed, and $x_e = 0$ for the reverse of the dashed arcs (which we do not draw here). The vector x is a 2-SCO but it cannot be decomposed into 2 strongly connected orientations.

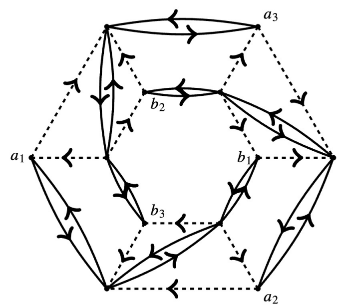

Fig. 2. The solid arcs with weight 1 and dashed arcs with weight 2 cannot be decomposed into 2 SCO's O_1, O_2. Assume for a contradiction that O_1, O_2 exist. The dashed arcs have their orientation fixed in both O_i. Three paths consisting of solid arcs in between $a_i b_i$ have to be directed paths in both O_i, otherwise there is a trivial dicut along the paths in some O_i. Both O_i need to enter the inner hexagon from the outer hexagon, which means each O_i should have at least one directed path oriented as $a_i \to b_i$. Thus, one O_i has exactly one directed path oriented as $b_i \to a_i$ and two oriented as $a_i \to b_i$. Assume O_1 has orientation $b_1 \to a_1, a_2 \to b_2$ and $a_3 \to b_3$. This leaves no arc to go from the left half to the right half of the graph, a contradiction to O_1 being an SCO.

However, slightly revising the statement, we obtain an equivalent form of Woodall's conjecture 2, which is still open.

Theorem 10. *Woodall's Conjecture 2 is true if and only if for every undirected graph G and integer $\tau > 0$, every nowhere-zero τ-SCO can be decomposed into τ SCO's.*

We will first prove Theorem 10 and modify the proof to prove Theorem 9. Our proof of Theorem 10 is inspired by Schrijver's Theorem 8. Schrijver's reformulation essentially covers the special case when $\bar{w}^D \in \mathbb{Z}^{E^+ \cup E^-}$ with $\bar{w}^D_{e^+} = \tau - 1, \forall e^+ \in E^+$ and $\bar{w}^D_{e^-} = 1, \forall e^- \in E^-$ in Theorem 10. One can also easily verify that $\bar{w}^D$ is a nowhere-zero τ-SCO. We generalize the weights to be any nowhere-zero τ-SCO of D, and thus give a stronger consequence of Woodall's conjecture. By allowing the entries of a τ-SCO to take 0 values, we give an equivalent statement of the Edmonds-Giles conjecture in Theorem 9, showing a contrast between the two conjectures.

Proof of Theorem 10. We first prove the "if" direction. Let $D = (V, A)$ be a digraph (e.g. Figure 3-(1)) whose underlying undirected graph is $G = (V, E)$. Let τ be the size of a minimum dicut of D. We assume $\tau \geq 2$ w.l.o.g. and this implies that the size of minimum cut of D is also greater than or equal to 2. By making two copies of each edge of G and orienting them oppositely, we obtain $\vec{G} = (V, E^+ \cup E^-)$. For convenience we will assume that e^+ and e^- are defined according to their direction in D, i.e. $e = (u, v) \in A$ iff $e^+ = (u, v)$ and $e^- = (v, u)$. In other words, $E^+ = A$ and $E^- = A^{-1}$. Take $x \in \mathbb{Z}^{E^+ \cup E^-}$ such that $x_{e^+} = \tau - 1$, $x_{e^-} = 1$ for every $e \in E$ (as shown in Figure 3-(2)). We claim that $x \in P_1^\tau$. The only nontrivial constraint to prove is $x(\delta^+_{\vec{G}}(U)) \geq \tau$ for every $U \subsetneq V, U \neq \emptyset$. If $\delta^-_D(U)$ is a dicut such that $\delta^+_D(U) = \emptyset$, then $x(\delta^+_{\vec{G}}(U)) = x(\delta^+_{E^-}(U)) = |\delta^-_D(U)| \geq \tau$. Otherwise, $\delta^+_D(U) \neq \emptyset$

and thus $\delta^+_{\vec{G}}(U)$ contains at least one arc in E^+. Moreover, since $|\delta_D(U)| \geq 2$, one has $x(\delta^+_{\vec{G}}(U)) = x(\delta^+_{E^+}(U)) + x(\delta^+_{E^-}(U)) \geq (\tau - 1) + 1 = \tau$. Thus, $x \in P_1^\tau$. By the assumption, $x = \sum_{i=1}^{\tau} \chi_{O_i}$ where each O_i is a strongly connected orientation. Take $J_i = \{e^+ \in E^+ \mid \chi_{O_i}(e^-) = 1, e^- \in E^-\}$. Note that $(A \setminus J_i) \cup (J_i^{-1}) = O_i$. Therefore, $(A \setminus J_i) \cup (J_i^{-1})$ is strongly connected, which means J_i is a strengthening of D, and thus a dijoin of D. Since $x_{e^-} = 1$ for each $e \in E$, J_i's are disjoint. Thus we get τ disjoint dijoins of D. We now prove the "only if" direction. Given an undirected graph $G = (V, E)$,

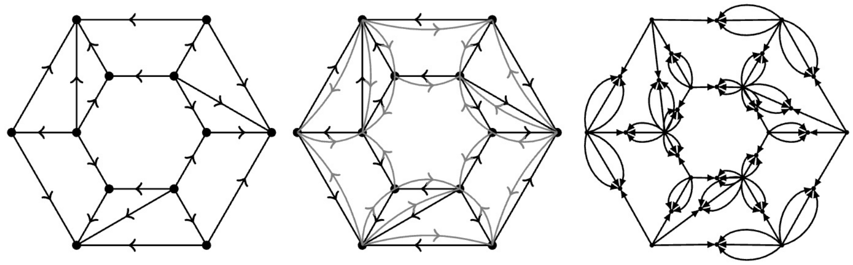

Fig. 3. (1) is a digraph with minimum dicut size 4. In (2) the weights of black arcs are 3 and the weights of gray arcs are 1. This figure illustrates how to convert from a digraph D (1) to a weighted digraph $\vec{G}$ (2) in the first part of the proof of Theorem 10 and how to convert from a weighted digraph $\vec{G}$ (2) to a digraph D (3) in the second part of the proof of Theorem 10.

consider the corresponding directed graph $\vec{G} = (V, E^+ \cup E^-)$ with each edge of E copied and oppositely oriented. For an integral $x \in P_1^\tau$, (e.g. Figure 3-(2)) construct a new digraph D from G in the following way. For each edge $e = (u, v) \in E$ where $e^+ = (u, v)$ and $e^- = (v, u)$, add a node w_e, add $x_{e^+} \geq 1$ arcs from u to w_e and $x_{e^-} \geq 1$ arcs from v to w_e, and delete e (as shown in Figure 3-(3)). We claim that the size of a minimum dicut of D is τ. Every vertex w_e induces a dicut $\delta^-_D(\{w_e\})$ of size $x_{e+}+x_{e^-} = \tau$. Thus, we only need to show that the size of every dicut of D is at least τ. Given U such that $\delta^-_D(U) = \emptyset$, if there exists $e = (u, v) \in E$ such that $u, v \in U$ but $w_e \notin U$, then $|\delta^+_D(U)| \geq |\delta^-_D(\{w_e\})| \geq \tau$. Thus, we may assume w.l.o.g. that for every $e = (u, v) \in E$ such that $u, v \in U$, we also have $w_e \in U$. Since $\delta^-_D(U) = \emptyset$, for every $e = (u, v) \in E$ such that $u, v \notin U$, we also have $w_e \notin U$. Moreover, for every $e = (u, v) \in E$ such that $u \in U$, $v \notin U$, since there is at least an arc from v to w_e but $\delta^-_D(U) = \emptyset$, we infer that $w_e \notin U$. Thus, $\delta^+_D(U) = \{uw_e \mid e = (u, v) \in E, u \in U, v \notin U\}$. Thus, by the way we construct D, $|\delta^+_D(U)| \geq x(\delta^+_{\vec{G}}(U)) \geq \tau$. Therefore, D has minimum dicut size τ. By Woodall's conjecture, there exists τ disjoint dijoins $J_1, ..., J_\tau$ in D. In particular, each dijoin intersects dicut $\delta^-_D(\{w_e\})$ exactly once since $|\delta^-_D(\{w_e\})| = \tau$. Let O_i be an orientation defined by $O_i := \{e^+ \mid uw_e \in J_i\} \cup \{e^- \mid vw_e \in J_i\}$. Note that O_i is indeed an orientation since exactly one of uw_e and vw_e is in J_i, for every $e \in E$. We claim that each O_i is a strongly connected orientation of G. Assume not. Then there exists $U \subseteq V$, such that $\delta^+_{\vec{G}}(U) \cap O_i = \emptyset$. Let $U' := U \cup \{w_e \mid e = (u, v) \in E, u, v \in U\}$. It is easy to see that U' is a dicut of D such that $\delta^-_D(U') = \emptyset$. It follows from $\delta^+_{\vec{G}}(U) \cap O_i = \emptyset$ that $\delta^+_D(U') \cap J_i = \emptyset$, a contradiction to J_i being a dijoin of D. Moreover, by the way we construct D, for each $e^+ = (u, v)$, $\sum_{i=1}^{\tau} \chi_{O_i}(e^+) = |\{J_i \mid uw_e \in J_i\}| = x_{e^+}$. For each $e^- = (v, u)$, $\sum_{i=1}^{\tau} \chi_{O_i}(e^-) = |\{J_i \mid vw_e \in J_i\}| = x_{e^-}$. Therefore, $\sum_{i=1}^{\tau} \chi_{O_i} = x$. This ends the proof of this direction. □

To prove Theorem 9, we need a structural lemma.

Lemma 2. *Let $D = (V, A)$ be a digraph with weight $w \in \{0, 1\}^A$ and assume that the minimum weight of a dicut is $\tau \geq 2$. Let $e \in A$ be some arc such that $w_e = 1$. If there exists a cut $\delta_D(U)$ such that $\delta^+_D(U) = \{e\}$ and $w(\delta^-_D(U)) = 0$, then e is not contained in any minimum dicut of D.*

Proof. Suppose not. Then there exists a dicut $\delta^-_D(W)$ such that $w(\delta^-_D(W)) = \tau$ and $e \in \delta^-_D(W)$. Let D' be obtained from D by deleting e. Then $\delta^-_{D'}(U)$ becomes a dicut of D'. Therefore, $\delta^-_{D'}(U \cap W)$ and $\delta^-_{D'}(U \cup W)$ are both dicuts of D'. However, since e leaves U and enters W, e goes from $U \setminus W$ to $W \setminus U$. Thus, $e \notin \delta_D(U \cap W)$ and $e \notin \delta_D(U \cup W)$. Therefore, both $\delta^-_D(U \cap W)$ and $\delta^-_D(U \cup W)$ are dicuts of D. Moreover, $w(\delta^-_D(U \cap W)) + w(\delta^-_D(U \cup W)) = w(\delta^-_D(U)) + w(\delta^-_D(W)) - 1 = \tau - 1$. It follows that $w(\delta^-_D(U \cap W)) \leq \tau - 1$ and $w(\delta^-_D(U \cup W)) \leq \tau - 1$. Notice that either $U \cap W \neq \emptyset$ or $U \cup W \neq V$. Otherwise, e is a bridge of D, contradicting to $\tau \geq 2$. Therefore, either $\delta^-_D(U \cap W)$ or $\delta^-_D(U \cup W)$ violates the assumption that the size of a minimum dicut is τ, contradiction. □

Proof of Theorem 9. We modify the proof of Theorem 10 to prove Theorem 9. We first prove the "if" direction. Let $D = (V, A)$ be a digraph with weight $w \in \{0, 1\}^A$ and minimum dicut $\tau \geq 2$. We can assume there is no arc $e \in A$ with weight 1 such that there exists a cut $\delta_D(U)$ such that $\delta^+_D(U) = \{e\}$ and $w(\delta^-_D(U)) = 0$. For otherwise, by Lemma 2, e is not contained in any minimum dicut, which means we can set the weight of e to be 0 without decreasing the size of a minimum dicut. Any packing of τ dijoins in the new graph will be a valid packing of τ dijoins of the old graph.

Let G be the underlying undirected graph of D and $\vec{G} = (V, E^+ \cup E^-)$ be defined as before such that $E^+ = A$ and $E^- = A^{-1}$. Define $x \in \mathbb{Z}^{E^+ \cup E^-}$ as follows. For weight 1 arcs $e^+ \in A$, we define $x_{e^+} = \tau - 1$ and $x_{e^-} = 1$ as before. For the weight 0 arcs $e^+ \in A$, we define $x_{e^+} = \tau$ and $x_{e^-} = 0$. To argue that $x(\delta^+_{\vec{G}}(U)) \geq \tau$ for every $U \subsetneq V, U \neq \emptyset$, if $\delta^-_D(U)$ is a dicut it follows in the same way as in the proof of Theorem 10. Therefore, without loss of generality, we assume there exists at least one arc $e^+ \in E^+$ in $\delta^+_D(U)$. If there exists such an arc with $w(e^+) = 0$, then $x(\delta^+_{\vec{G}}(U)) \geq x_{e^+} \geq \tau$. Otherwise, all the arcs in $\delta^+_D(U)$ have weight 1. If there exist at least 2 arcs of weight 1 in $\delta_D(U)$, we follow the same argument as in the earlier proof. The only case left is when $\delta^+_D(U)$ is a single arc of weight 1 and all the arc in $\delta^-_D(U)$ has weight 0, which has been excluded in the beginning. Therefore, we have proved $x(\delta^+_{\vec{G}}(U)) \geq \tau$ for every $U \subsetneq V, U \neq \emptyset$, which implies $x \in P^{\tau}_0$. By the assumption, $x = \sum_{i=1}^{\tau} \chi_{O_i}$ where each O_i is a strongly connected orientation. We define the dijoins in the same way as the other proof. Note that the dijoins are disjoint and never use weight 0 arcs. Therefore, we find τ dijoins that is a valid packing of graph D with weight w.

Next, we prove the "only if" direction. Given an undirected graph $G = (V, E)$, the corresponding $\vec{G} = (V, E^+ \cup E^-)$, and an integral $x \in P_0^\tau$, we construct weighted digraph D as follows. For an edge $e = (u, v)$ such that $x_{e^+}, x_{e^-} \geq 1$, we construct node w_e and arcs uw_e, vw_e in the same way as in the proof of Theorem 10. For an edge $e^+ = (u, v)$ such that $x_{e^+} = \tau$, $x_{e^-} = 0$, we add node w_e, add τ arcs of weight 1 from u to w_e and add a weight 0 arc from v to w_e. Similarly, for $e^+ = (u, v)$ with $x_{e^+} = 0$, $x_{e^-} = \tau$, we add node w_e, add a weight 0 arc from u to w_e and τ arcs of weight 1 from v to w_e. The same argument applies to see the minimum dicut size of D is τ. By Edmonds-Giles' conjecture, we can find τ disjoint dijoins in the weighted digraph D. As before, we can find τ strongly connected orientations accordingly that sum up to x. □

4 Conclusions and Discussions

We showed that every digraph with minimum dicut size τ can pack $\lfloor \frac{\tau}{6} \rfloor$ dijoins, or $\left\lfloor \frac{\tau p}{2p+1} \right\rfloor$ dijoins when the digraph is $6p$-edge-connected. The existence of nowhere-zero circular k-flow for a smaller k (< 6) when special structures are imposed on the underlying undirected graphs would lead to a better ratio, i.e. $\lfloor \frac{\tau}{k} \rfloor$, approximate packing of dijoins for those digraphs. The limitation of this approach is that we cannot hope that nowhere-zero 2-flows always exist because this is equivalent to the graph being Eulerian. Thus, bringing the number up to $\lfloor \frac{\tau}{2} \rfloor$ disjoint dijoins would be challenging using this approach. However, it is necessary for Woodall's conjecture to be true that every digraph with minimum dicut size τ contains two disjoint $\lfloor \frac{\tau}{2} \rfloor$-dijoins. Therefore, new ideas are needed to prove or disprove whether such a decomposition exists.

The careful reader may have noticed that the approach only works for the unweighted case. Yet, by a slight modification of the argument, it extends to the weighted case when the underlying undirected graph of the weight 1 arcs is 2-edge-connected. In this case, we can find a nowhere-zero k-flow on the weight 1 arcs and construct the decomposition of weight 1 arcs the same way as in Theorem 6. However, unfortunately, in general the underlying graph of weight 1 arcs may have bridges or be disconnected, in which case the above argument does not work. Studying a proper analogue of nowhere-zero flows in mixed graphs could be helpful in resolving the question in weighted digraphs.

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References

1. Ahmad Abdi, Gérard Cornuéjols, and Giacomo Zambelli. Arc connectivity and submodular flows in digraphs. 2023. https://www.andrew.cmu.edu/user/gc0v/webpub/connectedflip.pdf.

2. Ahmad Abdi, Gérard Cornuéjols, and Michael Zlatin. On packing dijoins in digraphs and weighted digraphs. *SIAM Journal on Discrete Mathematics*, 37(4):2417–2461, 2023.
3. Jørgen Bang-Jensen and Matthias Kriesell. Disjoint sub (di) graphs in digraphs. *Electronic Notes in Discrete Mathematics*, 34:179–183, 2009.
4. Jørgen Bang-Jensen and Anders Yeo. Decomposing k-arc-strong tournaments into strong spanning subdigraphs. *Combinatorica*, 24:331–349, 2004.
5. Maria Chudnovsky, Katherine Edwards, Ringi Kim, Alex Scott, and Paul Seymour. Disjoint dijoins. *Journal of Combinatorial Theory, Series B*, 120:18–35, 2016.
6. Matthew Devos. Woodall's conjecture. `http://www.openproblemgarden.org/op/woodalls_conjecture`.
7. Jack Edmonds. Edge-disjoint branchings. *Combinatorial Algorithms*, pages 91–96, 1973.
8. Jack Edmonds and Rick Giles. A min-max relation for submodular functions on graphs. In *Annals of Discrete Mathematics*, volume 1, pages 185–204. Elsevier, 1977.
9. Luis A Goddyn. Some open problems I like. `https://www.sfu.ca/~goddyn/Problems/problems.html`.
10. Luis A Goddyn, Michael Tarsi, and Cun-Quan Zhang. On (k, d)-colorings and fractional nowhere-zero flows. *Journal of Graph Theory*, 28(3):155–161, 1998.
11. François Jaeger. On nowhere-zero flows in multigraphs. *Proceedings, Fifth British Combinatorial Conference, Aberdeen*, pages 373–378, 1975.
12. François Jaeger. Flows and generalized coloring theorems in graphs. *Journal of Combinatorial Theory, series B*, 26(2):205–216, 1979.
13. François Jaeger. On circular flows in graphs. In *Finite and Infinite Sets*, pages 391–402. Elsevier, 1984.
14. François Jaeger. Nowhere-zero flow problems. *Selected Topics in Graph Theory*, 3:71–95, 1988.
15. Tamás Király. A result on crossing families of odd sets. 2007. `https://egres.elte.hu/tr/egres-07-10.pdf`.
16. László Lovász. *Combinatorial problems and exercises*, volume 361. American Mathematical Soc., 2007.
17. László Miklós Lovász, Carsten Thomassen, Yezhou Wu, and Cun-Quan Zhang. Nowhere-zero 3-flows and modulo k-orientations. *Journal of Combinatorial Theory, Series B*, 103(5):587–598, 2013.
18. András Mészáros. A note on disjoint dijoins. *Combinatorica*, 38(6):1485–1488, 2018.
19. Alexander Schrijver. Observations on Woodall's conjecture. `https://homepages.cwi.nl/~lex/files/woodall.pdf`.
20. Alexander Schrijver. A counterexample to a conjecture of Edmonds and Giles. *Discrete Mathematics*, 32:213–214, 1980.
21. Alexander Schrijver. Total dual integrality from directed graphs, crossing families, and sub- and supermodular functions. In *Progress in Combinatorial Optimization*, pages 315–361. Elsevier, 1984.
22. Paul D Seymour. Nowhere-zero 6-flows. *Journal of Combinatorial Theory, series B*, 30(2):130–135, 1981.
23. Bruce Shepherd and Adrian Vetta. Visualizing, finding and packing dijoins. *Graph Theory and Combinatorial Optimization*, pages 219–254, 2005.
24. Carsten Thomassen. The weak 3-flow conjecture and the weak circular flow conjecture. *Journal of Combinatorial Theory, Series B*, 102(2):521–529, 2012.
25. William Thomas Tutte. A contribution to the theory of chromatic polynomials. *Canadian Journal of Mathematics*, 6:80–91, 1954.
26. Douglas R Woodall. Menger and König systems. *Theory and Applications of Graphs: Proceedings, Michigan May 11–15, 1976*, pages 620–635, 1978.
27. D.H. Younger. Integer flows. *Journal of Graph Theory*, 7(3):349–357, 1983.