Approximately Packing Dijoins via Nowhere-Zero Flows

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Abstract. In a digraph, a dicut is a cut where all the arcs cross in one direction. A dijoin is a subset of arcs that intersects each dicut. Woodall conjectured in 1976 that in every digraph, the minimum size of a dicut is equal to the maximum number of disjoint dijoins. However, prior to our work, it was not even known whether at least 3 disjoint dijoins exist in a digraph whose minimum dicut size is arbitrarily large.

By building connections with nowhere-zero (circular) k-flows, we prove that every digraph with minimum dicut size τ contains $\left\lfloor \frac{\tau}{k} \right\rfloor$ disjoint dijoins if the underlying undirected graph admits a nowhere-zero (circular) k-flow. The existence of nowhere-zero 6-flows in 2-edge-connected graphs (Seymour 1981) directly leads to the existence of $\left\lfloor \frac{\tau}{6} \right\rfloor$ disjoint dijoins in any digraph with minimum dicut size τ, which can be found in polynomial time as well. The existence of nowhere-zero circular $(2+\frac{1}{p})$-flows in $6p$-edge-connected graphs (Lovász et al 2013) directly leads to the existence of $\left\lfloor \frac{\tau p}{2p+1} \right\rfloor$ disjoint dijoins in any digraph with minimum dicut size τ whose underlying undirected graph is $6p$-edge-connected.

1 Introduction

Given a digraph $D = (V, A)$, for any $U \subsetneq V, U \neq \emptyset$, denote by $\delta^+_D(U)$ and $\delta^-_D(U)$ the arcs leaving and entering U in D, resp. And denote by $\delta_D(U) := \delta^+_D(U) \cup \delta^-_D(U)$ the cut induced by U. A *dicut* is an arc subset of the form $\delta^+(U)$ such that $\delta^-(U) = \emptyset$, or the other way. A *dijoin* is a subset $J \subseteq A$ that intersects every dicut at least once. We say that a dijoin is *minimal* if it is not contained in another dijoin. A *k-dijoin* is a subset $J \subseteq A$ that intersects every dicut at least k times. If D is a weighted digraph where the weight is $w : A \to \mathbb{Z}_+$, then we say that D can *pack* k dijoins if there exist k dijoins $J_1, \ldots, J_k$ of D such that no arc e is contained in more than $w(e)$ of those k dijoins. And in this case we say that $J_1, \ldots, J_k$ is a *packing* of D under weight w. The *value* of the packing is the number of dijoins which is k. We can assume without loss of generality that $w \in \{0,1\}^A$ because we can always replace an arc e with weight $w(e) > 1$ by $w(e)$ parallel arcs. Edmonds and Giles [9] conjectured the following.

Conjecture 1 (Edmonds-Giles). For any digraph $D = (V, A)$ with weight $w \in \{0,1\}^A$, if the minimum weight of a dicut is τ, then D can pack τ dijoins.

This was disproved by Schrijver [23]. However, the following unweighted version of the Edmonds-Giles conjecture where all arc weights are 1, which was proposed by Woodall [31], is still open.

Conjecture 2 (Woodall). For every digraph, the size of the minimum dicut equals the maximum number of disjoint dijoins.

Several weakenings of Woodall's conjecture have been made in the literature. It has been conjectured that there exists some integer $\tau \geq 3$ such that every digraph with minimum dicut size at least τ contains 3 disjoint dijoins [7]. Shepherd and Vetta [28] raised the following question. Let $f(\tau)$ be the maximum value such that every weighted digraph whose minimum weight of a dicut is at least τ, contains a weighted packing of $f(\tau)$ dijoins. They conjectured that $f(\tau)$ is order $\Omega(\tau)$. In Section 3, we give an affirmative answer to this conjecture in the *unweighted* case. The main results in this paper are the following approximate versions of Woodall's Conjecture.

Theorem 1. *Any digraph $D = (V, A)$ with minimum dicut size τ can be decomposed into $\left\lfloor \frac{\tau}{6} \right\rfloor$ disjoint dijoins in polynomial time.*

Theorem 2. *Let p be a positive integer. Any digraph $D = (V, A)$ with minimum dicut size τ and with the property that its underlying undirected graph is $6p$-edge-connected can be decomposed into $\left\lfloor \frac{\tau p}{2p+1} \right\rfloor$ disjoint dijoins.*

Both the above are consequences of the following main theorem.

Theorem 3. *For any digraph $D = (V, A)$ with minimum dicut size τ, if the underlying undirected graph admits a nowhere-zero (circular) k-flow, where $k \geq 2$ is a rational, then D can be decomposed into $\left\lfloor \frac{\tau}{k} \right\rfloor$ disjoint dijoins.*

Our approach to proving the above results is to first augment the input digraph D by adding reverse arcs for all input arcs and assigning weights τ to the original arcs and 1 to the newly added reverse arcs. Denote the augmented digraph by $\vec{G}$ with weight w^D. Define a *τ-strongly-connected digraph (τ-SCD)* to be a weighted digraph such that the arcs leaving every cut have weight at least τ. In particular, a *strongly connected digraph (SCD)* is a 1-SCD. It is not hard to see that for a digraph D with minimum dicut size τ, the augmented digraph $\vec{G}$ with weight w^D is τ-strongly-connected. We then show that decomposing the original graph into some $\tau' \leq \tau$ dijoins is equivalent to decomposing the augmented weighted digraph $\vec{G}$ into τ' strongly connected digraphs (Proposition 2).

We then draw a connection to nowhere-zero flows: For any positive integer k, a *nowhere-zero k-flow* in an *undirected* graph is an assignment of nonzero integral flow values to some orientation of all its edges such that the flow is conserved at every node, and the flow values are integers between 1 and $k-1$. This concept extends to any rational number $k > 2$. A *nowhere-zero circular k-flow* in an undirected graph is defined by relaxing the integrality requirement on the flow values to be any rational numbers in $[1, k-1]$. There is a rich literature on the existence of nowhere zero k-flows from which we will use two important results: Seymour [27] showed that there is always a nowhere-zero 6-flow in 2-edge-connected graphs. Younger [32] gave a polynomial time algorithm to construct a nowhere-zero 6-flow in 2-edge-connected graphs. Lovász et al. [19] proved that for any positive integer p, there is always a nowhere-zero circular $(2 + \frac{1}{p})$-flow in $6p$-edge-connected graphs.

Returning to dijoins and our augmented graph $\vec{G}$, we need to decompose this augmented graph into some $\tau' \leq \tau$ strongly connected digraphs. Decomposing a digraph into strongly connected digraphs is a notoriously hard problem. It is not known whether there exists any integer τ such that a τ-strongly-connected digraph can be decomposed into 2 disjoint strongly connected digraphs [4]. Finding a minimum subdigraph that is strongly connected is NP-hard as well [10]. Hence a common approximate approach to finding strongly connected subgraphs is to find a pair of in and out r-arborescences from the same root r and take their union. However, even finding such a pair of arborescences in a given digraph is still hard. For instance, it is still open whether there exists some integer τ such that a τ-strongly-connected digraph can pack one in-arborescence and one out-arborescence [3].

To get around this difficulty, we use the approximation approach, and reduce our goal to finding *two* subdigraphs of $\vec{G}$, each of which can be decomposed into τ' in or out r-arborescences for some fixed root r. We crucially argue (in Theorem 7) that if the underlying undirected graph of D admits a nowhere-zero k-flow, then the weight w^D can be decomposed into two disjoint $\lfloor \frac{\tau}{k} \rfloor$-SCD's. Using Edmonds' disjoint arborescences theorem [8], we can now extract $\lfloor \frac{\tau}{k} \rfloor$ different in r-arborescences from the first and the same number of out r-arborescences from the second. Pairing them up gives us the final set of $\lfloor \frac{\tau}{k} \rfloor$ strongly connected digraphs. Our results then follow from the prior theorems about the existence of nowhere-zero flows.

In Section 4, we give equivalent forms of Woodall's conjecture and the Edmonds-Giles conjecture, respectively, in terms of packing strongly connected orientations, which are of independent interest. Given any undirected graph $G = (V, E)$, let $\vec{G} = (V, E^+ \cup E^-)$ be a digraph obtained by making two copies of each edge $e \in E$ and directing them oppositely, one being $e^+ \in E^+$ and the other being $e^- \in E^-$. A *τ-strongly connected orientation (τ-SCO)* of G is a multi-subset of arcs from $E^+ \cup E^-$ picking exactly τ arcs (possibly with repetitions) for each e such that at least τ arcs leave every cut. In particular, a *strongly connected orientation (SCO)* is a 1-SCO. We prove in Theorem 9 that the Edmonds-Giles conjecture is true for all digraph if and only if for any undirected graph G, a τ-SCO can be decomposed into τ disjoint SCO's. By contrast, we define w to be a *nowhere-zero τ-SCO* if it is a τ-SCO and $w_e \geq 1$ for any arc e. In Theorem 10, we prove that Woodall's conjecture is true for all digraphs if and only if for any undirected graph G, a *nowhere-zero* τ-SCO can be decomposed into τ disjoint *SCO*'s.

In Section 5, we extend decomposing the special weight w^D into decomposing any integral weight w that is a *nowhere-zero τ-SCD*, i.e. $w_e \geq 1$ for any arc e. We show that if the underlying undirected graph admits a nowhere-zero (circular) k-flow for some rational number $k \geq 2$, then any nowhere-zero τ-SCD can pack $\lfloor \frac{\tau}{k+1} \rfloor$ strongly connected digraphs.

Related Work

Shepherd and Vetta [28] raised the question of approximately packing dijoins. They introduced the idea of adding reverse arcs to make the digraph τ-strongly-connected and combining solutions from in and out r-arborescences. They use this idea to get a half integral packing of value $\frac{\tau}{2}$ in any digraph with minimum dicut size τ. It is conjectured by Király [17] that any digraph with minimum dicut size τ contains two disjoint $\frac{\tau}{2}$ dijoins, see also in [1]. One might notice that if this conjecture is true, together with the approach of combining in and out r-arborescences, one can show that there exist $\lfloor \frac{\tau}{2} \rfloor$

disjoint dijoins in every digraph with minimum dicut size τ. Abdi et al. [2] proved that any digraph can be decomposed into a dijoin and a $(\tau-1)$-dijoin. Abdi et al. [1] showed that any digraph with minimum dicut size τ can be decomposed into a k-dijoin and a $(\tau-k)$-dijoin for any integer $k \in \{1, ..., \tau-1\}$ under the condition that the underlying undirected graph is τ-edge-connected. Mészáros [20] proved that when the underlying undirected graph is $(q-1,1)$-partition-connected for some prime power q, the digraph can be decomposed into q disjoint dijoins. However, none of these approaches tell us how to decompose a digraph with minimum dicut size τ into a large number of disjoint dijoins without connectivity requirements. We also refer to the papers that view the problem from the perspective of reorienting the directions of a subset of arcs to make the graph strongly connected, such as [22,1,6]. For background on nowhere-zero k-flows, we refer the reader to [13,15,14,27,29,19,32] and to the excellent survey by Jaeger [16].

2 Technical Background

Notation. Given a graph $G=(V,E)$, for any $U \subseteq V$, let $\delta_G(U)$ denote the cut induced by U. For any edge subset $F \subseteq E$, let $\delta_F(U) := \delta_G(U) \cap F$ denote the edges in F that are also in the cut induced by U. Given a digraph $D=(V,A)$, for any $U \subseteq V$, denote by $\delta^+_D(U)$ and $\delta^-_D(U)$ the arcs leaving and entering U in D, respectively. Denote by $\delta_D(U) := \delta^+_D(U) \cup \delta^-_D(U)$ the cut induced by U. Similarly, for any arc subset $B \subseteq A$, denote by $\delta^+_B(U) := \delta^+_D(U) \cap B$ and $\delta^-_B(U) := \delta^-_D(U) \cap B$ the arcs in B that leave and enter U, respectively, and $\delta_B(U) := \delta_D(U) \cap B$ the arcs in B that are also in the cut induced by U. Denote by e^{-1} the reverse of arc $e \in A$ and B^{-1} the arcs obtained from reversing the directions of the arcs in $B \subseteq A$. Given any undirected graph $G=(V,E)$, make two copies of each edge $e=(u,v) \in E$ and direct one to be $e^+=(u,v)$ and the other to be $e^-=(v,u)$ (the assignment of (u,v) and (v,u) to e^+ and e^- is arbitrary). Denote by $E^+ := \cup_{e \in E}\{e^+\}$ and $E^- := \cup_{e \in E}\{e^-\}$. Each of E^+ and E^- is an *orientation* of G satisfying $E^- = (E^+)^{-1}$. Denote the resulting digraph by $\vec{G} = (V, E^+ \cup E^-)$. For any $f : E \to \mathbb{R}$, denote by $f(F) := \sum_{e \in F} f(e)$, $\forall F \subseteq E$. For any $F \subseteq E$, denote by $\chi_F \in \{0,1\}^E$ the characteristic vector of F.

2.1 Strongly Connected Digraphs

Let $G=(V,E)$ be any undirected graph and $\vec{G} = (V, E^+ \cup E^-)$ be the digraph obtained by copying each edge of G twice and directing them oppositely. A *strongly connected (sub)digraph (SCD)* of G is a subset of arcs $F \subseteq E^+ \cup E^-$ such that F spans V and the digraph induced by F is strongly connected. Or equivalently, for any $U \subsetneq V, U \neq \emptyset$, $|F \cap \delta^+_{\vec{G}}(U)| \geq 1$. For any integral vector $x \in \mathbb{Z}^{E^+ \cup E^-}$, we say x is a *k-strongly connected digraph (k-SCD)* of G if it satisfies that $|x(\delta^+_{\vec{G}}(U))| \geq k, \forall U \subsetneq V, U \neq \emptyset$.

We further recall a classical theorem about decomposing digraphs into arborescences. In a digraph $D=(V,A)$, when fixing any root r, an *out (in) r-arborescence* is a directed spanning tree such that each vertex in $V \setminus \{r\}$ has exactly one arc entering (leaving) it. When the root is not fixed it is called an *out (in) arborescence*. Edmonds's disjoint arborescences theorem [8] states that when fixing a root r, any *rooted-τ-connected* digraph, i.e. $|\delta^+_D(U)| \geq \tau, \forall U \subsetneq V, r \in U$, can be decomposed into τ disjoint out r-arborescences. Furthermore, this decomposition can be done in polynomial time.

Theorem 4 ([8]). *Given any digraph D and a root r, if D is rooted-τ-connected, then D can be decomposed into τ disjoint out r-arborescences in polynomial time.*

A τ-strongly-connected digraph is in particular rooted-τ-connected. Therefore, fixing any root $r \in V$, a τ-strongly-connected digraph can be decomposed into τ disjoint out r-arborescences. If we switch the direction, we can readily see any τ-strongly-connected digraph can also be decomposed into τ disjoint in r-arborescences.

Corollary 1. *Given any τ-strongly-connected digraph D and a root r, D can be decomposed into τ disjoint out (in) r-arborescences in polynomial time.*

2.2 Nowhere-Zero Flows in Undirected Graphs

Let $G = (V, E)$ be an undirected graph and k an integer such that $k \geq 2$. Introduced by Tutte [30], a *nowhere-zero k-flow* of G is an orientation E^+ and $f : E^+ \to \{1, 2, ..., k-1\}$ such that $\sum_{e \in \delta^+_{E^+}(v)} f(e) = \sum_{e \in \delta^-_{E^+}(v)} f(e)$ for every vertex $v \in V$. Another way to look at it is to let E^- be the orientation obtained by reversing the arcs in E^+ and let $\vec{G} = (V, E^+ \cup E^-)$. Moreover, let $f(e) = -f(e^{-1})$ for each arc $e \in E^-$. f is a nowhere-zero k-flow if and only if $\sum_{e \in \delta^+_{\vec{G}}(v)} f(e) = 0, \forall v \in V$. Lots of efforts have been made to find the smallest k such that there exists a nowhere-zero k-flow (see e.g. in [16]). The following is the famous Tutte's 3-flow conjecture.

Conjecture 3 (Tutte, 3-flow). Every 4-edge-connected graph has a nowhere-zero 3-flow.

Goddyn et al. [12] introduced the fractional version of nowhere-zero k-flow. Let p, q be two integers such that $0 < p < q$. A *nowhere-zero circular $(1 + \frac{q}{p})$-flow* of G is an orientation E^+ and $f : E^+ \to \{p, p+1, ..., q\}$, such that $\sum_{e \in \delta^+_{E^+}(v)} f(e) = \sum_{e \in \delta^-_{E^+}(v)} f(e)$. From the integrality of the flow polytope, it is easy to see that this definition is equivalent to allowing f to take any rational number in $[p, q]$, or equivalently defining $f : E^+ \to [1, \frac{q}{p}]$. It has been shown that if G admits a nowhere-zero circular α-flow, then it also admits a nowhere-zero circular β-flow for any $\beta \geq \alpha$, where α and β are rational numbers [12].

Jaeger [15] generalizes Tutte's 3-flow conjecture to the following weak circular flow conjecture.

Conjecture 4 (Jaeger, weak circular flow [15]). Let p be any positive integer. Every $4p$-edge-connected graph has a nowhere-zero circular $(2 + \frac{1}{p})$-flow.

There are several relaxations of Conjecture 3 and 4 that have been proved. Jaeger [14,13] proved that for any 2-edge-connected graph there exists a nowhere-zero 8-flow, which was improved by Seymour [27] showing that there is always a nowhere-zero 6-flow in 2-edge-connected graphs. Younger [32] gave an algorithmic version of this result.

Theorem 5 ([27,32]). *Every* 2*-edge-connected graph admits a nowhere-zero* 6*-flow which can be found in polynomial time.*

Thomassen [29] proved a weaker version of Conjecture 4 with higher connectivity requirements (quadratic in p). Lovász et al. [19] improved the argument to $6p$-edge-connected graphs, i.e.

Theorem 6 ([19]). *Every* $6p$*-edge-connected graph admits a nowhere-zero circular* $(2+\frac{1}{p})$*-flow.*

As a consequence, Conjecture 3 is proved to be true for 8-edge-connected graphs by [29] and has been improved to 6-edge-connected graphs by [19].

As pointed out in different places (e.g. see in [12], [29], [11]), nowhere-zero integer flow and circular flow are closely related to nearly balanced orientations. We summarize the key fact we are going to use in the following lemma. Since we will reuse this fact we also give the proof here.

Lemma 1. *Given any undirected graph* $G=(V,E)$ *that admits a nowhere-zero (circular)* k*-flow* E^+ *and* $f: E^+ \to [1, k-1]$*, where* $k \geq 2$ *is a rational number. Let* $E^- = (E^+)^{-1}$*. Then, for any* $U \subsetneq V, U \neq \emptyset$*,*

$$\frac{1}{k}|\delta_G(U)| \leq |\delta^+_{E^+}(U)|, |\delta^+_{E^-}(U)| \leq \frac{k-1}{k}|\delta_G(U)|.$$

Proof. f satisfies flow conservation, i.e. $f(\delta^+_{E^+}(U)) = f(\delta^-_{E^+}(U)), \forall U \subsetneq V, U \neq \emptyset$. Thus, one has $1 \cdot |\delta^+_{E^+}(U)| \leq f(\delta^+_{E^+}(U)) = f(\delta^-_{E^+}(U)) \leq (k-1) \cdot |\delta^-_{E^+}(U)|$. It follows from the fact $|\delta_G(U)| = |\delta^+_{E^+}(U)| + |\delta^-_{E^+}(U)|$ that $\frac{1}{k}|\delta_G(U)| \leq |\delta^+_{E^+}(U)| \leq \frac{k-1}{k}|\delta_G(U)|$. The inequalities of $|\delta^+_{E^-}(U)|$ follows from $|\delta^+_{E^+}(U)| + |\delta^+_{E^-}(U)| = |\delta_G(U)|$ and the inequalities of $|\delta^+_{E^+}(U)|$. □

3 An Approximate Packing of Dijoins

In this section we prove our main Theorem 3. We begin by observing that the existence of a nowhere-zero (circular) flow implies that there is a nearly balanced orientation in the sense that, for each cut, the number of arcs entering it differs by a constant factor of the number of arcs leaving it (Lemma 1). This implies both the subdigraph consisting of the arcs that are in the same orientation as the nowhere-zero (circular) flow and its complement intersect every dicut by a large proportion of its size. This gives us a way to decompose the digraph into two subdigraphs each intersecting every dicut with a large number of arcs (see an example in Figure 1).

Proposition 1. *For any digraph* $D=(V,A)$ *with minimum dicut size* τ*, if the underlying undirected graph admits a nowhere-zero (circular)* k*-flow for some rational number* $k \geq 2$*, then* D *can be decomposed into two disjoint* $\lfloor \frac{\tau}{k} \rfloor$*-dijoins.*

Proof. Let E^+ and $f: E^+ \to [1, k-1]$ be a nowhere-zero (circular) k-flow of the underlying undirected graph G of D. By Lemma 1, $\frac{1}{k}|\delta_G(U)| \leq |\delta^+_{E^+}(U)| \leq \frac{k-1}{k}|\delta_G(U)|$

for any $U \subsetneq V, U \neq \emptyset$. Take $J = A \cap E^+$ to be the arcs that have the same directions in A and E^+. Then, for any dicut $\delta^+_D(U)$ such that $\delta^-_D(U) = \emptyset$, we have $|J \cap \delta^+_D(U)| = |\delta^+_{E^+}(U)| \geq \frac{1}{k}|\delta_G(U)| = \frac{1}{k}|\delta^+_D(U)| \geq \frac{\tau}{k}$ and $|(A \setminus J) \cap \delta^+_D(U)| = |\delta^+_D(U)| - |\delta^+_{E^+}(U)| \geq |\delta^+_D(U)| - \frac{k-1}{k}|\delta_G(U)| = |\delta^+_D(U)| - \frac{k-1}{k}|\delta^+_D(U)| \geq \frac{\tau}{k}$. Thus, both J and $A \setminus J$ are $\lfloor \frac{\tau}{k} \rfloor$-dijoins. $\square$

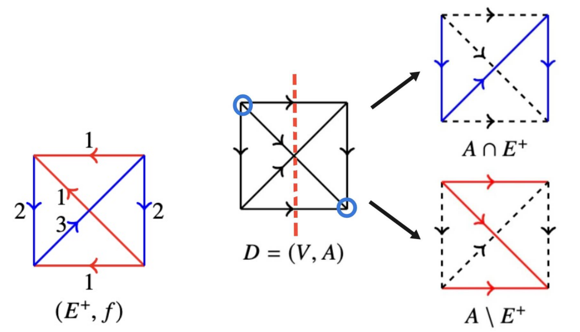

Fig. 1. (E^+, f) is a nowhere-zero 4-flow of $G = K_4$. $D = (V, A)$ can be decomposed into a dijoin $A \cap E^+$ and a 2-dijoin $A \setminus E^+$.

In a digraph with minimum dicut value τ, although Proposition 1 says the arcs of the graph can be decomposed into two $\lfloor \frac{\tau}{k} \rfloor$-dijoins, there is no guarantee that the graph induced by each $\lfloor \frac{\tau}{k} \rfloor$-dijoin has minimum dicut size at least $\lfloor \frac{\tau}{k} \rfloor$. This is because some non-dicut can become a dicut when we delete arcs, which can potentially have very small size, even though each subgraph is a $\lfloor \frac{\tau}{k} \rfloor$-dijoin of the original digraph. This is a general difficulty of doing induction in decomposing the graph into dijoins.

The key observation here is switching to the setting of strongly connected digraphs, we can address this issue. Given a digraph $D = (V, A)$ with minimum dicut size τ, let G be the underlying undirected graph of D and $\vec{G} = (V, E^+ \cup E^-)$ be the digraph obtained by copying each edge of G twice and directing them oppositely. For convenience, we let $E^+ = A$ and $E^- = A^{-1}$. We augment the digraph D by adding reverse arcs for all arcs of D and assigning weights τ to the original arcs and 1 to the reverse arcs. In other words, define the *weights associated with* D to be $w^D \in \mathbb{Z}^{E^+ \cup E^-}$ such that $w^D_{e^+} = \tau, \forall e^+ \in E^+$ and $w^D_{e^-} = 1, \forall e^- \in E^-$. It is easy to see that w^D is a τ-SCD of G. Indeed, for any $U \subsetneq V, U \neq \emptyset$ such that $\delta^+_D(U) \neq \emptyset$, there exists some arc $e^+ \in E^+$ such that $e^+ \in \delta^+_{\vec{G}}(U)$, and thus $w^D(\delta^+_{\vec{G}}(U)) \geq w^D_{e^+} = \tau$. Otherwise, $\delta^+_D(U) = \emptyset$ which means $\delta^-_D(U)$ is a dicut. Therefore, $w^D(\delta^+_{\vec{G}}(U)) = w^D(\delta^+_{E^-}(U)) = |\delta^-_D(U)| \geq \tau$. This means the augmentation makes $\vec{G}$ with weight w^D τ-strongly-connected. We first reformulate packing dijoins in D into packing strongly connected digraphs in $\vec{G}$ under weight w^D and then prove a decomposition result regarding k-strongly connected digraphs with the help of nowhere-zero (circular) flows.

Proposition 2. *For any integer $k \le \tau$, the digraph D can be decomposed into k disjoint dijoins if and only if $\vec{G}$ with weight w^D can pack k strongly connected digraphs.*

Proof. Let $F_1, ..., F_k$ be k strongly connected digraphs of G that is a packing of $\vec{G}$ under weight w^D. Define $J_i := \{e^+ \in E^+ \mid \chi_{F_i}(e^-) = 1, e^- \in E^-\}$. We claim each J_i is a dijoin of D. Suppose not. Then there exists some dicut $\delta^-_D(U)$ such that $J_i \cap \delta^-_D(U) = \emptyset$. This implies $F_i \cap \delta^+_{\vec{G}}(U) = \emptyset$, contradicting the fact that F_i is a strongly connected digraph of $\vec{G}$. Moreover, since $w^D_{e^-} = 1$, at most one of $F_1, ..., F_k$ uses e^-, $\forall e^- \in E^-$. Thus, at most one of $J_1, ..., J_k$ uses $e^+, \forall e^+ \in E^+ = A$. Therefore, $J_1, ..., J_k$ are disjoint dijoins of D. Conversely, let $J_1, ..., J_k$ be a packing of dijoins in D under weight w. W.l.o.g. we can assume each J_i is a minimal dijoin. A. Frank showed (see e.g. in [18] Chapter 6) that each minimal dijoin is a strengthening, i.e. $(A \setminus J_i) \cup J_i^{-1}$ is a strongly connected digraph. Let $F_i := (A \setminus J_i) \cup J_i^{-1}$, $\forall i$. The same argument applies to see $F_1, ..., F_k$ is a valid packing of strongly connected digraphs in $\vec{G}$ under weight w^D. □

Theorem 7. *Let $D = (V, A)$ be a digraph with minimum dicut size τ. If the underlying undirected graph $G = (V, E)$ admits a nowhere-zero (circular) k-flow for some rational number $k \ge 2$, then the weight w^D associated with D can be decomposed into two disjoint $\lfloor \frac{\tau}{k} \rfloor$-SCD's.*

Proof. Let E^+ and $f : E^+ \to \{1, ..., k-1\}$ be a nowhere-zero k-flow of G. Let E^- be obtained by reversing the arcs of E^+. Let G be the underlying undirected graph of D and $\vec{G} = (V, E^+ \cup E^-)$. Construct $x \in \mathbb{Z}^{E^+ \cup E^-}$ as follows.

$$x_e = \begin{cases} \lceil \frac{\tau}{2} \rceil, & e \in A \cap E^+ \\ \lfloor \frac{\tau}{2} \rfloor, & e \in A \cap E^- \\ 1, & e \in A^{-1} \cap E^+ \\ 0, & e \in A^{-1} \cap E^- \end{cases}, \quad \text{and equivalently} \quad (w^D - x)_e = \begin{cases} \lfloor \frac{\tau}{2} \rfloor, & e \in A \cap E^+ \\ \lceil \frac{\tau}{2} \rceil, & e \in A \cap E^- \\ 0, & e \in A^{-1} \cap E^+ \\ 1, & e \in A^{-1} \cap E^- \end{cases}.$$

We prove that both x and $(w^D - x)$ are $\lfloor \frac{\tau}{k} \rfloor$-SCD's. We discuss two cases.

If $|\delta_G(U)| \ge \tau$, then the weight x_e of all the arcs $e \in \delta^+_{E^+}(U)$ will be rounded up to $\left\lceil \frac{w^D_e}{2} \right\rceil \ge \lceil \frac{1}{2} \rceil = 1$. Therefore, $x(\delta^+_{\vec{G}}(U)) \ge x(\delta^+_{E^+}(U)) \ge |\delta^+_{E^+}(U)| \ge \frac{1}{k}|\delta_G(U)| \ge \frac{\tau}{k}$, where the third inequality follows from Lemma 1. Symmetrically, the weight $(w^D - x)_e$ of all the arcs $e \in \delta^+_{E^-}(U)$ will be rounded up to $\left\lceil \frac{w^D_e}{2} \right\rceil \ge \lceil \frac{1}{2} \rceil = 1$. Therefore, $(w^D - x)(\delta^+_{\vec{G}}(U)) \ge (w^D - x)(\delta^+_{E^-}(U)) \ge |\delta^+_{E^-}(U)| \ge \frac{1}{k}|\delta_G(U)| \ge \frac{\tau}{k}$.

Otherwise, $|\delta_G(U)| < \tau$ and thus $\delta_D(U)$ is not a dicut. Then, $\delta^+_{\vec{G}}(U) \cap A \ne \emptyset$. Therefore, $x(\delta^+_{\vec{G}}(U)) \ge x(\delta^+_{\vec{G}}(U) \cap A) \ge \lfloor \frac{\tau}{2} \rfloor \ge \lfloor \frac{\tau}{k} \rfloor$ since $k \ge 2$. And $(w^D - x)(\delta^+_{\vec{G}}(U)) \ge (w^D - x)(\delta^+_{\vec{G}}(U) \cap A) \ge \lfloor \frac{\tau}{2} \rfloor \ge \lfloor \frac{\tau}{k} \rfloor$ since $k \ge 2$. Therefore, both x and $(w^D - x)$ are $\lfloor \frac{\tau}{k} \rfloor$-SCD's. □

Proof of Theorem 3

From Proposition 2, given a digraph D, we can reduce packing dijoins of D into packing strongly connected digraphs of the augmented digraph $\vec{G}$ with weight w^D which is

τ-strongly-connected. Note that using Corollary 1, we can decompose this digraph into τ in r-arborescences, or into τ out r-arborescences. If we pair each in r-arborescence with an out r-arborescence, we will obtain τ strongly connected digraphs. However, each arc can be used in both in and out r-arborescences. Shepherd and Vetta [28] use this idea to obtain a half integral packing of dijoins of value $\frac{\tau}{2}$. Yet, finding disjoint in and out arborescences together is quite challenging. As we noted earlier, it is open whether there exists any τ such that a τ-strongly-connected digraph can even pack one in-arborescence and one out-arborescence [3].

Theorem 7 paves a way to approximately packing disjoint in and out arborescences. Fixing any root r, if we are able to decompose the graph into two τ'-strongly-connected graphs and thereby find τ' disjoint in r-arborescences in the first graph and τ' disjoint out r-arborescences in the second graph, then we can combine them to get a strongly connected digraph.

Proof (Theorem 3). By Proposition 2, it suffices to prove w^D can pack $\lfloor\frac{\tau}{k}\rfloor$ strongly connected digraphs. By Theorem 7, $\vec{G}$ with weight w^D can be decomposed into weighted digraphs J_1 and J_2 such that each of them is $\lfloor\frac{\tau}{k}\rfloor$-strongly-connected. Fixing any root r, by Corollary 1, J_1 can be decomposed into $\lfloor\frac{\tau}{k}\rfloor$ disjoint out r-arborescences $S_1, ..., S_{\lfloor\frac{\tau}{k}\rfloor}$ and J_2 can be decomposed into $\lfloor\frac{\tau}{k}\rfloor$ disjoint in r-arborescences $T_1, ..., T_{\lfloor\frac{\tau}{k}\rfloor}$. Let $F_i := S_i \cup T_i$, for $i = 1, ..., \lfloor\frac{\tau}{k}\rfloor$. Each F_i is a strongly connected digraph. This is because every out r-cut $\delta^+_{\vec{G}}(U), r \in U$ is covered by S_i and every in r-cut $\delta^+_{\vec{G}}(U), r \notin U$ is covered by T_i and thus every cut $\delta^+_{\vec{G}}(U)$ is covered by F_i. Therefore, $F_1, ..., F_{\lfloor\frac{\tau}{k}\rfloor}$ forms a packing of strongly connected digraphs under weight w^D. □

Theorem 1 now follows by combining Theorem 3 and Theorem 5 and noting that the underlying undirected graph of any digraph with minimum dicut size $\tau \geq 2$ is 2-edge-connected. By Theorem 5, the nowhere-zero 6-flow can be found in polynomial time, and thus the decomposition described in Theorem 7 can apparently be done in polynomial time. Moreover, further decomposing J_1 and J_2 into in and out r-arborescences can also be done in polynomial time due to Corollary 1. Thus in the end we can find a decomposition of $\lfloor\frac{\tau}{6}\rfloor$ disjoint dijoins in polynomial time. Theorem 2 now follows by combining Theorem 3 and Theorem 6. However, as far as we know there is no constructive versions of Theorem 6, which means Theorem 2 cannot be made algorithmic directly.

4 A Reformulation of Woodall's Conjecture by Strongly Connected Orientations

In this section, we discuss the relation between packing dijoins, strongly connected orientations and strongly connected digraphs. We also discuss another reformulation of Woodall's Conjecture in terms of strongly connected orientations.

Given an undirected graph $G = (V, E)$, let $\vec{G} = (V, E^+ \cup E^-)$ be the digraph obtained by copying each edge of G twice and orienting them oppositely. An *orientation* of G is a subset of $E^+ \cup E^-$ consisting of exactly one of e^+ and e^- for each $e \in E$. Note that E^+ itself is an orientation, so is E^-. For an orientation O, let $\chi_O \in \{0, 1\}^{E^+ \cup E^-}$ be the characteristic vector of O. For any $U \subseteq V$, denote by $\delta^+_O(U) := \delta^+_{\bar{G}}(U) \cap O$ the arcs in the orientation O that leave U. An orientation O is a *strongly connected orientation (SCO)* if $|\delta^+_O(U)| \geq 1$ for any $U \subsetneq V, U \neq \emptyset$. Note that each strongly connected orientation is a strongly connected digraph. Let

$$SCO(G) := \left\{x \in \{0,1\}^{E^+ \cup E^-} \middle| x = \chi_O \text{ for some strongly connected orientation } O \text{ of } G\right\}. \tag{1}$$

Given a directed graph $D = (V, A)$, a *strengthening* is a subset $J \subseteq A$ such that by flipping J the graph becomes strongly connected [22]. Note that a strengthening is necessarily a dijoin.

Schrijver observed the following reformulation of Woodall's conjecture in terms of strengthenings in his unpublished note ([22] Section 2).

Theorem 8 ([22]). *Woodall's conjecture is true for all digraphs if and only if every digraph with minimum dicut size τ can be partitioned into τ strengthenings.*

Another way to look at $SCO(G)$ is to fix a direction E^+ and view it as a lift of the set of strengthenings of $G^+ = (V, E^+)$. Indeed, given any strengthening $J \subseteq E^+$, $(E^+ \setminus J) \cup J^{-1}$ is a strongly connected orientation of G. And given any strongly connected orientation $O \subseteq E^+ \cup E^-$, $E^+ \setminus O$ is a strengthening of G^+. The characteristic vectors of the strengthenings of G^+ are the 0, 1 vectors in the following polytope:

$$\begin{aligned}\Big\{x \in \mathbb{R}^{E^+} \Big|\, & 0 \leq x_{e^+} \leq 1,\ \forall e \in E,\\ & x(\delta^-(U)) - x(\delta^+(U)) \leq |\delta^-(U)| - 1, \forall U \subsetneq V, U \neq \emptyset\Big\}.\end{aligned}$$

Due to Edmonds and Giles' theorem of submodular flow [9], this is an integral polytope and thus it describes the convex hull of the set of strengthenings of G^+, see also in [24]. As we illustrated above, $SCO(G)$ is a linear transformation of the set of strengthenings of G^+, and thus the convex hull of $SCO(G)$ can be expressed as

$$\begin{aligned}\operatorname{conv}(SCO(G)) = \Big\{x \in \mathbb{R}^{E^+ \cup E^-} \Big|\, & x_{e^+} \geq 0,\ x_{e^-} \geq 0,\ \forall e \in E,\\ & x_{e^+} + x_{e^-} = 1,\ \forall e \in E,\\ & x(\delta^+(U)) \geq 1, \forall U \subsetneq V, U \neq \emptyset\Big\}.\end{aligned}$$

As a generalization, we define the *k-strongly-connected orientation (k-SCO)* of G, where k is a positive integer, to be the integral vectors of the following polytope:

$$\begin{aligned}P_0^k := \Big\{x \in \mathbb{R}^{E^+ \cup E^-} \Big|\, & x_{e^+} \geq 0,\ x_{e^-} \geq 0,\ \forall e \in E,\\ & x_{e^+} + x_{e^-} = k,\ \forall e \in E,\\ & x(\delta^+(U)) \geq k,\ \forall U \subsetneq V, U \neq \emptyset\Big\}.\end{aligned} \tag{2}$$

Note that $P_0^1 = \operatorname{conv}(SCO(G))$. $P_0^k = k \operatorname{conv}(SCO(G))$ is just a scaling of $\operatorname{conv}(SCO(G))$ and it is integral if k is integral. Thus, P_0^k describes the convex hull of k-SCO's. Taking the union of P_0^k for all nonnegative k, we obtain

$$\begin{aligned}P_0 := \Big\{x \in \mathbb{R}^{E^+ \cup E^-} \Big|\, & \exists k,\ s.t.\ x_{e^+} \geq 0,\ x_{e^-} \geq 0,\ \forall e \in E,\\ & x_{e^+} + x_{e^-} = k,\ \forall e \in E,\\ & x(\delta^+(U)) \geq k,\ \forall U \subsetneq V, U \neq \emptyset\Big\}.\end{aligned} \tag{3}$$

Note that $P_0 = \mathrm{cone}(SCO(G))$ is the convex cone generated by all the strongly connected orientations of G. One might be curious if P_0^1 has the integer decomposition property [5], i.e. any integral vector in P_0^k is a sum of k integral vectors in P_0^1. In other words, if $SCO(G)$ forms a Hilbert basis (see e.g. in [25]). A finite set of integral vectors $x_1, ..., x_n$ forms a *Hilbert basis* if any integral vector in $\mathrm{cone}(\{x_1, ..., x_n\})$ is a nonnegative integral combination of $x_1, ..., x_n$.

Our first observation is the following.

Theorem 9. *The Edmonds-Giles conjecture 1 is true for all digraphs if and only if $SCO(G)$ defined by* (1) *forms a Hilbert basis for all undirected graphs G.*

Since the Edmonds-Giles conjecture 1 is disproved by eg: Schrijver's counterexample [23], the answer is no. $SCO(G)$ does not necessarily form a Hilbert basis. However, slightly revising the statement, we obtain an equivalent form of Woodall's conjecture 2, which is still of interest. Define the *nowhere-zero k-strongly connected orientation (nowhere-zero k-SCO)* to be the integral vectors of the following polytope:

$$P_1^k := \Big\{x \in \mathbb{R}^{E^+ \cup E^-} \Big| \, x_{e^+} \geq 1,\ x_{e^-} \geq 1,\ \forall e \in E, \quad x_{e^+} + x_{e^-} = k,\ \forall e \in E, \quad x(\delta^+(U)) \geq k,\ \forall U \subsetneq V, U \neq \emptyset \Big\}. \tag{4}$$

The only difference between P_0^k and P_1^k is that each entry of the integral vectors in P_1^k is nonzero. Similarly, define

$$P_1 := \Big\{x \in \mathbb{R}^{E^+ \cup E^-} \Big| \, \exists k,\ s.t.\ x_{e^+} \geq 1,\ x_{e^-} \geq 1,\ \forall e \in E, \quad x_{e^+} + x_{e^-} = k,\ \forall e \in E, \quad x(\delta^+(U)) \geq k,\ \forall U \subsetneq V, U \neq \emptyset \Big\}. \tag{5}$$

We have the following reformulation of Woodall's conjecture.

Theorem 10. *Woodall's conjecture 2 is true for all digraphs if and only if given any undirected graph G, any integral vector in P_1 defined by* (5) *is a nonnegative integral combination of vectors in $SCO(G)$ defined by* (1).

We will first prove Theorem 10 and modify the proof to prove Theorem 9. There is some overlap between this form (Theorem 10) and the one proposed by Schrijver (restated in Theorem 8). But our form extends Schrijver's form in different angles and we will see how it connects to packing strongly connected digraphs.

Proof (Theorem 10). We first prove the "if" direction. Let $D = (V, A)$ be a digraph (e.g. Figure 2-(1)) whose underlying undirected graph is $G = (V, E)$. Let τ be the size of a minimum dicut of D. We assume $\tau \geq 2$ w.l.o.g. and this implies the size of minimum cut of D is also greater than or equal to 2. By making two copies of each edge of G and orienting them oppositely, we obtain $\vec{G} = (V, E^+ \cup E^-)$. For convenience we will assume that e^+ and e^- are defined according to their direction in D, i.e. $e = (u, v) \in A$ iff $e^+ = (u, v)$ and $e^- = (v, u)$. In other words, $E^+ = A$ and $E^- = A^{-1}$. Take $x \in \mathbb{Z}^{E^+ \cup E^-}$

such that $x_{e^+} = \tau - 1$, $x_{e^-} = 1$ for any $e \in E$ (as shown in Figure 2-(2)). We claim that $x \in P_1^\tau$. The only nontrivial constraint to prove is $x(\delta^+_{\vec{G}}(U)) \geq \tau$ for any $U \subsetneq V, U \neq \emptyset$. If $\delta^-_D(U)$ is a dicut such that $\delta^+_D(U) = \emptyset$, then $x(\delta^+_{\vec{G}}(U)) = x(\delta^+_{E^-}(U)) = |\delta^-_D(U)| \geq \tau$. Otherwise, $\delta^+_D(U) \neq \emptyset$ and thus $\delta^+_{\vec{G}}(U)$ contains at least one arc in E^+. Moreover, since $|\delta_D(U)| \geq 2$, one has $x(\delta^+_{\vec{G}}(U)) = x(\delta^+_{E^+}(U)) + x(\delta^+_{E^-}(U)) \geq (\tau - 1) + 1 = \tau$. Thus, $x \in P_1^\tau \subseteq P_1$. By the assumption, $x = \sum_{i=1}^\tau \chi_{O_i}$ where each O_i is a strongly connected orientation. Since for each O_i, $\chi_{O_i}(e^+) + \chi_{O_i}(e^-) = 1$, and $x(e^+) + x(e^-) = \tau$, $\forall e \in E$, there should be τ O_i's summing up to x. Take $J_i = \{e^+ \in E^+ \mid \chi_{O_i}(e^-) = 1, e^- \in E^-\}$. Note that $(A \setminus J_i) \cup (J_i^{-1}) = O_i$. Therefore, $(A \setminus J_i) \cup (J_i^{-1})$ is strongly connected, which means J_i is a strengthening of D, and thus a dijoin of D. Since $x_{e^-} = 1$ for each $e \in E$, J_i's are disjoint. Thus we get τ disjoint dijoins of D.

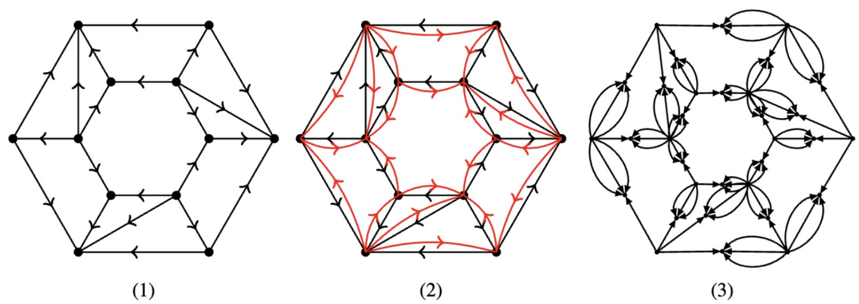

Fig. 2. (1) is a digraph with minimum dicut size 4. In (2) the weights of black arcs are 3 and the weights of red arcs are 1. This figure illustrates how to convert from a digraph D (1) to a weighted digraph $\vec{G}$ (2) in the first part of the proof of Theorem 10 and how to convert from a weighted digraph $\vec{G}$ (2) to a digraph D (3) in the second part of the proof of Theorem 10.

We now prove the "only if" direction. Given any undirected graph $G = (V, E)$, consider the corresponding directed graph $\vec{G} = (V, E^+ \cup E^-)$ with each edge of E copied and oppositely oriented. For any integral $x \in P_1$, there exists some $\tau \in \mathbb{Z}^+$ such that $x \in P_1^\tau$ by definition (e.g. Figure 2-(2)). Construct a new digraph D from G in the following way. For each edge $e = (u, v) \in E$ where $e^+ = (u, v)$ and $e^- = (v, u)$, add a node w_e, add $x_{e^+} \geq 1$ arcs from u to w_e and $x_{e^-} \geq 1$ arcs from v to w_e, and delete e (as shown in Figure 2-(3)). We claim that the size of minimum dicut of D is τ. Every vertex w_e induces a dicut $\delta^-_D(\{w_e\})$ of size $x_{e+} + x_{e^-} = \tau$. Thus, we only need to show the size of any dicut of D is at least τ. Given any U such that $\delta^-_D(U) = \emptyset$, if there exists $e = (u, v) \in E$ such that $u, v \in U$ but $w_e \notin U$, then $|\delta^+_D(U)| \geq |\delta^-_D(\{w_e\})| \geq \tau$. Thus, we may assume w.l.o.g. that for any $e = (u, v) \in E$ such that $u, v \in U$, we also have $w_e \in U$. Since $\delta^-_D(U) = \emptyset$, for any $e = (u, v) \in E$ such that $u, v \notin U$, we also have $w_e \notin U$. Moreover, for any $e = (u, v) \in E$ such that $u \in U$, $v \notin U$, since there is at least an arc from v to w_e but $\delta^-_D(U) = \emptyset$, we infer that $w_e \notin U$. Thus, $\delta^+_D(U) = \cup\{uw_e \mid e = (u, v) \in E, u \in U, v \notin U\}$. Thus, by the way we construct D, $|\delta^+_D(U)| \geq x(\delta^+_{\vec{G}}(U)) \geq \tau$. Therefore, D has minimum dicut of size

τ. By Woodall's conjecture, there exists τ disjoint dijoins $J_1, ..., J_\tau$ in D. In particular, since dicut $|\delta_D^-(\{w_e\})| = \tau$, each dijoin intersects $\delta_D^-(\{w_e\})$ exactly once. Let O_i be an orientation defined by $O_i := \{e^+ \mid uw_e \in J_i\} \cup \{e^- \mid vw_e \in J_i\}$. Note that O_i is indeed an orientation since exactly one of uw_e and vw_e is in J_i, for any $e \in E$. We claim that each O_i is a strongly connected orientation of G. Assume not. Then there exists $U \subseteq V$, such that $\delta_{\vec{G}}^+(U) \cap O_i = \emptyset$. Let $U' := U \cup \{w_e \mid e = (u, v) \in E, u, v \in U\}$. It is easy to see U' is a dicut of D such that $\delta_D^-(U') = \emptyset$. It follows from $\delta_{\vec{G}}^+(U) \cap O_i = \emptyset$ that $\delta_D^+(U') \cap J_i = \emptyset$, a contradiction to J_i being a dijoin of D. Moreover, by the way we construct D, for each $e^+ = (u, v)$, $\sum_{i=1}^{\tau} \chi_{O_i}(e^+) = |\{J_i \mid uw_e \in J_i\}| = x_{e^+}$. For each $e^- = (v, u)$, $\sum_{i=1}^{\tau} \chi_{O_i}(e^-) = |\{J_i \mid vw_e \in J_i\}| = x_{e^-}$. Therefore, $\sum_{i=1}^{\tau} \chi_{O_i} = x$. This ends the proof of this direction. □

To prove Theorem 9, we need a structural lemma.

Lemma 2. *Let $D = (V, A)$ be a digraph with weight $w \in \{0, 1\}^A$ and the minimum weight of a dicut $\tau \geq 2$. Let $e \in A$ be some arc such that $w_e = 1$. If there exists a cut $\delta_D(U)$ such that $\delta_D^+(U) = \{e\}$ and $w(\delta_D^-(U)) = 0$, then e is not contained in any minimum dicut of D.*

Proof (Proof of Lemma 2). Suppose not. Then there exists a dicut $\delta_D^-(W)$ such that $w(\delta_D^-(W)) = \tau$ and $e \in \delta_D^-(W)$. Let D' be obtained from D by deleting e. Then $\delta_{D'}^-(U)$ becomes a dicut of D'. Therefore, $\delta_{D'}^-(U \cap W)$ and $\delta_{D'}^-(U \cup W)$ are both dicuts of D'. However, since e leaves U and enters W, e goes from $U \setminus W$ to $W \setminus U$. Thus, $e \notin \delta_D(U \cap W)$ and $e \notin \delta_D(U \cup W)$. Therefore, both $\delta_D^-(U \cap W)$ and $\delta_D^-(U \cup W)$ are dicuts of D. Moreover, $w(\delta_D^-(U \cap W)) + w(\delta_D^-(U \cup W)) = w(\delta_D^-(U)) + w(\delta_D^-(W)) - 1 = \tau - 1$. It follows that $w(\delta_D^-(U \cap W)) \leq \tau - 1$ and $w(\delta_D^-(U \cup W)) \leq \tau - 1$. Notice that either $U \cap W \neq \emptyset$ or $U \cup W \neq V$. Otherwise, e is a bridge of D, contradicting to $\tau \geq 2$. Therefore, either $\delta_D^-(U \cap W)$ or $\delta_D^-(U \cup W)$ violates the assumption that the size of a minimum dicut is τ. Contradiction. □

We are now ready to prove Theorem 9.

Proof (Theorem 9). We modify the proof of Theorem 10 to prove Theorem 9. We first prove the "if" direction. Let $D = (V, A)$ be a digraph with weight $w \in \{0, 1\}^A$ and minimum dicut $\tau \geq 2$. We can assume there is no arc $e \in A$ with weight 1 such that there exists a cut $\delta_D(U)$ such that $\delta_D^+(U) = \{e\}$ and $w(\delta_D^-(U)) = 0$. For otherwise, by Lemma 2, e is not contained in any minimum dicut, which means we can set the weight of e to be 0 without decreasing the size of minimum dicut. Any packing of τ dijoins in the new graph will be a valid packing of τ dijoins of the old graph. Let G be the underlying undirected graph of D and $\vec{G} = (V, E^+ \cup E^-)$ be defined as before such that $E^+ = A$ and $E^- = A^{-1}$. Define $x \in \mathbb{Z}^{E^+ \cup E^-}$ as follows. For weight 1 arcs $e^+ \in A$, we define $x_{e^+} = \tau - 1$ and $x_{e^-} = 1$ as before. For the weight 0 arcs $e^+ \in A$, we define $x_{e^+} = \tau$ and $x_{e^-} = 0$. To argue that $x(\delta_{\vec{G}}^+(U)) \geq \tau$ for any $U \subsetneq V, U \neq \emptyset$, if $\delta_D^-(U)$ is a dicut it follows the same way as in the proof of Theorem 10. So w.l.o.g. we assume there exists at least one arc $e^+ \in E^+$ in $\delta_D^+(U)$. If there exists such an arc with $w(e^+) = 0$, then $x(\delta_{\vec{G}}^+(U)) \geq x_{e^+} \geq \tau$. Otherwise, all the arcs in $\delta_D^+(U)$ have weight 1. If there exist at least 2 arcs of weight 1 in $\delta_D(U)$, it also follows the same way as in the other proof. The only case left is when $\delta_D^+(U)$ is a single arc of weight 1 and all the arc in

$\delta^-_D(U)$ has weight 0, which has been excluded in the beginning. Therefore, we proved $x(\delta^+_{\vec{G}}(U)) \geq \tau$ for any $U \subsetneq V, U \neq \emptyset$, which implies $x \in P_0^\tau \subseteq P_0$. By the assumption, $x = \sum_{i=1}^{\tau} \chi_{O_i}$ where each O_i is a strongly connected orientation. We define the dijoins in the same way as the other proof. Note that the dijoins are disjoint and never use weight 0 arcs. Therefore, we find τ dijoins that is a valid packing of graph D with weight w.

Next, we prove the "only if" direction. Given any undirected graph $G = (V, E)$, corresponding $\vec{G} = (V, E^+ \cup E^-)$, and an integral $x \in P_0^\tau$, we construct weighted digraph D as follows. For any edge $e = (u, v)$ such that $x_{e^+}, x_{e^-} \geq 1$, we construct node w_e and arcs uw_e, vw_e in the same way as in the proof of Theorem 10. For any edge $e^+ = (u, v)$ such that $x_{e^+} = \tau, x_{e^-} = 0$, we add node w_e, add τ arcs of weight 1 from u to w_e and add a weight 0 arc from v to w_e. Similarly, for $e^+ = (u, v)$ with $x_{e^+} = 0, x_{e^-} = \tau$, we add node w_e, add a weight 0 arc from u to w_e and τ arcs of weight 1 from v to w_e. The same argument applies to see the minimum dicut size of D is τ. By Edmonds-Giles' conjecture, we can find τ disjoint dijoins in the weighted digraph D. And similarly, we can find τ strongly connected orientations accordingly that sum up to x. This holds for any $x \in P_0^\tau$ with any τ, which implies $SCO(G)$ forms a Hilbert basis. □

To see how Schrijver's counterexample [23] to the Edmonds-Giles conjecture can be translated to a counterexample to the statement that $SCO(G)$ forms a Hilbert basis, we give the following example shown in Figure 3. Let $x \in \mathbb{Z}^{E^+ \cup E^-}$ be defined by $x_e = 1$ if e is blue and $x_e = 2$ if e is black, $x_e = 0$ otherwise, for each $e \in E^+ \cup E^-$. It satisfies that $x(\delta^+_{\vec{G}}(U)) \geq 2$ but x cannot be decomposed into 2 strongly connected orientations.

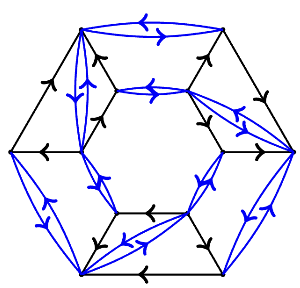

Fig. 3. The blue arcs with weight 1 and black arcs with weight 2 cannot be decomposed into 2 strongly connected orientations.

A direct consequence of Theorem 10 is the following, which has been noted by A. Frank (see e.g. in [26] Theorem 56.3).

Corollary 2. *Woodall's conjecture is true for $\tau = 2$.*

Proof. It is implicit in the proof of Theorem 10 that the equivalence also holds true when restricting to a specific τ: Woodall's conjecture is true for all graphs with minimum dicut τ if and only if for all undirected graph G any vector $x \in P_1^\tau$ can be decomposed into τ strongly connected orientations. Thus, it suffices to prove any vector $x \in P_1^2$ can be decomposed into 2 strongly connected orientations. Yet, the all one vector **1** is the only possible vector in P_1^2. Clearly, $\mathbf{1} \in P_1^2$ if and only if G is 2-edge-connected. By the classical result of Robbins [21], any 2-edge-connected graph has a strongly connected orientation O. Therefore, $\mathbf{1} = \chi_O + \chi_{O^{-1}}$ is a valid decomposition. □

5 Strongly Connected Digraphs and Decomposition

In this section, we will revisit strongly connected digraphs and study their polytopes. We will see the difference between the polytopes of strongly connected orientations and strongly connected digraphs. We also give a decomposition result of any strongly connected digraphs if its weights satisfy the "nowhere-zero" condition.

5.1 Strongly Connected Digraph Polytopes

Given any undirected graph $G = (V, E)$ and its $\vec{G} = (V, E^+ \cup E^-)$ obtained by copying and oppositely orienting each edge of G. Let $SCD(G)$ be the polytope of strongly connected digraphs of G.

$$SCD(G) := \left\{ x \in \{0,1\}^{E^+ \cup E^-} \middle| x = \chi_F \text{ for some strongly connected digraph } F \text{ of } G \right\}. \tag{6}$$

And $SCD(G)$ coincides with the 0, 1 vectors in the following polytope:

$$\Big\{ x \in \mathbb{R}^{E^+ \cup E^-} \Big| \; x_{e^+} \geq 0, \; x_{e^-} \geq 0, \; \forall e \in E \\ x(\delta^+(U)) \geq 1, \forall U \subsetneq V, U \neq \emptyset \Big\}.$$

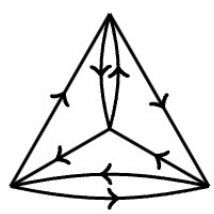

Fig. 4. $\frac{1}{2}$ every arc drawn and 0 else is a half-integral extreme point.

In contrast to $SCO(G)$, this polytope is not integral (e.g. the complete graph K_4 has non-integral extreme points such as the one in Figure 4.). For any positive integer k, the k-strongly-connected (sub)digraphs (k-SCD's) are the integral vectors in

$$Q_0^k := \Big\{ x \in \mathbb{R}^{E^+ \cup E^-} \Big| \; x_{e^+} \geq 0, \; x_{e^-} \geq 0, \; \forall e \in E \\ x(\delta^+(U)) \geq k, \; \forall U \subsetneq V, U \neq \emptyset \Big\} \tag{7}$$

and define the *nowhere-zero k-strongly-connected (sub)digraphs (nowhere-zero k-SCD's)* to be the integral vectors in

$$Q_1^k := \Big\{x \in \mathbb{R}^{E^+ \cup E^-} \,\Big|\, x_{e^+} \geq 1,\ x_{e^-} \geq 1,\ \forall e \in E \\ x(\delta^+(U)) \geq k,\ \forall U \subsetneq V, U \neq \emptyset\Big\}. \tag{8}$$

Note that Q_0^k (Q_1^k) is obtained from P_0^k (P_1^k) by dropping the constraint $x_{e^+} + x_{e^-} = k, \forall e \in E$. In particular, in a strongly connected digraph, for each edge $e \in E$, we can use both e^+ and e^- or neither of them.

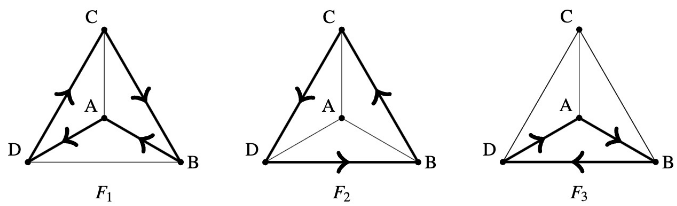

Fig. 5. A counterexample to packing 3 strongly connected digraphs with weight in Q_1^3.

Remark. In general, it is not true that any integral vector in Q_1^k can be decomposed into k disjoint strongly connected digraphs. A counterexample is given in Figure 5. The underlying graph G is the complete graph K_4. Let the weight be the all one vector $w = \mathbf{1}$. It satisfies $w(\delta_{\vec{G}}(U)) \geq 3, \forall U \subsetneq V, U \neq \emptyset$ and thus $w \in Q_1^3$. Assume w can be decomposed into 3 strongly connected digraphs F_1, F_2, F_3. There are 3 arc-disjoint directed paths from D to B: $D \to B$, $D \to A \to B$ and $D \to C \to B$, so each strongly connected digraph should use exactly one of them. So should the 3 arc-disjoint directed paths from B to D. If $D \to B$ and $B \to D$ are used in the same strongly connected digraph F_1, then F_2 and F_3 will use all the arcs connecting $\{A, C\}$ to $\{B, D\}$ which leaves $\{A, C\}$ disconnected to $\{B, D\}$ in F_1, a contradiction. Thus, exactly one of them, say F_1, uses length 2 paths in both directions, and the other two, F_2, F_3, use length 1 path in one direction and length 2 path in the other (see e.g. Figure 5). Then there will be a length 3 directed cycle in each of F_2 and F_3. The only arcs left are $A \to C$ and $C \to A$. Both F_2 and F_3 need both arcs to strongly connect the remaining isolated node to the directed cycle. This is impossible.

5.2 Decomposing Nowhere-Zero τ-SCD's

We provide a counterpart of Theorem 7 about decomposing any nowhere-zero τ-SCD's.

Theorem 11. *For any undirected graph $G = (V, E)$, if it admits a nowhere-zero (circular) k-flow for some rational number $k \geq 2$, then any nowhere-zero τ-SCD associated to G can be decomposed into two disjoint $\left\lfloor \frac{\tau}{k+1} \right\rfloor$-SCD's.*

Proof. Let E^+ and $f : E^+ \to [1, k-1]$ be a nowhere-zero (circular) k-flow of G. Let E^- be obtained by reversing the arcs of E^+. Let G be the underlying undirected graph of D and $\vec{G} = (V, E^+ \cup E^-)$. Given any integral $w \in Q_1^\tau$ a nowhere-zero τ-SCD, we construct $x \in \mathbb{Z}^{E^+ \cup E^-}$ in the following way:

$$x_e = \begin{cases} \lceil \frac{w_e}{2} \rceil, & e \in E^+ \\ \lfloor \frac{w_e}{2} \rfloor, & e \in E^- \end{cases}, \quad \text{and equivalently} \quad (w-x)_e = \begin{cases} \lfloor \frac{w_e}{2} \rfloor, & e \in E^+ \\ \lceil \frac{w_e}{2} \rceil, & e \in E^- \end{cases}.$$

We claim that x and $(w - x)$ are both $\frac{\tau}{k+1}$-SCD's. We discuss two cases.

If $|\delta_G(U)| \geq \frac{k}{k+1}\tau$, then the weights x_e of all the arcs $e \in \delta^+_{E^+}(U)$ will be rounded up to $\lceil \frac{w_e}{2} \rceil \geq \lceil \frac{1}{2} \rceil = 1$. Therefore, $x(\delta^+_{\vec{G}}(U)) \geq x(\delta^+_{E^+}(U)) \geq |\delta^+_{E^+}(U)| \geq \frac{1}{k}|\delta_G(U)| \geq \frac{\tau}{k+1}$, where the third inequality follows from Lemma 1. Symmetrically, the weights $(w-x)_e$ of all the arcs $e \in \delta^+_{E^-}(U)$ will be rounded up to $\lceil \frac{w_e}{2} \rceil \geq \lceil \frac{1}{2} \rceil = 1$. Therefore, $(w-x)(\delta^+_{\vec{G}}(U)) \geq (w-x)(\delta^+_{E^-}(U)) \geq |\delta^+_{E^-}(U)| \geq \frac{1}{k}|\delta_G(U)| \geq \frac{\tau}{k+1}$.

Otherwise, $|\delta_G(U)| < \frac{k}{k+1}\tau$. Then, $x(\delta^+_{\vec{G}}(U))$ will be deviating from (less than) $\frac{1}{2}w(\delta^+_{\vec{G}}(U))$ by at most $\frac{1}{2}|\delta^+_{E^-}(U)|$ since only the weights x_e of arcs $e \in \delta^+_{E^-}(U)$ can be rounded down. And the weight x_e of each arc $e \in \delta^+_{E^-}(U)$ can be deviating from (less than) $\frac{w_e}{2}$ by at most $\frac{1}{2}$. Therefore, $x(\delta^+_{\vec{G}}(U)) \geq \frac{1}{2}w(\delta^+_{\vec{G}}(U)) - \frac{1}{2}|\delta^+_{E^-}(U)| \geq \frac{1}{2}w(\delta^+_{\vec{G}}(U)) - \frac{1}{2} \cdot \frac{k-1}{k}|\delta_G(U)| \geq \frac{1}{2}\tau - \frac{1}{2} \cdot \frac{k-1}{k} \cdot \frac{k}{k+1}\tau = \frac{\tau}{k+1}$, where the second inequality follows from Lemma 1 and the third inequality uses the fact that $w \in Q_1^k$. By symmetry, $(w-x)(\delta^+_{\vec{G}}(U)) \geq \frac{1}{2}w(\delta^+_{\vec{G}}(U)) - \frac{1}{2}|\delta^+_{E^+}(U)| \geq \frac{1}{2}w(\delta^+_{\vec{G}}(U)) - \frac{1}{2} \cdot \frac{k-1}{k}|\delta_G(U)| \geq \frac{1}{2}\tau - \frac{1}{2} \cdot \frac{k-1}{k} \cdot \frac{k}{k+1}\tau = \frac{\tau}{k+1}$. □

The above theorem along with the technique of pairing in and out r-arborescences gives the following result.

Theorem 12. *For any graph $G = (V, E)$, if it admits a nowhere-zero (circular) k-flow for some rational number $k \geq 2$, then the weighted digraph $\vec{G} = (V, E^+ \cup E^-)$ with weight $w \in Q_1^\tau$ can pack $\lfloor \frac{\tau}{k+1} \rfloor$ many strongly connected digraphs.*

6 Conclusions and Discussions

This paper shows that any digraph with minimum dicut size τ can be decomposed into $\lfloor \frac{\tau}{6} \rfloor$ disjoint dijoins, and $\left\lfloor \frac{\tau p}{2p+1} \right\rfloor$ disjoint dijoins when the digraph is $6p$-edge-connected. The existence of a nowhere-zero (circular) k-flow for a smaller k (< 6) when special structures are imposed on the underlying undirected graph would lead to a better ratio, i.e. $\lfloor \frac{\tau}{k} \rfloor$, approximate packing of dijoins for those digraphs. The limitation of this approach is that we cannot hope that nowhere-zero 2-flows always exist because this is equivalent to the graph being Eulerian. Thus, bringing the number up to $\lfloor \frac{\tau}{2} \rfloor$ disjoint dijoins would be a barrier for this approach. However, for Woodall's conjecture to be true, it is necessary that we should be able to decompose the digraph into two disjoint $\lfloor \frac{\tau}{2} \rfloor$-dijoins. Therefore, new ideas need to be developed to prove or disprove whether this decomposition exists.

The careful reader may have noticed that the approach only works for the unweighted case. Yet, with a slight modification, the argument extends to the weighted case when the underlying undirected graph induced by the weight 1 arcs is 2-edge-connected. In this case, we can find a nowhere-zero k-flow only on the weight 1 arcs and construct the decomposition of weight 1 arcs the same way as in Theorem 7. However, unfortunately the underlying graph of weight 1 arcs may have bridges or be disconnected, and then the above argument does not work. A proper analogue of nowhere-zero flows in mixed graphs should be developed, which motivate us to study the following open problems:

Given a mixed graph $M = (V, E \cup A)$, where A is directed and E is undirected, a *pseudo dicut* is an edge subset of the form $\delta_E(U)$ such that $\delta_A^-(U) = \emptyset$. Let τ be the size of a minimum pseudo dicut. Is there any constant $k > 2$ such that for any mixed graph $M = (V, E \cup A)$, there always exists an orientation E^+ of the edges E such that $|\delta^+_{E^+}(U)|, |\delta^-_{E^+}(U)| \geq \left\lfloor \frac{1}{k}|\delta_E(U)| \right\rfloor$ for any pseudo dicut $\delta_E(U)$? Note that the incorrectness of Edmonds-Giles conjecture implies $k \neq 2$. Or even weaker, is there always an orientation E^+ such that $|\delta^+_{E^+}(U)|, |\delta^-_{E^+}(U)| \geq 1$ for any pseudo dicut $\delta_E(U)$ when τ goes to infinity? The affirmative answer to the first question would immediately lead to a $\left\lfloor \frac{\tau}{k} \right\rfloor$-approximate packing of dijoins in the weighted case.

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