The Strong Perfect Graph Conjecture

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Abstract

A graph is *perfect* if, in all its induced subgraphs, the size of a largest clique is equal to the chromatic number. Examples of perfect graphs include bipartite graphs, line graphs of bipartite graphs and the complements of such graphs. These four classes of perfect graphs will be called *basic*. In 1960, Berge formulated two conjectures about perfect graphs, one stronger than the other. The weak perfect graph conjecture, which states that a graph is perfect if and only if its complement is perfect, was proved in 1972 by Lovász. This result is now known as the perfect graph theorem. The strong perfect graph conjecture (SPGC) states that a graph is perfect if and only if it does not contain an odd hole or its complement. The SPGC has attracted a lot of attention. It was proved recently (May 2002) in a remarkable sequence of results by Chudnovsky, Robertson, Seymour and Thomas. The proof is difficult and, as of this writing, they are still checking the details. Here we give a flavor of the proof. Let us call *Berge graph* a graph that does not contain an odd hole or its complement. Conforti, Cornuéjols, Robertson, Seymour, Thomas and Vušković (2001) conjectured a structural property of Berge graphs that implies the SPGC: Every Berge graph G is basic or has a skew partition or a homogeneous pair, or G or its complement has a 2-join. A *skew partition* is a partition of the vertices into nonempty sets A, B, C, D such that every vertex of A is adjacent to every vertex of B and there is no edge between C and D. Chvátal introduced this concept in 1985 and conjectured that no minimally imperfect graph has a skew partition. This conjecture was proved recently by Chudnovsky and Seymour (May 2002). Cornuéjols and Cunningham introduced 2-joins in 1985 and showed that they cannot occur in a minimally imperfect graph different from an odd hole. Homogeneous pairs were introduced in 1987 by Chvátal and Sbihi, who proved that they cannot occur in minimally imperfect graphs. Since skew partitions, 2-joins and homogeneous pairs cannot occur in minimally imperfect Berge graphs, the structural property of Berge graphs stated above implies the SPGC. This structural property was proved: (i) When G contains the line graph of a bipartite subdivision of a 3-connected graph (Chudnovsky, Robertson, Seymour and Thomas (September 2001)); (ii) When G contains a stretcher (Chudnovsky and Seymour (January 2002)); (iii) When G contains no proper wheels, stretchers or their complements (Conforti, Cornuéjols and Zambelli (May 2002)); (iv) When G contains a proper wheel, but no stretchers or their complements (Chudnovsky and Seymour (May 2002)). (ii), (iii) and (iv) prove the SPGC.

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1 Introduction

In this paper, all graphs are simple (no loops or multiple edges) and finite. The vertex set of graph G is denoted by $V(G)$ and its edge set by $E(G)$. A *stable set* is a set of vertices no two of which are adjacent. A *clique* is a set of vertices every pair of which are adjacent. The cardinality of a largest clique in graph G is denoted by $\omega(G)$. The cardinality of a largest stable set is denoted by $\alpha(G)$. A *k-coloring* is a partition of the vertices into k stable sets (these stable sets are called *color classes*). The *chromatic number* $\chi(G)$ is the smallest value of k for which there exists a k-coloring. Obviously, $\omega(G) \leq \chi(G)$ since the vertices of a clique must be in distinct color classes of the k-coloring. An *induced subgraph* of G is a graph with vertex set $S \subseteq V(G)$ and edge set comprising all the edges of G with both ends in S. It is denoted by $G(S)$. The graph $G(V(G)-S)$ is denoted by $G \setminus S$. A graph G is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraphs H of G. A graph is *minimally imperfect* if it is not perfect but all its proper induced subgraphs are.

A *hole* is a graph induced by a chordless cycle of length at least 4. A hole is *odd* if it contains an odd number of vertices. Odd holes are not perfect since their chromatic number is 3 whereas the size of their largest clique is 2. It is easy to check that odd holes are minimally imperfect. The complement of a graph G is the graph $\bar{G}$ with the same vertex set as G, and uv is an edge of $\bar{G}$ if and only if it is not an edge of G. The odd holes and their complements are the only known minimally imperfect graphs. In 1960 Berge [3] proposed the following conjecture, known as the *Strong Perfect Graph Conjecture*.

Conjecture 1.1 (Strong Perfect Graph Conjecture) (Berge [3]) *The only minimally imperfect graphs are the odd holes and their complements.*

At the same time, Berge also made a weaker conjecture, which states that a graph G is perfect if and only if its complement $\bar{G}$ is perfect. This conjecture was proved by Lovász [29] in 1972 and is known as the *Perfect Graph Theorem*.

Theorem 1.2 (Perfect Graph Theorem) (Lovász [29]) *Graph G is perfect if and only if graph $\bar{G}$ is perfect.*

Proof: Lovász [30] proved the following stronger result.

Claim 1: A graph G is perfect if and only if, for every induced subgraph H, the number of vertices of H is at most $\alpha(H)\omega(H)$.

Since $\alpha(H) = \omega(\bar{H})$ and $\omega(H) = \alpha(\bar{H})$, Claim 1 implies Theorem 1.2.

Proof of Claim 1: We give a proof of this result due to Gasparyan [25]. First assume that G is perfect. Then, for every induced subgraph H, $\omega(H) = \chi(H)$. Since the number of vertices of H is at most $\alpha(H)\chi(H)$, the inequality follows.

Conversely, assume that G is not perfect. Let H be a minimally imperfect subgraph of G and let n be the number of vertices of H. Let $\alpha = \alpha(H)$ and $\omega = \omega(H)$. Then H satisfies

$$\omega = \chi(H\backslash v) \text{ for every vertex } v \in V(H)$$

$$\text{and } \omega = \omega(H\backslash S) \text{ for every stable set } S \subseteq V(H).$$

Let A_0 be an α-stable set of H. Fix an ω-coloring of each of the α graphs $H\backslash s$ for $s \in A_0$, let $A_1, \ldots, A_{\alpha\omega}$ be the stable sets occuring as a color class in one of these colorings and let $\mathcal{A} := \{A_0, A_1, \ldots, A_{\alpha\omega}\}$. Let $\mathbf{A}$ be the corresponding stable set versus vertex incidence matrix. Define $\mathcal{B} := \{B_0, B_1, \ldots, B_{\alpha\omega}\}$ where B_i is an ω-clique of $H\backslash A_i$. Let $\mathbf{B}$ be the corresponding clique versus vertex incidence matrix.

Claim 2: Every ω-clique of H intersects all but one of the stables sets in $\mathcal{A}$.

Proof of Claim 2: Let $S_1, \ldots, S_\omega$ be any ω-coloring of $H \setminus v$. Since any ω-clique C of H has at most one vertex in each S_i, C intersects all S_i's if $v \notin C$ and all but one if $v \in C$. Since C has at most one vertex in A_0, Claim 2 follows.

In particular, it follows that $\mathbf{AB}^T = J - I$. Since $J - I$ is nonsingular, $\mathbf{A}$ and $\mathbf{B}$ have at least as many columns as rows, that is $n \geq \alpha\omega + 1$. This completes the proof of Claim 1. □

2 Four Basic Classes of Perfect Graphs

Bipartite graphs are perfect since, for any induced subgraph H, the bipartition implies that $\chi(H) \leq 2$ and therefore $\omega(H) = \chi(H)$.

A graph L is the *line graph* of a graph G if $V(L) = E(G)$ and two vertices of L are adjacent if and only if the corresponding edges of G are adjacent.

Proposition 2.1 *Line graphs of bipartite graphs are perfect.*

Proof: If G is bipartite, $\chi'(G) = \Delta(G)$ by a theorem of König [28], where χ' denotes the edge-chromatic number and Δ the largest vertex degree.

If L is the line graph of a bipartite graph G, then $\chi(L) = \chi'(G)$ and $\omega(L) = \Delta(G)$. Therefore $\chi(L) = \omega(L)$. Since induced subgraphs of L are also line graphs of bipartite graphs, the result follows. □

Since bipartite graphs and line graphs of bipartite graphs are perfect, it follows from Lovász's perfect graph theorem (Theorem 1.2) that the complements of bipartite graphs and of line graphs of bipartite graphs are perfect. This can also be verified directly, without using the perfect graph theorem. To summarize, in this section we have introduced four basic classes of perfect graphs:

- bipartite graphs and their complements, and
- line graphs of bipartite graphs and their complements.

3 2-Join

A graph G has a *2-join* if its vertices can be partitioned into sets V_1 and V_2, each of cardinality at least three, with nonempty disjoint subsets $A_1, B_1 \subseteq V_1$ and $A_2, B_2 \subseteq V_2$, such that all the vertices of A_1 are adjacent to all the vertices of A_2, all the vertices of B_1 are adjacent to all the vertices of B_2 and these are the only adjacencies between V_1 and V_2. There is an $O(|V(G)|^2|E(G)|^2)$ algorithm to find whether a graph G has a 2-join [23].

When G contains a 2-join, we can decompose G into two blocks G_1 and G_2 defined as follows.

Definition 3.1 *If A_2 and B_2 are in different connected components of $G(V_2)$, define* block G_1 *to be* $G(V_1 \cup \{p_1, q_1\})$*, where $p_1 \in A_2$ and $q_1 \in B_2$. Otherwise, let P_1 be a shortest path from A_2 to B_2 and define* block G_1 *to be $G(V_1 \cup V(P_1))$. Block G_2 is defined similarly.*

Next we show that the 2-join decomposition preserves perfection (Cornuéjols and Cunningham [23]; see also Kapoor [27] Chapter 8). Earlier, Bixby [4] had shown that the simpler join decomposition preserves perfection.

Theorem 3.2 *Graph G is perfect if and only if its blocks G_1 and G_2 are perfect.*

Proof: By definition, G_1 and G_2 are induced subgraphs of G. It follows that, if G is perfect, so are G_1 and G_2. Now we prove the converse: If G_1 and G_2 are perfect, then so is G. Let G^* be an induced subgraph of G. We must show

$$(*) \quad \omega(G^*) = \chi(G^*).$$

For $i = 1, 2$, let $V_i^* = V_i \cap V(G^*)$. The proof of $(*)$ is based on a coloring argument, combining $\omega(G^*)$-colorings of the perfect graphs $G(V_1^*)$ and $G(V_2^*)$ (Claim 3) into an $\omega(G^*)$-coloring of G^* (Claim 4). To prove Claim 3, we will use the following results.

Claim 1: *(Lovász's Replication Lemma [29])* Let Γ be a perfect graph and $v \in V(\Gamma)$. Create a new vertex v' adjacent to v and to all the neighbors of v. Then the resulting graph Γ' is perfect.

Proof of Claim 1: It suffices to show that $\omega(\Gamma') = \chi(\Gamma')$ since, for induced subgraphs, the proof follows similarly. We distinguish two cases. Suppose first that v is contained in some $\omega(\Gamma)$-clique of Γ. Then $\omega(\Gamma') = \omega(\Gamma) + 1$. Since at most one new color is needed in Γ', $\omega(\Gamma') = \chi(\Gamma')$ follows.

Now suppose that v is not contained in any $\omega(\Gamma)$-clique of Γ. Consider any $\omega(\Gamma)$-coloring of Γ and let A be the color class containing v. Then, $\omega(\Gamma \setminus (A - \{v\})) = \omega(\Gamma) - 1$, since every $\omega(\Gamma)$-clique of Γ meets $A - \{v\}$. By the perfection of Γ, the graph $\Gamma \setminus (A - \{v\})$ can be colored with $\omega(\Gamma) - 1$ colors. Using one additional color for the vertices $(A - \{v\}) \cup \{v'\}$, we obtain an $\omega(\Gamma)$-coloring of Γ'. This proves Claim 1.

We say that Γ' is obtained from Γ by *replicating* v. Replication can be applied recursively. We say that v is *replicated k times* if k copies of v are made, including v.

Claim 2: Let Γ be a graph and uv an edge of Γ such that the vertices u and v have no common neighbor. Let Γ' be the graph obtained from Γ by replicating vertex v into v'. Let H be the graph obtained from Γ' by deleting edge uv'. Then Γ is perfect if and only if H is perfect.

Proof of Claim 2: If H is perfect, then so is Γ since Γ is an induced subgraph of H.

Conversely, suppose that Γ is perfect and H is not. Let H^* be a minimally imperfect subgraph of H. Let Γ^* be the subgraph of Γ' induced by the vertices of H^*. Since Γ^* is perfect but H^* is not, $V(H^*)$ must contain vertices u and v'. Also $\chi(\Gamma^*) = \chi(H^*)$ and $\omega(\Gamma^*) = \omega(H^*) + 1$. Therefore uvv' is the unique maximum clique in Γ^* and $\omega(H^*) = 2$. The only neighbor of v in H^* is u since otherwise v, v' would be in a clique of cardinality three in H^*. Now v' is a vertex of degree 1 in H^*, a contradiction to the assumption that H^* is minimally imperfect. This proves Claim 2.

For $i = 1, 2$, let $A_i^* = A_i \cap V(G^*)$, $B_i^* = B_i \cap V(G^*)$, $a_i = \omega(A_i^*)$ and $b_i = \omega(B_i^*)$. Let $G_i^* = G_i \setminus (V_i - V_i^*)$ and $\omega \geq \omega(G_i^*)$. In an ω-coloring of G_i^*, let $C(A_i^*)$ and $C(B_i^*)$ denote the sets of colors in A_i^* and B_i^* respectively.

Claim 3: There exists an ω-coloring of V_i^* such that $|C(A_i^*)| = a_i$ and $|C(B_i^*)| = b_i$. Furthermore, if G_i contains path P_i and

(i) if P_i has an odd number of edges, then $|C(A_i^*) \cap C(B_i^*)| = \max(0, a_i + b_i - \omega)$,

(ii) if P_i has an even number of edges, then $|C(A_i^*) \cap C(B_i^*)| = \min(a_i, b_i)$.

Proof of Claim 3: First assume that block G_i is induced by $V_i \cup \{p_i, q_i\}$. In G_i^*, replicate p_i $\omega - a_i$ times and q_i $\omega - b_i$ times. By Claim 1, this new graph H is perfect and $\omega(H) = \omega$. Therefore an ω-coloring of H exists. This coloring induces an ω-coloring of V_i^* with $|C(A_i^*)| = a_i$ and $|C(B_i^*)| = b_i$. Now assume that G_i contains path P_i. We consider two cases.

(i) P_i has an odd number of edges.

Let $P_i = x_1, \ldots, x_{2k}$. In G_i^*, replicate vertex x_{2k} into x'_{2k} and remove edge $x_{2k-1}x'_{2k}$. By Claim 2, the new graph is perfect. For i odd, $1 \leq i < 2k$, replicate vertex x_i $\omega - a_i$ times. For i even, $1 < i \leq 2k - 2$, replicate vertex x_i a_i times.

If $a_i + b_i < \omega$, replicate x_{2k} a_i times and replicate x'_{2k} $\omega - a_i - b_i$ times. By Claim 1, this new graph H is perfect. Since $\omega(H) = \omega$, H has an ω-coloring. Note that $|C(A_i^*)| = a_i$ and $|C(B_i^*)| = b_i$ and every vertex of P_i belongs to two cliques of size ω. So the colors that appear in the replicates of x_{2k} are precisely $C(A_i)$. Therefore B_i^* is colored with colors that do not appear in $C(A_i^*)$. Thus $|C(A_i^*) \cap C(B_i^*)| = 0$.

If $a_i + b_i \geq \omega$, replicate x_{2k} $\omega - b_i$ times and remove x'_{2k}. The new graph H is perfect and $\omega(H) = \omega$. Therefore H has an ω-coloring. Again $|C(A_i^*)| = a_i$ and $|C(B_i^*)| = b_i$, and the $\omega - b_i$ colors that appear in the replicates of x_{2k} belong to $C(A_i^*)$. Since these colors cannot appear in $C(B_i^*)$, the number of common colors in $C(A_i^*)$ and $C(B_i^*)$ is $a_i + b_i - \omega$.

(ii) P_i has an even number of edges.

Assume w.l.o.g. that $a_i \leq b_i$. Let $P_i = x_1, \ldots, x_{2k+1}$. In G_i^*, replicate vertex x_i $\omega - a_i$ times for i odd, $1 \leq i \leq 2k - 1$, and replicate vertex x_i a_i times for i even, $1 < i \leq 2k$. Finally, replicate x_{2k+1} $\omega - b_i$ times. By Claim 1, the new graph H is perfect and $\omega(H) = \omega$. In an ω-coloring of H, $|C(A_i^*)| = a_i$ and $|C(B_i^*)| = b_i$ and the colors that appear in the replicates of x_{2k} are precisely $C(A_i^*)$. But then these colors do not appear in the replicates of x_{2k+1} and consequently they must appear in $C(B_i)$. Thus $|C(A_i) \cap C(B_i)| = \min(a_i, b_i)$. This proves Claim 3.

Claim 4: G^* has an $\omega(G^*)$-coloring.

Proof of Claim 4:
Let $\omega = \omega(G^*)$. Clearly, $\omega \geq a_1 + a_2$ and $\omega \geq b_1 + b_2$. To prove the claim, we will combine ω-colorings of V_1^* and V_2^*.

If at least one of the sets $A_1^*, A_2^*, B_1^*, B_2^*$ is empty, one can easily construct the desired ω-coloring of G^*. So we assume now that these sets are nonempty. This implies that $\omega \geq \omega(G_1^*)$ and $\omega \geq \omega(G_2^*)$. By Claim 3, there exist ω-colorings of V_i^* such that $|C(A_i^*)| = a_i$ and $|C(B_i^*)| = b_i$. Thus, if A_2^* and B_2^* are in different connected components of $G(V_2^*)$, an ω-coloring of V_1^* can be combined with ω-colorings of the components of $G(V_2^*)$ into an ω-coloring of G^*. So we can assume that both P_1 and P_2 exist. Since G_1 contains no odd hole, every chordless path from A_1 to B_1 has the same parity as P_1. It follows from the definition of 2-join decomposition that P_1 and P_2 have the same parity.

(i) P_1 and P_2 both have an odd number of edges.

Then by Claim 3 (i), there exists an ω-coloring of V_i^* with $|C(A_i^*) \cap C(B_i^*)| = \max(0, a_i + b_i - \omega)$. In the coloring of V_1^*, label by 1 through a_1 the colors that occur in A_1^* and by ω through $\omega - b_1 + 1$ the colors that occur in B_1^*. In the coloring of V_2^*, label by ω through $\omega - a_2 + 1$ the colors that occur in A_2^* and by 1 through b_2 the colors that occur in B_2^*. If this is not an ω-coloring of G^*, there must exist a common color in A_1^* and A_2^* or in B_1^* and B_2^*. But then either $a_1 \geq \omega - a_2 + 1$ or $b_2 \geq \omega - b_1 + 1$, a contradiction.

(ii) P_1 and P_2 both have an even number of edges.

Then by Claim 3 (ii), there exists an ω-coloring of V_i^* with $|C(A_i^*) \cap C(B_i^*)| = \min(a_i, b_i)$. In the coloring of V_1^*, label by 1 through a_1 the colors that occur in A_1^* and by 1 through b_1 the colors that occur in B_1^*. In the coloring of V_2^*, label by ω through $\omega - a_2 + 1$ the colors that occur in A_2^* and by ω through $\omega - b_2 + 1$ the colors that occur in B_2^*. If this is not an ω-coloring of G, there must exist a common color in A_1^* and A_2^* or in B_1^* and B_2^*. But then either $a_1 \geq \omega - a_2 + 1$ or $b_1 \geq \omega - b_2 + 1$, a contradiction. □

Corollary 3.3 *If a minimally imperfect graph G has a 2-join, then G is an odd hole.*

Proof: Since G is not perfect, Theorem 3.2 implies that block G_1 or G_2 is not perfect, say G_1. Since G_1 is an induced subgraph of G and G is minimally imperfect, it follows that $G = G_1$. Since $|V_2| \geq 3$, V_2 induces a chordless path. Thus G is a minimally imperfect graph with a vertex of degree 2. This implies that G is an odd hole [32]. □

We end this section with another decomposition that preserves perfection. A graph G has a *6-join* if $V(G)$ can be partitioned into eight nonempty sets $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4$ with the property that, for any $x_i \in X_i$ $(i = 1, 2, 3)$ and $y_j \in Y_j$ $(j = 1, 2, 3)$, the graph induced by $x_1, y_1, x_2, y_2, x_3, y_3$ is a 6-hole and these kinds of edges are the only adjacencies between $X = X_1 \cup X_2 \cup X_3 \cup X_4$ and $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$.

Theorem 3.4 (Aossey and Vušković [2]) *No minimally imperfect graph contains a 6-join.*

If G contains a 6-join, define *blocks* G_X and G_Y as follows. G_X is the graph induced by $X \cup \{y_1, y_2, y_3\}$ where $y_j \in Y_j$ $(j = 1, 2, 3)$. Similarly G_Y is the graph induced by $Y \cup \{x_1, x_2, x_3\}$ where $x_i \in X_i$ $(i = 1, 2, 3)$. It can be shown [1] that G is perfect if and only if its blocks G_X and G_Y are perfect.

4 Skew Partition and Homogeneous Pair

A graph has a *skew partition* if its vertices can be partitioned into four nonempty sets A, B, C, D such that there are all the possible edges between A and B and no edges from C to D. It is easy to verify that the odd holes and their complements do not have a skew partition. Chvátal [6] conjectured that no minimally imperfect graph has a skew partition.

Theorem 4.1 (Skew Partition Theorem) (Chudnovsky and Seymour [13]) *No minimally imperfect graph has a skew partition.*

Chudnovsky and Seymour obtained this result as a consequence of their proof of the SPGC. In order to prove the SPGC, they first proved the following weaker result.

Theorem 4.2 (Chudnovsky and Seymour [12]) *A minimally imperfect Berge graph with smallest number of vertices does not have a skew partition.*

We do not give the proof of this difficult theorem here. Instead, we prove results due to Hoàng [26] on two special skew partitions called T-cutset and U-cutset respectively.

Assume that G is a minimally imperfect graph with skew partition A, B, C, D. Let $a = \omega(A)$, $b = \omega(B)$, $\omega = \omega(G)$ and $\alpha = \alpha(G)$. The vertex sets $A \cup B \cup C$ and $A \cup B \cup D$ induce perfect graphs G_1 and G_2 respectively and both of these graphs contain an ω-clique. Indeed, each vertex of a minimally imperfect graph belongs to ω ω-cliques [32] and, for $u \in C$, these ω-cliques are contained in G_1. For $u \in D$, they are contained in G_2.

Lemma 4.3 (Hoàng [26]) *Let $\mathcal{C}_i$ be an ω-coloring of G_i, for $i = 1, 2$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ cannot have the same number of colors in A.*

Proof: Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same number of colors in A and assume w.l.o.g. that these colors are $1, 2, \ldots, k$. Let K be the subgraph of G induced by the vertices with colors $1, 2, \ldots, k$ and let $H = G \setminus K$. Since every ω-clique of G is in G_1 or G_2, the largest clique in K has size k and the largest clique in H has size $\omega - k$. The graphs H and K are perfect since they are proper subgraphs of G. Color K with k colors and H with $\omega - k$ colors. Now G is colored with ω colors, a contradiction to the assumption that G is minimally imperfect. □

Lemma 4.4 *No ω-clique is contained in $A \cup B$.*

Proof: Suppose that a ω-clique were contained in $A \cup B$. Then any ω-coloring of G_i, for $i = 1, 2$, would contain a colors in A and $b = \omega - a$ colors in B, contradicting Lemma 4.3. □

Lemma 4.5 *Every α-stable set intersects $A \cup B$.*

Proof: By Lemma 4.4 applied to the complement graph, no α-stable set is contained in $C \cup D$. □

Lemma 4.6 *If some $u \in A$ has no neighbor in C, then there exists an ω-coloring of G_1 with b colors in B.*

Proof: Let $\mathcal{C}_1$ be an ω-coloring of G_1 with minimum number k of colors in B and suppose that this number is strictly greater than b. Consider the subgraph H of G_1 induced by the vertices colored with the colors of $\mathcal{C}_1$ that appear in B. The graph $H \cup u$ can be colored with k colors since it is perfect and has no clique of size greater than k. Keeping the other colors of $\mathcal{C}_1$ in $G_1 \setminus (H \cup u)$, we get an ω-coloring of G_1 with fewer colors on B than $\mathcal{C}_1$, a contradiction. □

Lemma 4.7 *If some $u \in A$ has no neighbor in C, then every vertex of A has a neighbor in D and every vertex of B has a neighbor in C.*

Proof: By Lemma 4.6, there exists an ω-coloring of G_1 with b colors in B. Thus, by Lemma 4.3, there exists no ω-coloring of G_2 with b colors in B. By Lemma 4.6, this implies that every vertex of A has a neighbor in D.

Suppose that $v \in B$ has no neighbor in C. In the complement graph, u and v are adjacent to all the vertices of C. By Lemma 4.3, $|A| \geq 2$ and $|B| \geq 2$. So $A' = A - u$, $B' = B - v$, $C' = C$, $D' = D \cup \{u, v\}$ form a skew partition. But u has no neighbor in B and v has no neighbor in A, contradicting the first part of the lemma. So every $v \in B$ has a neighbor in C. □

A *T-cutset* is a skew partition with $u \in C$ and $v \in D$ such that every vertex of A is adjacent to both u and v.

Lemma 4.8 (Hoàng [26]) *No minimally imperfect graph contains a T-cutset.*

Proof: In the complement, u and v contradict Lemma 4.7. □

A *U-cutset* is a skew partition with $u, v \in C$ such that every vertex of A is adjacent to u and every vertex of B is adjacent to v.

Lemma 4.9 (Hoàng [26]) *No minimally imperfect graph contains a U-cutset.*

Proof: In the complement, u and v contradict Lemma 4.7. □

We conclude this section with the notion of homogeneous pair introduced by Chvátal and Sbihi [8]. A graph G has a *homogeneous pair* if $V(G)$ can be partitioned into subsets A_1, A_2 and B, such that:

- $|A_1| + |A_2| \geq 3$ and $|B| \geq 2$.
- If a node of B is adjacent to a node of A_1 (A_2) then it is adjacent to all the nodes of A_1 (A_2).

Theorem 4.10 (Chvátal and Sbihi [8]) *No minimally imperfect graph contains a homogeneous pair.*

5 Decomposition of Berge Graphs

A graph is a *Berge graph* if it does not contain an odd hole or its complement. Clearly, all perfect graphs are Berge graphs. The SPGC states that the converse is also true.

Conjecture 5.1 (Conforti, Cornuéjols, Robertson, Seymour, Thomas and Vušković (2001)) **(Decomposition Conjecture)** *Every Berge graph G is basic or has a skew partition or a homogeneous pair, or G or $\bar{G}$ has a 2-join.*

This conjecture implies the SPGC. Indeed, suppose that the Decomposition Conjecture holds but not the SPGC. Then there exists a minimally imperfect graph G distinct from an odd hole or its complement. Choose G with the smallest number of vertices. G is a Berge graph and it cannot have a skew partition by Theorem 4.2. G cannot have an homogeneous pair by Theorem 4.10. Neither G nor $\bar{G}$ can have a 2-join by Corollary 3.3. So G must be basic by the Decomposition Conjecture. Therefore G is perfect, a contradiction.

Note that there are other decompositions that cannot occur in minimally imperfect Berge graphs, such as 6-joins (Theorem 3.4) or universal 2-amalgams [15] (universal 2-amalgams generalize both 2-joins and homogeneous pairs). These decompositions could be added to the statement of Conjecture 5.1 while still implying the SPGC. However they do not appear to be needed. Paul Seymour commented that homogeneous pairs might not be necessary either. In fact, we had initially formulated Conjecture 5.1 without homogeneous pairs. I added them to the statement to be on the safe side since they currently come up in the proof of the SPGC (see below).

Several special cases of Conjecture 5.1 are known. For example, it holds when G is a Meyniel graph (Burlet and Fonlupt [5] in 1984), when G is claw-free (Chvatal and Sbihi [9] in 1988 and Maffray and Reed [31] in 1999), diamond-free (Fonlupt and Zemirline [24] in 1987), bull-free (Chvátal and Sbihi [8] in 1987), or dart-free (Chvátal, Fonlupt, Sun and Zemirline [7] in 2000). All these results involve special types of skew partitions (such as star cutsets) and, in some cases, homogeneous pairs [8]. A special case of 2-join called augmentation of a flat edge appears in [31]. In 1999, Conforti and Cornuéjols [14] used more general 2-joins to prove Conjecture 5.1 for WP-free Berge graphs, a class of graphs that contains all bipartite graphs and all line graphs of bipartite graphs. This paper was the precursor of a sequence of decomposition results involving 2-joins:

Theorem 5.2 (Conforti, Cornuéjols and Vušković [18]) *A square-free Berge graph is bipartite, the line graph of a bipartite graph, or has a 2-join or a star cutset.*

Theorem 5.3 (Chudnovsky, Robertson, Seymour and Thomas [10]) *If G is a Berge graph that contains the line graph of a bipartite subdivision of a 3-connected graph, then G has a skew partition, or G or $\bar{G}$ has a 2-join or is the line graph of a bipartite graph.*

Given two vertex disjoint triangles a_1, a_2, a_3 and b_1, b_2, b_3, a *stretcher* is a graph induced by three chordless paths, $P^1 = a_1, \ldots, b_1$, $P^2 = a_2, \ldots, b_2$ and $P^3 = a_3, \ldots, b_3$, at least one of which has length greater than one, such that P^1, P^2, P^3 have no common vertices and the only adjacencies between the vertices of distinct paths are the edges of the two triangles. The next result is a real tour-de-force and a key step in the proof of the SPGC.

Theorem 5.4 (Chudnovsky and Seymour [12]) *If G is a Berge graph that contains a stretcher, then G is the line graph of a bipartite graph or G has a skew partition or a homogeneous pair, or G or $\bar{G}$ has a 2-join.*

A *wheel* (H, v) consists of a hole H together with a vertex v, called the *center*, with at least three neighbors in H. If v has k neighbors in H, the wheel is called a *k-wheel*. A *line wheel* is a 4-wheel (H, v) that contains exactly two triangles and these two triangles have only the center v in common. A *twin wheel* is a 3-wheel containing exactly two triangles. A *universal wheel* is a wheel (H, v) where the center v is adjacent to all the vertices of H. A *triangle-free wheel* is a wheel containing no triangle. A *proper wheel* is a wheel that is not any of the above four types. These concepts were first introduced in [14]. The following theorem generalizes an earlier result by Conforti, Cornuéjols and Zambelli [21] and Thomas [35].

Theorem 5.5 (Conforti, Cornuéjols and Zambelli [22]) *If G is a Berge graph that contains no proper wheels, stretchers or their complements, then G is basic or has a skew partition.*

The last step in proving the SPGC is the following difficult theorem.

Theorem 5.6 (Chudnovsky and Seymour [13]) *If G is a Berge graph that contains a proper wheel, but no stretchers or their complements, then G has a skew partition, or G or $\bar{G}$ has a 2-join.*

A monumental paper containing these results is forthcoming [11]. Independently, Conforti, Cornuéjols, Vušković and Zambelli [20] proved that the Decomposition Conjecture holds for Berge graphs containing a large class of proper wheels but, as of May 2002, they could not prove it for all proper wheels. Theorems 5.4, 5.5 and 5.6 imply that Conjecture 5.1 holds, and therefore the SPGC is true.

Corollary 5.7 (Strong Perfect Graph Theorem) *The only minimally imperfect graphs are the odd holes and their complements.*

Conforti, Cornuéjols and Vušković [19] proved a weaker version of the Decomposition Conjecture where "skew partition" is replaced by "double star cutset". A *double star* is a vertex set S that contains two adjacent vertices u, v and a subset of the vertices adjacent to u or v. Clearly, if G has a skew partition, then G has a double star cutset: Take $S = A \cup B$, $u \in A$ and $v \in B$. Although the decomposition result in [19] is weaker than Conjecture 5.1 for Berge graphs, it holds for a larger class of graphs than Berge graphs: By changing the decomposition from "skew partition" to "double star cutset", the result can be obtained for all odd-hole-free graphs instead of just Berge graphs.

Theorem 5.8 (Conforti, Cornuéjols and Vušković [19]) *If G is an odd-hole-free graph, then G is a bipartite graph or the line graph of a bipartite graph or the complement of the line graph of a bipartite graph, or G has a double star cutset or a 2-join.*

One might try to use Theorem 5.8 to construct a polynomial time recognition algorithm for odd-hole-free graphs. Conforti, Cornuéjols, Kapoor and Vušković [17] obtained a polynomial time recognition algorithm for the class of even-hole-free graphs. This algorithm is based on the decomposition of even-hole-free graphs by 2-joins, double star and triple star cutsets obtained in [16].

A useful tool for studying Berge graphs is due to Roussel and Rubio [34]. This lemma was proved independently by Robertson, Seymour and Thomas [33], who popularized it and named it *The Wonderful Lemma*. It is used repeatedly in the proofs of Theorems 5.3-5.6.

Lemma 5.9 (The Wonderful Lemma) (Roussel and Rubio [34]) *Let G be a Berge graph and assume that $V(G)$ can be partitioned into a set S and an odd chordless path $P = u, u', \ldots, v', v$ of length at least 3 such that u, v are both adjacent to all the vertices in S and $\bar{G}(S)$ is connected. Then one of the following holds:*

(i) An odd number of edges of P have both ends adjacent to all the vertices in S.

(ii) P has length 3 and $\bar{G}(S \cup \{u', v'\})$ contains an odd chordless path between u' and v'.

(iii) P has length at least 5 and there exist two nonadjacent vertices x, x' in S such that $(V(P) \setminus \{u, v\}) \cup \{x, x'\}$ induces a path.

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